Chapter 17

Thinking about Chance

Random means although an individual outcome is unknown, there is still a regular distribution of outcomes in a large number of repetitions. In particular, repeatedly sampling, sample proportion of an outcome will approach and stay close to expected population probability, a number between 0 and 1. This long run result is called *law of averages*. *Personal probability* is also a number between 0 and 1 that expresses an individual's belief in likelihood of an outcome. *Bayes' theorem* is a formal way of adjusting for personal probabilities.

Exercise 17.1 (Thinking about Chance)

1. *Probability or personal probability?*

   (a) **probability / personal probability**
   Chance of tossing a head.

   (b) **probability / personal probability**
   Chance of drawing an ace from a deck of cards.

   (c) **probability / personal probability**
   Joe's opinion on chance Chicago Cubs go to World Series next year.

   (d) **probability / personal probability**
   Alice's opinion on chance it will rain on Friday.

   (e) **probability / personal probability**
   Chance of sinking a free throw in basketball.

   (f) **probability / personal probability**
   Newscaster's opinion on chance Shaq O'Neil's of sinking next free throw.

   (g) **probability / personal probability**
   Chance of car accident.

   (h) **probability / personal probability**
   Your opinion on chance of car accident.

2. *Chance error and % chance error and law of averages: coin tossing* Results of coin tossing experiment given in table and figure below.

no of tosses	no of heads	expected heads	difference	% difference
10	4	5	-1	$-\frac{1}{10} \times 100 \approx -10.0\%$
50	25	25	0	$\frac{0}{50} \times 100 \approx 0.0\%$
100	44	50	-6	$-\frac{6}{100} \times 100 \approx -6.0\%$
500	255	250	5	$\frac{5}{500} \times 100 \approx 1.0\%$
1000	502	500	2	$\frac{2}{1000} \times 100 \approx 0.2\%$
5000	2533	2500	33	$\frac{33}{5000} \times 100 \approx 0.7\%$
10000	4938	5000	-62	$-\frac{62}{10000} \times 100 \approx -0.6\%$

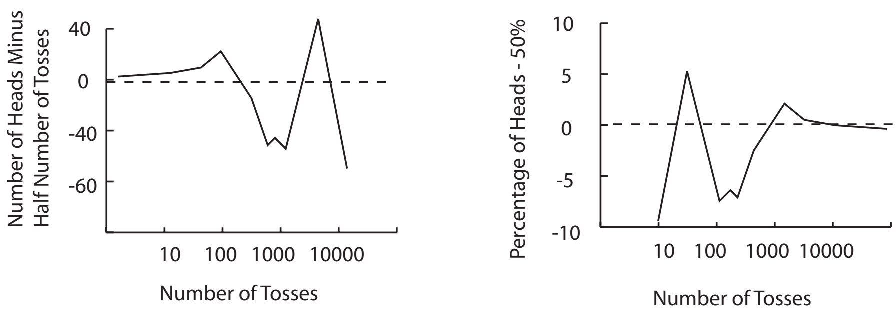


Figure 17.1 (Law of averages: coin tossing)

(a) After 10 tosses, **4** / **5** / **6** came up heads;
after 10000 tosses, **4938** / **5000** / **5009** came up heads.

(b) If fair, after 10 tosses, we *expect* **4** / **5** / **6** heads;
after 10000 tosses, we expect **4938** / **5000** / **5009** heads.

(c) *Chance error* after 10 tosses $4 - 5 =$ **−1** / **0** / **1** heads;
after 10000 tosses, chance error $4938 - 5000 =$ **−62** / **−5** / **38**.

(d) Chance error **small** / **large** / **same** for a small number of tosses but **smaller** / **same** / **larger** for a large number of tosses. See left figure.

(e) *Percentage of chance error*, 10 tosses, is $\frac{4-5}{10} \times 100 =$ **−10%** / **0%** / **10%**;
after 10000 tosses, $\frac{4938-5000}{10000} \times 100 \approx$ **−0.6%** / **0.2%** / **2.0%**.

(f) Percentage of chance error **increases** / **decreases** / **remains same** as number of tosses increases. This is called *law of averages*. See right figure.

3. *Chance error, % chance error and law of averages: die rolling* Results of die rolling experiment given in table below.

no of rolls	no of 4s	expected 4s	difference	% difference
60	9	10	-1	$-\frac{1}{60} \times 100 \approx -1.7\%$
120	21	20	1	$\frac{1}{120} \times 100 \approx 0.8\%$
180	34	30	4	$\frac{4}{180} \times 100 \approx 2.2\%$
240	42	40	2	$\frac{2}{240} \times 100 \approx 0.8\%$
600	110	100	10	$\frac{10}{600} \times 100 \approx 1.7\%$
1200	234	200	34	$\frac{34}{1200} \times 100 \approx 2.8\%$
2400	390	400	-10	$-\frac{10}{2400} \times 100 \approx -0.4\%$

(a) After 60 rolls, **6** / **7** / **9** are 4s;
after 2400 rolls, **390** / **400** / **409** are 4s.

(b) If fair, after 60 rolls, we *expect* $\frac{60}{6}$ = **4** / **5** / **10** are 4s;
after 2400 rolls, we expect $\frac{2400}{6}$ = **390** / **400** / **409** are 4s.

(c) *Chance error* after 60 rolls $9 - 10 =$ **−1** / **0** / **1** are 4s;
after 2400 rolls, chance error $390 - 400 =$ **−10** / **−5** / **10**.

(d) Chance error **small** / **large** / **same** for a small number of rolls but **smaller** / **same** / **larger** for a large number of rolls.

(e) *Percentage of chance error*, 60 rolls, is $\frac{9-10}{60} \times 100 =$ **−1.7%** / **0%** / **1.7%**;
after 2400 rolls, $\frac{390-400}{2400} \times 100 \approx$ **−0.4%** / **−0.0%** / **2.0%**.

(f) Percentage of chance error **increases** / **decreases** / **remains same** as number of rolls increases: *observed variability* around $\frac{1}{6}$ decreases.

(g) So *observed* proportion of 4s **converges towards** / **diverges from** the *expected* proportion of 4s, $\frac{1}{6}$, by the law of large numbers.

(h) Law of large numbers **does** / **does not** describe the *expected variability* of chance error around $\frac{1}{6}$, but is known to be a bell-shaped histogram.

4. *More on chance error and % chance error: rolling dice.* A die is rolled a number of times. In sixty rolls, you would expect "four" to come up in 10 rolls; in 6000 rolls, you would expect "four" to come up 1000 times.

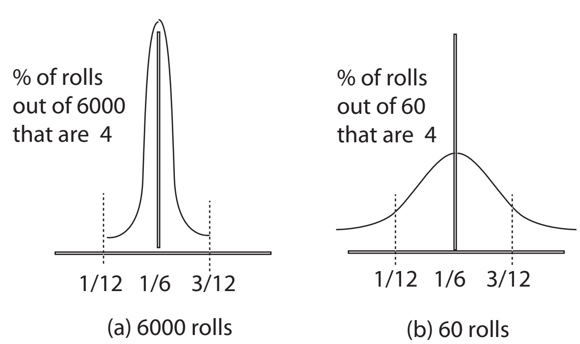


Figure 17.2 (Law of averages: rolling dice)

(a) Percentage of chance error for 6000 rolls smaller than for 60 rolls because of law of averages, so **larger** / **smaller** chance getting *around* $\frac{1}{6}$th of 6000 rolls as "fours" than getting *around* $\frac{1}{6}$th of 60 rolls as "fours".

(b) **Larger** / **smaller** chance of *not* getting around $\frac{1}{6}$th of 6000 rolls as "fours" than of *not* getting around $\frac{1}{6}$th of 60 rolls as "fours".

(c) **Larger** / **smaller** chance of getting more than $\frac{3}{12}$th of 6000 rolls as "fours" than getting more than $\frac{3}{12}$th of 60 rolls as "fours".

(d) Chance error for 6000 rolls *larger* than for 60 rolls, so **larger** / **smaller** chance getting *exactly* $\frac{1}{6}$th of 6000 rolls as "fours" than getting *exactly* $\frac{1}{6}$th of 60 rolls as "fours".

Chapter 18

Probability Models

We look at terminology related to *probability models* and then a number of related rules, given below.

- Probability of any event, E, must be between 0 and 1.
- Sum of probability of all outcomes equals 1.
- Probability event does *not* occur equals 1 (one) minus it does occur.
- If no outcomes in common, probability one or other event occurs equals sum of individual probabilities.

We also look at odds. On the one hand, *odds "a to b" for event E* converted into probabilities

$$P(E) = \frac{a}{a+b}, \qquad P(\mathit{not}\, E) = \frac{b}{a+b}.$$

On the other hand, probability $P(E)$ converted into *odds "a to b" for event E*

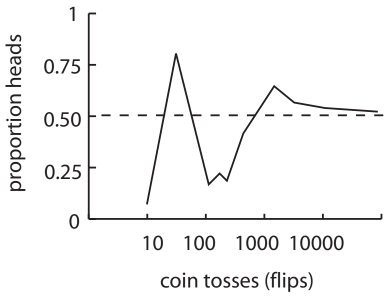
$$\frac{P(E)}{P(\mathit{not}\, E)} = \frac{a}{b}, \quad \text{or "}a\text{ to }b\text{"}$$

and *odds against event E* by $\frac{P(\mathit{not}\, E)}{P(E)}$. We then look at a *sampling distribution*, a probability model of a statistic.

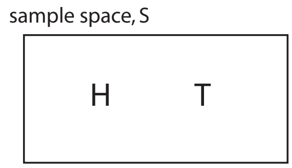
Exercise 18.1 (Probability Models)

1. *Terminology.*

   (a) *Coin tossing.*



coin	frequency	empirical probability
heads	5065	5065/10000 = 0.5065
tails	4935	4935/10000 = 0.4935



coin	number outcomes	classical probability
heads	1	1/2 = 0.5
tails	1	1/2 = 0.5

Figure 18.1 (Terminology: coin tossing)

i. **True** / **False**. As number of coin tosses increase, proportion of total tosses which are heads will approach and stay close to expected *probability* of tossing a head. This is an example of *law of averages*.

ii. *Approximating probability with empirical approach.*
Since 5065 tosses of 10000 tosses are heads we approximate probability of tossing a head by $P(H) \approx \frac{5065}{10000} = \mathbf{0.4935} \ / \ \mathbf{0.5} \ / \ \mathbf{0.5065}$.
Accuracy of empirical method depends on how well we can consistently flip the coin.

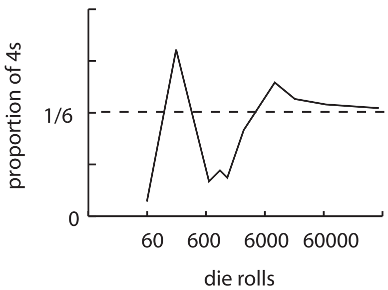
iii. *Calculating probability with classical method .*
Since a coin can be tossed only as a head (H) or tail (T) and *assuming equally* likely outcomes, $P(H) = \frac{1}{2} = \mathbf{0.4935} \ / \ \mathbf{0.5} \ / \ \mathbf{0.5065}$.
Accuracy of classical method depends on accuracy of equally likely outcomes assumption.

iv. *Classical method: experiment.*
Flipping a coin is a *probability experiment*. It is generally *un*known, when flipping a coin, whether coin comes up heads (Hs) or tails (Ts). However, it is known there are only two possible outcomes (choose one) **{H, T}** / **{H, H}** / **{T, T}** .

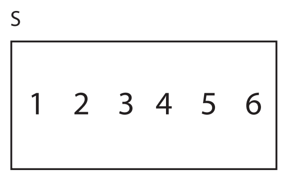
v. *Classical method: sample space .*
Sample space for flipping a coin is $S = \{\text{H, T}\}$. Sample space is (choose one) **set** / **subset** / **element** of all possible outcomes.
Displayed as a *Venn diagram* in figure.

vi. *Classical method: event, simple event.*
Flipping a head {H} is an example of an *event*, E. An event is a (choose one) **set** / **subset** / **element** of all possible outcomes. Since only one outcome, {H}, this event is a *simple* event, $E = e_1 = \{H\}$

(b) *Die rolling.*



die	frequency	empirical probability
1	9965	9965/60000 = 0.166
2	10035	10035/60000 = 0.167
3	10100	10100/60000 = 0.168
4	9900	9900/60000 = 0.165
5	10200	10200/60000 = 0.170
6	9800	9800/60000 = 0.163



die	number outcomes	classical probability
1	1	1/6
2	1	1/6
3	1	1/6
4	1	1/6
5	1	1/6
6	1	1/6

Figure 18.2 (Terminology: die rolling)

i. **True** / **False**. As number of die rolls increase, proportion of total rolls which are 4s will approach and stay close to expected probability of rolling a 4. This is an example of *law of averages*.

ii. *Approximating probability with empirical approach.*
Since 9900 tosses of 60000 rolls are 4s we approximate probability of rolling a 4 by $P(4) \approx \frac{9900}{60000} =$ (choose one) **0.163** / **0.164** / **0.165**.

iii. *Calculating probability with classical method.*
Since a die can be rolled either 1, 2, 3, 4, 5 or 6 and assuming equally likely outcomes, $P(4) = \frac{1}{6} \approx$ (choose one) **0.163** / **0.165** / **0.167**.

iv. *Classical method: experiment.*
Die rolling is *probability experiment*. Value to be rolled unknown, but six possible outcomes (choose one) **known** / **unknown**.

v. *Classical method: sample space.*
Sample space is (choose one) $\{\mathbf{1, 2, 3, 4, 5}\}$ / $\{\mathbf{1, 2, 3, 4, 5, 6}\}$.

vi. *Classical method: event, simple event.*
Examples of events are
(choose one or *more*!) $\{\mathbf{1}\}$ / $\{\mathbf{1, 2}\}$ / $\{\mathbf{1, 2, 3, 6}\}$.

2. *Rules of probability.*

(a) *Box of tickets.*
Box has six tickets. Each ticket has 1, 2 or 3 with one of three subscripts: *a*, *b* or *c*. One ticket drawn from box at random.

1_a 2_a 1_b 3_b 2_c 3_c

Probability ticket is

i. "1" is $\mathbf{\frac{1}{6}}$ / $\mathbf{\frac{2}{6}}$ / $\mathbf{\frac{3}{6}}$.
ii. "a" is $\mathbf{\frac{1}{6}}$ / $\mathbf{\frac{2}{6}}$ / $\mathbf{\frac{3}{6}}$.
iii. "1" *or* "2" is $\frac{2}{6} + \frac{2}{6} =$ $\mathbf{\frac{2}{6}}$ / $\mathbf{\frac{3}{6}}$ / $\mathbf{\frac{4}{6}}$.
iv. "1" *or* "2" *or* "3" is $\frac{2}{6} + \frac{2}{6} + \frac{2}{6} =$ $\mathbf{\frac{4}{6}}$ / $\mathbf{\frac{5}{6}}$ / $\mathbf{\frac{6}{6}}$.
v. "a" *or* "b" is $\frac{2}{6} + \frac{2}{6} =$ $\mathbf{\frac{2}{6}}$ / $\mathbf{\frac{3}{6}}$ / $\mathbf{\frac{4}{6}}$.
vi. *not* an "a" is $1 - \frac{2}{6} =$ $\mathbf{\frac{2}{6}}$ / $\mathbf{\frac{3}{6}}$ / $\mathbf{\frac{4}{6}}$.

(b) *Box of coins.*
Coins are sampled at random from box.

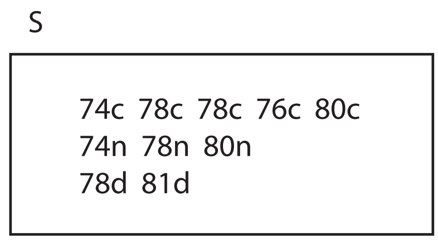


Figure 18.3 (Rules of probability: box of coins)

Chance coin is

i. cent (penny) is $\mathbf{\frac{3}{10}}$ / $\mathbf{\frac{4}{10}}$ / $\mathbf{\frac{5}{10}}$ / $\mathbf{\frac{6}{10}}$.
ii. 1978 is $\mathbf{\frac{3}{10}}$ / $\mathbf{\frac{4}{10}}$ / $\mathbf{\frac{5}{10}}$ / $\mathbf{\frac{6}{10}}$.
iii. cent *or* nickel is $\mathbf{\frac{5}{10}}$ / $\mathbf{\frac{6}{10}}$ / $\mathbf{\frac{7}{10}}$ / $\mathbf{\frac{8}{10}}$.
iv. cent *or* nickel *or* dime is $\mathbf{\frac{7}{10}}$ / $\mathbf{\frac{8}{10}}$ / $\mathbf{\frac{9}{10}}$ / $\mathbf{\frac{10}{10}}$.
v. *not* a dime $\mathbf{\frac{5}{10}}$ / $\mathbf{\frac{6}{10}}$ / $\mathbf{\frac{7}{10}}$ / $\mathbf{\frac{8}{10}}$.

3. *Odds and probability.*

(a) *Odds and Tennis.*

i. *Odds To Probability.*
Odds 1 to 3 for Federer winning a game of tennis same as $\frac{1}{1+3} =$ $\mathbf{\frac{1}{4}}$ / $\mathbf{\frac{1}{5}}$ / $\mathbf{\frac{1}{6}}$ chance Federer wins.
ii. *Odds To Probability.*
Odds 7 to 3 for Federer winning a game of tennis same as $\frac{7}{7+3} =$ $\mathbf{\frac{3}{7}}$ / $\mathbf{\frac{7}{3}}$ / $\mathbf{\frac{7}{10}}$ / $\mathbf{\frac{3}{10}}$ chance Federer wins.
iii. *Odds To Probability.*
Odds 7:100 for Federer winning a game of tennis same as $\mathbf{\frac{7}{99}}$ / $\mathbf{\frac{7}{100}}$ / $\mathbf{\frac{7}{103}}$ / $\mathbf{\frac{7}{107}}$ chance Federer wins.
iv. *Probability to Odds.*
Chance $\frac{3}{4}$ Federer wins same as odds $\frac{3/4}{1-3/4} = \frac{3}{1}$ or

A. 1 to 3 odds for Federer winning
B. 3 to 1 odds for Federer winning
C. 4 to 1 odds for Federer winning
D. 1 to 4 odds for Federer winning

v. *Probability to Odds.*
Chance $\frac{2}{11}$ Federer wins same as odds $\frac{2/11}{1-2/11} = \frac{2}{9}$ or
A. 2 to 11 odds for Federer winning
B. 11 to 9 odds for Federer winning
C. 2 to 9 odds for Federer winning
D. 9 to 11 odds for Federer winning

vi. *Probability to Odds.*
Chance $\frac{9}{13}$ Federer wins same as
A. 2 to 11 odds for Federer winning
B. 9 to 4 odds for Federer winning
C. 6 to 9 odds for Federer winning
D. 9 to 11 odds for Federer winning

(b) *Odds and Rental Car Agencies.*

i. *Probability to Odds.*
60% chance of renting from car agency A same as odds $\frac{0.6}{1-0.6} = \frac{6}{4}$ or
A. 2 to 11 odds for renting from A
B. 9 to 4 odds for renting from A
C. 3 to 2 odds for renting from A
D. 4 to 6 odds for renting from A

ii. *Probability to Odds.*
37% chance of renting from agency A same as odds $\frac{0.37}{1-0.37} = \frac{0.37}{0.63}$ or
A. 2 to 11 odds for renting from A
B. 37 to 63 odds for renting from A
C. 63 to 37 odds for renting from A
D. 4 to 6 odds for renting from A

iii. *Probability to Odds.*
37% chance of renting from A same as odds $\frac{1-0.37}{0.37} = \frac{0.63}{0.37}$ or
A. 2 to 11 odds *against* renting from A
B. 37 to 63 odds *against* renting from A
C. 63 to 37 odds *against* renting from A
D. 4 to 6 odds *against* renting from A

iv. *Odds To Probability.*
Odds 7 to 93 for renting from agency A same as
$\frac{7}{7+93} = \mathbf{\frac{7}{99}}$ / $\mathbf{\frac{7}{100}}$ / $\mathbf{\frac{7}{101}}$ / $\mathbf{\frac{7}{107}}$ chance of renting from A.

4. *Probability distribution: number of seizures*

number seizures	probability
0	0.17
2	0.21
4	0.18
6	0.11
8	0.16
10	0.17

(a) Chance person has 8 epileptic seizures is **0.14** / **0.15** / **0.16** / **0.17**.

(b) Chance person has *at most* 4 seizures is **0.17** / **0.21** / **0.56** / **0.67**.

(c) Chance person has *at least* 4 seizures is **0.21** / **0.38** / **0.56** / **0.62**.

(d) Probability distribution because probabilities add to **0.97** / **0.98** / **1**.

(e) Chance person has 2.1 seizures is **0** / **0.21** / **0.56** / **0.67**.

(f) *Probability graph, number of seizures.*

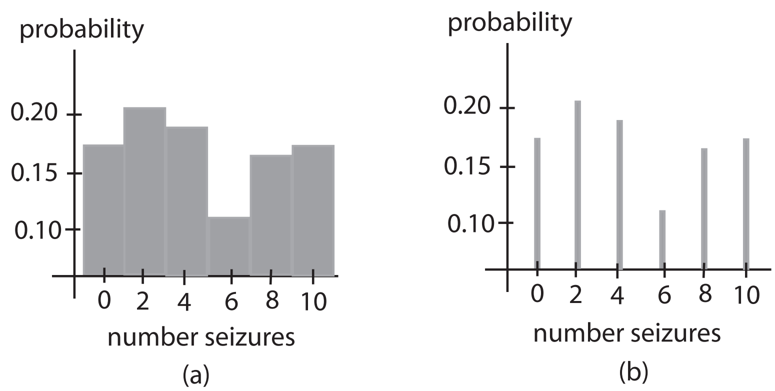


Figure 18.4 (Probability distribution: seizures)

Best graph for probability distribution of seizures: **(a)** / **(b)**

5. *Probability distribution: number of bikes on bike rack*

bikes	5	6	7	8	9
probability	$\frac{1}{5} = 0.2$	0.2	0.2	0.2	0.2

For example, there is a 20% chance 6 bikes are on bike rack.

(a) *Probability graph, number of bikes.*

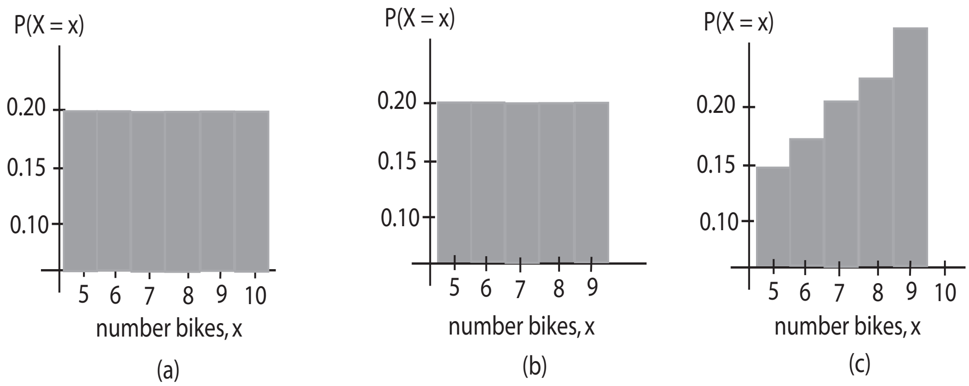


Figure 18.5 (Probability distribution of number of bikes.)

Probability distribution, number of bikes: **(a)** / **(b)** / **(c)**.

(b) Chance bike rack has 8 bicycles is $\mathbf{\frac{1}{5}}$ / $\mathbf{\frac{2}{5}}$ / $\mathbf{\frac{3}{5}}$ / $\mathbf{\frac{4}{5}}$.

(c) Chance bike rack has *at most* 6 bicycles is $\mathbf{\frac{1}{5}}$ / $\mathbf{\frac{2}{5}}$ / $\mathbf{\frac{3}{5}}$ / $\mathbf{\frac{4}{5}}$.

(d) Chance bike rack has *at least* 6 bicycles is $\mathbf{\frac{1}{5}}$ / $\mathbf{\frac{2}{5}}$ / $\mathbf{\frac{3}{5}}$ / $\mathbf{\frac{4}{5}}$.

(e) Chance bike rack has *more than* 6 bicycles is $\mathbf{\frac{1}{5}}$ / $\mathbf{\frac{2}{5}}$ / $\mathbf{\frac{3}{5}}$ / $\mathbf{\frac{4}{5}}$.

6. *Sampling distribution: proportion of wins.*
Lawyer estimates she wins 40% of her cases ($p = 0.4$). Let $\hat{p}$ represent proportion of wins of n cases. Use Normal with mean 0.4 and SD 0.07 to approximate various chances related to proportion $\hat{p}$ of wins.

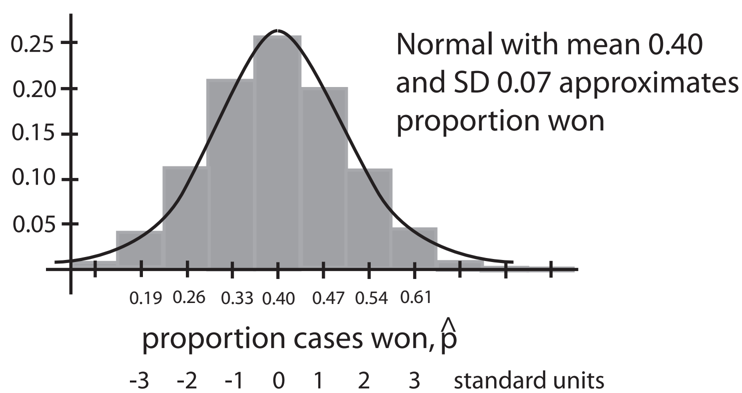


Figure 18.6 (Sampling distribution: proportion of wins.)

(a) The 68-95-99.7 rule says 95% chance proportion of wins within 2 SDs of mean proportion; that is, in interval $(0.40 - 2 \times 0.07, 0.40 + 2 \times 0.07) =$ $\mathbf{(0.33, 0.47)}$ / $\mathbf{(0.26, 0.54)}$ / $\mathbf{(0.12, 0.68)}$.

(b) Chance proportion of wins less than 0.26: $\frac{0.05}{2} =$ **2.5%** / **5%** / **7.5%**.

(c) Chance proportion of wins greater than 0.54: $\frac{0.05}{2} =$ **2.5%** / **5%** / **7.5%**.

(d) Using Table B, chance proportion of wins less than 0.20, since standard score is $\frac{0.20-0.40}{0.07} \approx -2.9$, is **0.18%** / **0.19%** / **0.20%**.

7. *Another sampling distribution: winning at the track.*
Bookie estimates Tip-Top wins 25% of his races. Use Normal with mean 0.25 and SD 0.05 to approximate various chances related to proportion $\hat{p}$ of wins.

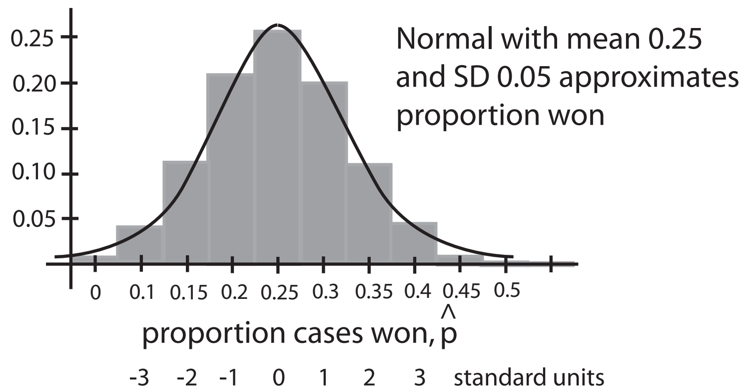


Figure 18.7 (Sampling distribution: proportion of wins.)

(a) The 68-95-99.7 rule says 68% chance proportion of wins within 1 SD of mean proportion; that is, in interval $(0.25 - 0.05, 0.25 + 0.05) =$ $\mathbf{(0.20, 0.30)}$ / $\mathbf{(0.15, 0.35)}$ / $\mathbf{(0.10, 0.40)}$.

(b) Chance proportion of wins less than 0.20: $\frac{0.32}{2} =$ **10%** / **12%** / **16%**.

(c) Chance proportion of wins greater than 0.30: $\frac{0.32}{2} =$ **10%** / **12%** / **16%**.

(d) Using Table B, chance proportion of wins less than 0.38, since standard score is $\frac{0.38-0.25}{0.05} \approx 2.6$, is **99.32%** / **99.53%** / **99.83%**.