Lecture Notes

For Mathematics 223

Introductory Analysis I

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by

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Preface

The point of this course is to acquaint a student with some of the ideas, definitions and concepts of calculus and its applications, particularly those connected to the derivative and exponential functions.

These lecture notes are a necessary component for a student to successfully complete this course. Without these lecture notes, a student will not be able to participate in the course.

- These lecture notes are *based* on the text.
- Although the material covered in each is very similar, the *presentation* of the material in the lecture notes is quite different from the presentation given in the text. The text consists essentially of definitions, formulas, worked out examples and exercises; these lecture notes, on the other hand, consists *solely* of exercises to be worked out by the student.
- A student is to use these lecture notes to follow along with during a lecture.
- There are different kinds of exercises, including multiple choice, true/false, matching and fill–in–the–blank.
- Each week, I recommend you read the text, answer the questions given here in the lecture notes and then do the online homework and quiz or test assignments, in that order.

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R. Algebra Reference

Topics in algebra relevant to calculus are reviewed in this chapter.

R.1 Polynomials

An example of a *term* is $4x^2$, where 4 is the *coefficient*, x is the *variable* and 2 is the *exponent*. An example of two *like* terms are $4q^2$ and $-9q^2$; two *unlike* terms are $4q^2$ and $-9q^3$. The term z^3 means $z \cdot z \cdot z$. A *polynomial* is a term or finite sum of terms where all variables have whole number exponents and no variable is in a denominator such as

$$3x^2 + 2x - 1, \qquad 3mn^2 + n^2, \qquad 8.$$

A *binomial* has exactly two terms, such as $3x + 4y$ or $5y - 1$. *Order of operations* include calculating:

- expressions inside *parentheses* first; expressions in numerators and denominators of fractions are treated as though in parentheses
- *powers* next, from left to right,
- *multiplications* and *divisions* next, left to right,
- *additions* and *subtractions* next, left to right.

Multiplying two binomials uses *FOIL* (first, outside, inside, last) rule. Properties of addition and multiplication operations include

- *commutative* property: $a + b = b + a$, $ab = ba$
- *associative* property: $(a + b) + c = a + (b + c)$, $(ab)c = a(bc)$
- *distributive* property: $a(b + c) = ab + ac$

Exercise R.1 (Polynomials)

1. *Properties of operations*

(a) **commutative / associative / distributive / none**
$3x + 5 = 5 + 3x$

(b) **commutative / associative / distributive / none**
$3x \cdot 5 \neq 5 + 3x$

(c) **commutative / associative / distributive / none**
$(3x + 5y) - 3 = 3x + (5y - 3)$

(d) **commutative / associative / distributive / none**
$3x(4y + 5) = 12xy + 15x$

(e) **commutative / associative / distributive / none**
$(3x)x = 3x^2$

(f) **commutative / associative / distributive / none**
$(x + 5)^2 = (x + 5)(x + 5) = x(x + 5) + 5(x + 5) = x^2 + 10x + 25$

2. *Adding, Subtracting and Multiplying Polynomials*

(a) $(3x^2 + 5x + 10) + (-2x^2 - 5x + 3) =$ (choose one)
(i) $\boldsymbol{x^2 + 13}$
(ii) $\boldsymbol{5x^2 + 10x + 7}$
(iii) $\boldsymbol{5x^2 + 10x}$

(b) $2m(4m^2 - 8m + 5) =$ (choose one)
(i) $\boldsymbol{8m^3 - 16m^2 + 10m}$
(ii) $\boldsymbol{8m^3 + 10m}$
(iii) $\boldsymbol{8m^2 - 16m + 10}$

(c) $6(2q^3 - 2q - 10) - (2q^3 + 4q - 5) =$ (choose one)
(i) $\boldsymbol{10q^3 - 10q - 65}$
(ii) $\boldsymbol{10q^3 - 16q - 55}$
(iii) $\boldsymbol{10q^3 + 16q + 55}$

(d) $0.27(1.6x^2 - 2.1x + 1) + (9x^2 - 4x - 1.321) =$ (choose one)
(i) $\boldsymbol{9.432x^2 - 4.567x - 1.051}$
(ii) $\boldsymbol{x^2 + x - 1}$
(iii) $\boldsymbol{9.432x^2 + 4.567x + 1.051}$

(e) $(k + 1)(k - 1) =$ (choose one)
(i) $\boldsymbol{k^2 - 2k - 1}$
(ii) $\boldsymbol{k^2 + 2k + 1}$
(iii) $\boldsymbol{k^2 - 1}$

(f) $(3c - 4t)(4c + 10t) =$ (choose one)
(i) $\boldsymbol{40t^2 + 12ct - 40t^2}$
(ii) $\boldsymbol{14c^2 + 12ct - 40t}$
(iii) $\boldsymbol{12c^2 + 14ct - 40t^2}$

(g) $\left(\frac{2}{3}y + \frac{1}{7}z\right)\left(\frac{4}{3}y - \frac{2}{7}z\right) =$ (choose one)
(i) $\mathbf{\frac{8}{9}y^2 - \frac{2}{49}z^2}$
(ii) $\mathbf{\frac{8}{9}y^2 - \frac{4}{21}yz - \frac{2}{49}z^2}$
(iii) $\mathbf{\frac{2}{49}y^2 - \frac{8}{9}z^2}$

(h) $(3a + 4)(8a^2 - 3a + 1) =$ (choose one)
(i) $\mathbf{24a^3 - 9a + 4}$
(ii) $\mathbf{24a^3 + 23a^2 - 9a + 4}$
(iii) $\mathbf{24a^3 + 23a + 4}$

(i) $(x + 2)(x + 4)(x + 6) =$ (choose one)
(i) $\mathbf{x^3 + 10x^2 + 44x + 48}$
(ii) $\mathbf{x^3 + 12x^2 + 44x + 48}$
(iii) $\mathbf{2x^3 + 12x^2 + 44x + 48}$

(j) $(5a + 5)^2 =$ (choose one)
(i) $\mathbf{50a^2 + 25a + 50}$
(ii) $\mathbf{25a^2 + 50a + 25}$
(iii) $\mathbf{50a^2 + 25a + 25}$

R.2 Factoring

Factoring is when a polynomial is written as the product of at least two other polynomials. For example, since $12 = 6 \cdot 2$, 6 and 2 are factors of the (simple) polynomial 12. Another example of factoring is given by the following binomial

$$4x + 20 = 4(x + 5),$$

where 4 is, in this case, the *greatest common factor*. Typically, $4x + 20$ would *not* be factored with anything less than the greatest common factor, such as 2, which would result in $2(2x + 10)$. Polynomials with more than two terms are often trickier to factor; for example, the trinomial,

$$y^2 + 5y + 6 = (y + 2)(y + 3),$$

where, notice, in this case (luckily) y^2 has coefficient 1, and both coefficient of y, 5, is the *sum* of 2 and 3, $5 = 2 + 3$ and 6 is the *product* of 2 and 3, $6 = 2 \cdot 3$. As another another example, trinomial

$$12c^2 + 14ct - 40t^2 = (3c - 4t)(4c + 10t),$$

because $12c^2$ has factors $3c$, $4c$ (but not, say, $4c$, $3c$ or $2c$, $6c$) and $-40t^2$ has factors $-4t$, $10t$ (but not, say, $4t$, $-10t$). Some *special factorizations* are

$$x^2 - y^2 = (x + y)(x - y) \quad \text{difference of two squares}$$

$$\begin{aligned} x^2+2xy+y^2 &= (x+y)^2 \quad \text{perfect square} \\ x^3-y^3 &= (x-y)(x^2+xy+y^2) \quad \text{difference of two cubes} \\ x^3+y^3 &= (x+y)(x^2-xy+y^2) \quad \text{sum of two cubes} \end{aligned}$$

Polynomials which cannot be factored, *except* by factor 1, are called *prime*.

Exercise R.2 (Factoring)

1. factoring, with largest common factor, $6b^3+12b^2=$ (choose one)
   (i) $\boldsymbol{6b(b^2+2)}$
   (ii) $\boldsymbol{3b^2(2b+4)}$
   (iii) $\boldsymbol{6b^2(b+2)}$

2. $8m^3-16m^2+10m=$ (choose one)
   (i) $\boldsymbol{m(4m^2-8m+5)}$
   (ii) $\boldsymbol{2m(2m^2-4m+10)}$
   (iii) $\boldsymbol{2m(4m^2-8m+5)}$

3. $x^2+9x+8=$ (choose one)
   (i) $\boldsymbol{(x+1)(x+8)}$
   (ii) $\boldsymbol{(x-1)(x+9)}$
   (iii) $\boldsymbol{(x+2)(x+4)}$

4. $x^2+8x-9=$ (choose one)
   (i) $\boldsymbol{(x+1)(x+8)}$
   (ii) $\boldsymbol{(x-1)(x+9)}$
   (iii) $\boldsymbol{(x+2)(x+4)}$

5. $2x^2+3xy-27y^2=$ (choose one)
   (i) $\boldsymbol{(2x+3y)(x-9y)}$
   (ii) $\boldsymbol{(x-3y)(2x+9y)}$
   (iii) $\boldsymbol{(x+3y)(2x-9y)}$

6. special case, difference of two squares: $k^2-m^2=$ (choose one)
   (i) $\boldsymbol{(k+m)^2}$
   (ii) $\boldsymbol{(k-m)(k+m)}$
   (iii) $\boldsymbol{(k+m)(k-m)}$

7. special case, difference of two squares: $k^2-1=k^2-1^2=$ (choose one)
   (i) $\boldsymbol{(k+1)(k-1)}$
   (ii) $\boldsymbol{(k+1)^2}$
   (iii) $\boldsymbol{(k+1)(k+2)}$

8. special case, difference of two squares: $16k^2-49m^2=(4k)^2-(7m)^2=$
   (i) $\boldsymbol{(4k+7m)^2}$

(ii) $\boldsymbol{(4k+7m)(4k-7m)}$
(iii) $\boldsymbol{(4k-7m)(4k-7m)}$

9. special case, difference of two cubes: $p^3 - q^3 =$
(i) $\boldsymbol{(p+q)(p^2-pq-q^2)}$
(ii) $\boldsymbol{(p+q)(p^2-pq+q^2)}$
(iii) $\boldsymbol{(p-q)(p^2+pq+q^2)}$

10. special case, sum of two cubes: $p^3 + q^3 =$
(i) $\boldsymbol{(p+q)(p^2-pq+q^2)}$
(ii) $\boldsymbol{(p-q)(p^2+pq+q^2)}$
(iii) $\boldsymbol{(p+q)(p^2-pq-q^2)}$

11. special case, sum of two cubes: $64p^3 + 27q^3 = (4p)^3 + (3q)^3 =$
(i) $\boldsymbol{(4p+3q)(4p^2-12pq+3q^2)}$
(ii) $\boldsymbol{(4p+3q)(4p^2-12pq-9q^2)}$
(iii) $\boldsymbol{(4p+3q)(16p^2-12pq+9q^2)}$

12. special case, perfect square: $z^2 + 4zx + 4x^2 = z^2 + 2(2x)z + (2x)^2 =$
(i) $\boldsymbol{(z+2x)^2}$
(ii) $\boldsymbol{(z+x)(z-x)}$
(iii) $\boldsymbol{(z+2x)(z+x)}$

13. $s^2 + t^2 =$ (choose one)
(i) $\boldsymbol{1 \cdot (s^2+t^2)}$ (prime)
(ii) $\boldsymbol{(s+t)^2}$
(iii) $\boldsymbol{(s-t)(s+t)}$

14. special case, difference of two squares, twice:
$s^4 - t^4 = (s^2)^2 - (t^2)^2 = (s^2 - t^2)(s^2 + t^2) =$
(i) $\boldsymbol{(s-t)(s+t)\,(s^2+t^2)}$
(ii) $\boldsymbol{(s^2+t^2)}$ (prime)
(iii) $\boldsymbol{(s-t)(s+t)(s+t)^2}$

R.3 Rational Expressions

Rational polynomials are quotients of polynomials with nonzero denominators, such as

$$\frac{5}{k-2}, \quad \frac{6b^3+12b^2}{b+2}, \quad \frac{6b^3+12b^2}{6b^2}.$$

Rational polynomials are *reduced* by dividing out common factors in the numerator and denominator of the quotient such as

$$\frac{6b^3+12b^2}{b+2} = \frac{6b^2(b+2)}{b+2} = 6b^2.$$

Combinations of rational polynomials $\frac{P}{Q}$ and $\frac{R}{Q}$ where $Q \neq 0$ and $S \neq 0$ include:

$$\begin{aligned}
\frac{P}{Q} &= \frac{PS}{QS} \quad \text{fundamental property} \\
\frac{P}{Q} + \frac{R}{Q} &= \frac{P+R}{Q} \quad \text{addition} \\
\frac{P}{Q} - \frac{R}{Q} &= \frac{P-R}{Q} \quad \text{subtraction} \\
\frac{P}{Q} \cdot \frac{R}{S} &= \frac{PR}{QS} \quad \text{multiplication} \\
\frac{P}{Q} \div \frac{R}{S} &= \frac{PS}{QR} \quad \text{division}
\end{aligned}$$

Also, $\frac{a^m}{a^n} = a^{m-n}$ and $a^m a^n = a^{m+n}$.

Exercise R.3 (Rational Expressions)

1. reduce rational expression,
$$\frac{64m^3}{8m} =$$
(i) $\boldsymbol{8m}$ (ii) $\boldsymbol{8m^2}$ (iii) $\boldsymbol{8}$

2. reduce rational expression,
$$\frac{6b^2(b+2)}{6b^2} =$$
(i) $\boldsymbol{b+2}$ (ii) $\boldsymbol{6b^2}$ (iii) $\boldsymbol{b+12b^2}$

3. reduce rational expression,
$$\frac{6b^3+12b^2}{6b^2} =$$
(i) $\boldsymbol{b+12b^2}$ (ii) $\boldsymbol{6b^2}$ (iii) $\boldsymbol{b+2}$

4. reduce rational expression,
$$\frac{8m^3-16m^2+10m}{2m} = \frac{2m(4m^2-8m+5)}{2m} =$$
(i) $\boldsymbol{4m^2-8m+5}$ (ii) $\boldsymbol{2m}$ (iii) $\boldsymbol{\frac{2m+1}{2m}}$

5. reduce rational expression,
$$\frac{x^2+9x+8}{x^2+8x+7} = \frac{(x+1)(x+8)}{(x+1)(x+7)} =$$
(i) $\boldsymbol{\frac{x+8}{x+7}}$ (ii) $\boldsymbol{\frac{(x+1)(x+8)}{(x+1)(x+7)}}$ (iii) $\boldsymbol{\frac{x+1}{x+1}}$

6. multiply,
$$\frac{64m^3}{7} \cdot \frac{7}{2m^2} =$$
(i) $\boldsymbol{64m}$ (ii) $\boldsymbol{32m}$ (iii) $\boldsymbol{m^2}$

7. divide,
$$\frac{64m^3}{7} \div \frac{7}{2m^2} = \frac{64m^3}{7} \cdot \frac{2m^2}{7} =$$
(i) $\boldsymbol{\frac{128m^6}{49}}$ (ii) $\boldsymbol{32m}$ (iii) $\boldsymbol{\frac{128m^5}{49}}$

8. multiply,
$$\frac{k^2-1}{k+2} \cdot \frac{2}{k+1} = \frac{(k+1)(k-1)}{k+2} \cdot \frac{2}{k+1} =$$
(i) $\boldsymbol{\frac{k-1}{k+2}}$ (ii) $\boldsymbol{\frac{2(k+1)}{k+2}}$ (iii) $\boldsymbol{\frac{2(k-1)}{k+2}}$

9. divide,
$$\frac{k^2-1}{k+2} \div \frac{2}{k+1} = \frac{(k+1)(k-1)}{k+2} \cdot \frac{k+1}{2} =$$
(i) $\boldsymbol{\frac{(k+1)^2(k-1)}{2(k+2)}}$ (ii) $\boldsymbol{\frac{2(k-1)}{k+2}}$ (iii) $\boldsymbol{\frac{k-1}{k+2}}$

10. multiply,
$$\frac{x^2+9x+8}{x^2+8x+7} \cdot \frac{x^2+6x-7}{x^2-1} = \frac{(x+1)(x+8)}{(x+1)(x+7)} \cdot \frac{(x-1)(x+7)}{(x+1)(x-1)} =$$
(i) $\boldsymbol{\frac{x+8}{x+1}}$ (ii) $\boldsymbol{\frac{x+1}{x-1}}$ (iii) $\boldsymbol{\frac{x-1}{x+7}}$

11. multiply,
$$\frac{3x^2+27x+24}{x^2+8x+7} \cdot \frac{x^2+6x-7}{x^2-1} = \frac{3(x+1)(x+8)}{(x+1)(x+7)} \cdot \frac{(x-1)(x+7)}{(x+1)(x-1)} =$$
(i) $\boldsymbol{\frac{3(x+8)}{x+1}}$ (ii) $\boldsymbol{\frac{3(x+1)}{x-1}}$ (iii) $\boldsymbol{\frac{3(x-1)}{x+7}}$

12. divide,
$$\frac{x^2+9x+8}{x^2+8x+7} \div \frac{x^2+6x-7}{x^2-1} = \frac{(x+1)(x+8)}{(x+1)(x+7)} \cdot \frac{(x+1)(x-1)}{(x-1)(x+7)} =$$
(i) $\boldsymbol{\frac{(x+1)(x+8)}{(x+7)^2}}$ (ii) $\boldsymbol{\frac{(x+1)(x-1)}{x+7}}$ (iii) $\boldsymbol{\frac{x-1}{x+7}}$

13. add,
$$\frac{62m^3}{7} + \frac{m^3}{7} = \frac{63m^3}{7} =$$
(i) $\boldsymbol{9m}$ (ii) $\boldsymbol{9m^2}$ (iii) $\boldsymbol{9m^3}$

14. subtract,

$$\frac{62m^3}{7} - \frac{m^3}{7} =$$

(i) $\boldsymbol{\frac{61m^3}{7}}$ (ii) $\boldsymbol{9m^3}$ (iii) $\boldsymbol{9m}$

15. add,

$$= \frac{62m^3}{7} + \frac{7}{m^3} = \frac{62m^3}{7} \cdot \frac{m^3}{m^3} + \frac{7}{m^3} \cdot \frac{7}{7} =$$

(i) $\boldsymbol{\frac{62m^6+49}{m^3}}$ (ii) $\boldsymbol{\frac{62m^6+49}{7m^3}}$ (iii) $\boldsymbol{\frac{62m^6+7}{7m^3}}$

16. add and subtract,

$$= \frac{62m^3}{7} + \frac{7}{m^3} - \frac{m^3}{7} = \frac{(62m^3 - m^3)}{7} \cdot \frac{m^3}{m^3} + \frac{7}{m^3} \cdot \frac{7}{7} =$$

(i) $\boldsymbol{\frac{61m^6+49}{7m^3}}$ (ii) $\boldsymbol{\frac{62m^6+49}{m^3}}$ (iii) $\boldsymbol{\frac{63m^6+7}{7m^3}}$

17. add,

$$\frac{(k-1)}{3(k+2)} + \frac{2}{k+2} = \frac{(k-1)}{3(k+2)} + \frac{2}{k+2} \cdot \frac{3}{3} =$$

(i) $\boldsymbol{\frac{k-1}{3(k+2)}}$ (ii) $\boldsymbol{\frac{3(k+5)}{k+2}}$ (iii) $\boldsymbol{\frac{k+5}{3(k+2)}}$

18. add,

$$\frac{x+7}{x^2+8x+7} + \frac{x^2+6x-7}{x^2-1} = \frac{(x+7)}{(x+1)(x+7)} \cdot \frac{(x-1)}{(x-1)} + \frac{(x-1)(x+7)}{(x+1)(x-1)} \cdot \frac{(x+7)}{(x+7)} =$$

(i) $\boldsymbol{\frac{x+8}{x+1}}$ (ii) $\boldsymbol{\frac{x+7}{x-1}}$ (iii) $\boldsymbol{\frac{x+8}{x+7}}$ (Hint: $1 + (x+7) = x+8$)

R.4 Equations

A *linear equation* is of the form $ax + b = 0$, a, b real and $a \neq 0$. Two properties of linear equations are if $a = b$, then $a + c = b + c$ (addition property) and also $ac = bc$ (multiplication property). Solving linear equations involves isolating x by itself; for example,

$$\begin{aligned} 4x + 20 &= 0 \\ 4x &= -20 \quad \text{(subtract 20 from both sides)} \\ \frac{4x}{4} &= \frac{-20}{4} \quad \text{(divide both sides by 4)} \\ x &= -5 \end{aligned}$$

A *quadratic equation* is of the standard form $ax^2 + bx + c = 0$, a, b, c real, $a \neq 0$. A quadratic equation can be solved by using the *zero property*: if $ab = 0$, a, b real, then $a = 0$ or $b = 0$ or both; for example, if

$$y^2 + 5y + 6 = (y + 2)(y + 3) = 0$$

then solutions are $y = -2$ or $y = -3$. Quadratic equations can also be solved using *quadratic formula*; for example, since, for $y^2 + 5y + 6$, $a = 1$, $b = 5$ and $c = 6$,

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-5 \pm \sqrt{5^2 - 4(1)(6)}}{2(1)} = \frac{-5 \pm \sqrt{25 - 24}}{2} = \frac{-5 \pm 1}{2} = -2, -3.$$

Some *rational* equations have *extraneous* solutions; for example,

$$\frac{x^2 + 9x + 8}{x^2 + 8x + 7} = \frac{(x + 1)(x + 8)}{(x + 1)(x + 7)} = 0$$

has solution $x = -8$ *only*. The other possible solution $x = -1$ is extraneous because the *denominator* of the equation is zero in this case, making the equation undefined.

Exercise R.4 (Equations)

1. solve linear equation

$$\begin{aligned} 2x + 10 &= 0 \\ 2x &= -10 \quad \text{(subtract 10 from both sides)} \\ \frac{2x}{2} &= \frac{-10}{2} \quad \text{(divide both sides by 2)} \end{aligned}$$

   So $x =$ (i) $\mathbf{-3}$ (ii) $\mathbf{-4}$ (iii) $\mathbf{-5}$

2. solve linear equation

$$\begin{aligned} 2x + 10 &= 3x - 5 \\ -x &= -15 \quad \text{(subtract } 3x \text{ from both sides, subtract 10 from both sides)} \end{aligned}$$

   So $x =$ (i) $\mathbf{-15}$ (ii) $\mathbf{15}$ (iii) $\mathbf{-30}$

3. solve linear equation

$$\begin{aligned} 2(m + 5) &= 5(m + 2) + 7 \\ 2m + 10 &= 5m + 10 + 7 \quad \text{(multiply polynomials)} \\ -3m &= 7 \quad \text{(subtract } 5m \text{ from both sides, subtract 10 from both sides)} \end{aligned}$$

   So $m =$ (i) $\mathbf{-\frac{7}{3}}$ (ii) $\mathbf{-\frac{3}{7}}$ (iii) $\mathbf{\frac{7}{3}}$

4. solve quadratic equation

$$\begin{aligned} k^2 - 1 &= 0 \\ (k-1)(k+1) &= 0 \quad \text{(factor)} \end{aligned}$$

So $k =$ (i) $\mathbf{1}$ (ii) $\mathbf{-1}$ (iii) $\mathbf{\pm 1}$

5. solve quadratic equation

$$\begin{aligned} x^2 + 9x + 8 &= 0 \\ (x+1)(x+8) &= 0 \quad \text{(factor)} \end{aligned}$$

So $x =$ (i) $\mathbf{-1, 8}$ (ii) $\mathbf{1, -8}$ (iii) $\mathbf{-1, -8}$

6. solve quadratic equation

$$\begin{aligned} x^2 + 9x + 5 &= -3 \\ x^2 + 9x + 8 &= 0 \quad \text{(add 3 both sides)} \\ (x+1)(x+8) &= 0 \quad \text{(factor)} \end{aligned}$$

So $x =$ (i) $\mathbf{-1, -8}$ (ii) $\mathbf{1, -8}$ (iii) $\mathbf{-1, 8}$

7. solve quadratic equation

$$\begin{aligned} 3x^2 + 27x + 21 &= -3 \\ 3x^2 + 27x + 24 &= 0 \quad \text{(add 3 both sides)} \\ 3(x+1)(x+8) &= 0 \quad \text{(factor)} \end{aligned}$$

So $x =$ (i) $\mathbf{-1, -8}$ (ii) $\mathbf{-3, -1, -8}$ (iii) $\mathbf{-1, 8}$

8. solve quadratic equation

$$\begin{aligned} (5a+5)^2 &= 0 \\ (5a+5)(5a+5) &= 0 \quad \text{(multiply)} \\ 5(a+1) \times 5(a+1) &= 0 \quad \text{(factor out 5)} \end{aligned}$$

So $a =$ (i) $\mathbf{1}$ (ii) $\mathbf{\pm 1}$ (iii) $\mathbf{-1}$

9. solve quadratic equation

$$\begin{aligned} p(p+8) &= 9 \\ p^2 + 8p - 9 &= 0 \quad \text{(multiply, subtract 9 both sides)} \\ (p-1)(p+9) &= 0 \quad \text{(factor)} \end{aligned}$$

So $p =$ (i) $\mathbf{-1, 9}$ (ii) $\mathbf{1, 9}$ (iii) $\mathbf{1, -9}$

10. solve quadratic equation $x^2 + 9x + 8 = 0$, where $a = 1$, $b = 9$, $c = 8$,

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{(quadratic formula)} \\ &= \frac{-(9) \pm \sqrt{(9)^2 - 4(1)(8)}}{2(1)} \\ &= \frac{-9 \pm \sqrt{49}}{2} \end{aligned}$$

So $x =$ (i) $\mathbf{-1, -8}$ (ii) $\mathbf{1, -8}$ (iii) $\mathbf{-1, 8}$

11. solve quadratic equation $4m^2 - 8m - 5 = 0$, where $a = 4$, $b = -8$, $c = -5$,

$$\begin{aligned} m &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{(quadratic formula)} \\ &= \frac{-(-8) \pm \sqrt{(-8)^2 - 4(4)(-5)}}{2(4)} \\ &= \frac{8 \pm \sqrt{144}}{8} \end{aligned}$$

So $m =$ (i) $\mathbf{-\frac{1}{2}, \frac{5}{2}}$ (ii) $\mathbf{\frac{1}{2}, \frac{5}{2}}$ (iii) $\mathbf{-\frac{1}{2}, -\frac{5}{2}}$

12. solve rational equation

$$\begin{aligned} \frac{3x-3}{7} &= \frac{x+4}{3} \\ \frac{3(3x-3)}{7(3)} &= \frac{7(x+4)}{3(7)} \quad \text{(common denominator)} \\ \frac{3(3x-3) - 7(x+4)}{21} &= 0 \quad \text{(subtract)} \\ \frac{2x-37}{21} &= 0 \quad \text{(simplify)} \end{aligned}$$

So $x =$ (i) $\mathbf{\frac{37}{2}}$ (ii) $\mathbf{-\frac{37}{2}}$ (iii) $\mathbf{\frac{2}{37}}$

13. solve rational equation

$$\begin{aligned} \frac{5}{k+1} + \frac{6}{k-1} &= \frac{4}{k^2-1} \\ \frac{5(k-1)}{(k+1)(k-1)} + \frac{6(k+1)}{(k+1)(k-1)} &= \frac{4}{(k+1)(k-1)} \quad \text{(common denominator)} \\ \frac{5(k-1) + 6(k+1) - 4}{(k+1)(k-1)} &= 0 \quad \text{(collect terms)} \\ \frac{11k-3}{(k+1)(k-1)} &= 0 \quad \text{(simplify)} \end{aligned}$$

So $k =$ (i) $\mathbf{-1, 1}$ (ii) $\mathbf{-1, 1, \frac{11}{3}}$ (iii) $\mathbf{\frac{3}{11}}$

14. solve rational equation,

$$\begin{aligned} \frac{k-1}{3(k+2)} &= -\frac{2}{k+2} \\ \frac{k-1}{3(k+2)} + \frac{2}{k+2} \cdot \frac{3}{3} &= 0 \quad \text{(common denominator)} \\ \frac{k+5}{3(k+2)} &= 0 \quad \text{(simplify)} \end{aligned}$$

So $k =$ (i) $\mathbf{-2, -5}$ (ii) $\mathbf{2, -5}$ (iii) $\mathbf{-5}$

15. solve rational equation,

$$\begin{aligned} \frac{k-4}{3(k+2)} &= -\frac{2}{k+2} \\ \frac{k-4}{3(k+2)} + \frac{2}{k+2} \cdot \frac{3}{3} &= 0 \quad \text{(common denominator)} \\ \frac{k+2}{3(k+2)} &= 0 \quad \text{(simplify)} \end{aligned}$$

So $k =$ (i) **no solution** (ii) $\mathbf{-2, -5}$ (iii) $\mathbf{2, -5}$

16. solve rational equation,

$$\begin{aligned} \frac{x^2+10x+16}{x^2+8x+7} &= 0 \\ \frac{(x+2)(x+8)}{(x+1)(x+7)} &= 0 \quad \text{(factor)} \end{aligned}$$

So $x =$ (i) $\mathbf{-2, -8}$ (ii) $\mathbf{-1, -7}$ (iii) **no solution**

R.5 Inequalities

An example of a *linear* inequality is $ax < b$ (or $>$, $\leq$, $\geq$). If $a < b$, then $a + c < b + c$; if $a < b$ and $c > 0$, $ac < bc$; if $a < b$ and $c < 0$, then $ac > bc$. An example of an *open interval*, $(-3, 1)$ or $-3 < x < 1$, is given in (a) of figure below; an example of a *closed interval*, $[-3, 1]$ or $-3 \leq x \leq 1$, is given in (b) of figure.

Figure R.1 (Open and closed intervals)

Linear inequality, $-6 < 6p \leq 18$, simplified by dividing by 6 as $-1 < p \leq 3$ or $(-1, 3]$, has graph given in (a) of figure below. *Quadratic inequality*, $p^2 + 5p + 6 > 0$, factored to $(p + 3)(p + 2) > 0$ and which has zeroes $-2, -3$ and which is positive when either $p < -3$ or $p > -2$ (equivalently $(-\infty, -3) \cup (-2, \infty)$) and has graph given in (b) of figure below.

Figure R.2 (Graphs of linear and quadratic inequalities)

An example of a *fraction inequality* is $\frac{k-1}{k} > 0$ where numerator zero is $k = 1$, denominator zero is $k = 0$ and numerator and denominator are both positive (so overall fraction is positive) when $k > 1$ and both are negative (so overall fraction is positive) when $k < 0$.

Exercise R.5 (Inequalities)

1. *Intervals and inequalities.* Consider the following intervals.

Figure R.3 (Graphs of two intervals)

(a) Figure (a) is (choose one)
(i) $\boldsymbol{x \geq -1}$
(ii) $\boldsymbol{x \leq -1}$
(iii) $\boldsymbol{1 \leq x \leq 3}$

(b) Figure (a) is (choose one)
(i) $\boldsymbol{[-1, \infty)}$
(ii) $\boldsymbol{(-1, \infty)}$
(iii) $\boldsymbol{[-1, 3)}$

(c) Figure (b) is (choose one)
(i) $\boldsymbol{y < -3}$ or $\boldsymbol{y < 0}$
(ii) $\boldsymbol{y < -3}$ and $\boldsymbol{-2 < y < 0}$
(iii) $\boldsymbol{y < -3}$ or $\boldsymbol{-2 < y < 0}$

(d) Figure (b) is (choose one)
(i) $\boldsymbol{(\infty, -3) \cup (-2, 0)}$
(ii) $\boldsymbol{(\infty, -3] \cup (-2, 0)}$
(iii) $\boldsymbol{(\infty, -3) \cup (0, -2)}$

2. *Solve linear inequalities.*

(a) $2x + 10 > 0$

$$\begin{aligned} 2x + 10 &> 0 \\ 2x &> -10 \quad \text{(subtract 10 from both sides)} \\ \frac{2x}{2} &> \frac{-10}{2} \quad \text{(divide both sides by 2)} \end{aligned}$$

(i) $\boldsymbol{[-5, \infty)}$ (ii) $\boldsymbol{(-5, \infty)}$ (iii) $\boldsymbol{(-\infty, -5]}$

(b) $2x + 10 < 0$
(i) $\boldsymbol{(-\infty, -5)}$ (ii) $\boldsymbol{(-\infty, -5]}$ (iii) $\boldsymbol{[-5, \infty)}$

(c) $2x + 10 \leq 0$
(i) $\boldsymbol{(-\infty, -5]}$ (ii) $\boldsymbol{(-\infty, -5)}$ (iii) $\boldsymbol{[-5, \infty)}$

(d) $2r + 10 \leq 3r - 5$

$$\begin{aligned} 2r + 10 &\leq 3r - 5 \\ -r &\leq -15 \quad \text{(subtract } 3r \text{ from both sides, subtract 10 from both sides)} \\ \frac{-r}{-1} &\geq \frac{-15}{-1} \quad \text{(divide both sides by } -1\text{, reverse inequality)} \end{aligned}$$

(i) $\boldsymbol{(-\infty, -15)}$ (ii) $\boldsymbol{(-\infty, -15]}$ (iii) $\boldsymbol{[15, \infty)}$

(e) $2(m + 5) > 5(m + 2) + 7$

$$\begin{aligned} 2(m + 5) &> 5(m + 2) + 7 \\ 2m + 10 &> 5m + 10 + 7 \quad \text{(multiply out polynomials)} \\ -3m &> 7 \quad \text{(subtract 10 both sides, subtract 5m both sides)} \\ \frac{-3m}{-3} &\leq \frac{7}{-3} \quad \text{(divide both sides by } -3\text{, reverse inequality)} \end{aligned}$$

(i) $\boldsymbol{\left(-\infty, -\frac{7}{3}\right]}$ (ii) $\boldsymbol{\left(-\infty, -\frac{7}{3}\right)}$ (iii) $\boldsymbol{(-5, -3)}$

(f) $10 < x + 10 < 20$

$$\begin{array}{rcccl} 10 & < & x+10 & < & 20 \\ 0 & < & x & < & 10 \quad \text{(subtract 10 three times)} \end{array}$$

(i) $\mathbf{(\infty, 10)}$ (ii) $\mathbf{(0, 10)}$ (iii) $\mathbf{(-10, 0)}$

(g) $10 < 2x + 10 < 20$

$$\begin{array}{rcccl} 10 & < & 2x+10 & < & 20 \\ 0 & < & 2x & < & 10 \quad \text{(subtract 10 three times)} \\ \frac{0}{2} & < & \frac{2x}{2} & < & \frac{10}{2} \quad \text{(divide by 2 three times)} \end{array}$$

(i) $\mathbf{(\infty, 5)}$ (ii) $\mathbf{(0, 5)}$ (iii) $\mathbf{(-5, 0)}$

(h) $10 < -2x + 10 < 20$

$$\begin{array}{rcccl} 10 & < & -2x+10 & < & 20 \\ 0 & < & -2x & < & 10 \quad \text{(subtract 10 three times)} \\ \frac{0}{-2} & > & \frac{-2x}{-2} & > & \frac{10}{-2} \quad \text{(divide by } -2 \text{ three times, reverse inequalities)} \end{array}$$

(i) $\mathbf{(-5, 0)}$ (ii) $\mathbf{(\infty, -5)}$ (iii) $\mathbf{(0, -5)}$

(Hint: subtract 10 three times, divide by -2 three times, remember to reverse both inequalities.)

(i) $10 < \frac{2x+10}{3} < 20$

$$\begin{array}{rcccl} 10 & < & \frac{2x+10}{3} & < & 20 \\ 30 & < & 2x+10 & < & 60 \quad \text{(multiply three times)} \\ 20 & < & 2x & < & 50 \quad \text{(subtract 10 three times)} \\ \frac{20}{2} & < & \frac{2x}{2} & < & \frac{50}{2} \quad \text{(divide by 2 three times)} \end{array}$$

(i) $\mathbf{(10, 25)}$ (ii) $\mathbf{(25, 10)}$ (iii) $\mathbf{(-25, -10)}$

(j) $10 < \frac{2x+10}{3} \leq 20$

(i) $\mathbf{(10, 25]}$ (ii) $\mathbf{(25, 10]}$ (iii) $\mathbf{[-25, -10)}$

(Hint: multiply by 3 three times, subtract 10 three times, divide by 2 three times.)

3. *Solve quadratic inequalities.*

(a) $(k - 1)(k + 1) > 0$

(i) $\mathbf{(-1, 1)}$

(ii) $\mathbf{(-\infty, 1) \cap (1, \infty)}$

(iii) $\mathbf{(-\infty, -1) \cup (1, \infty)}$

(Hint: Regions A and C in figure below.)

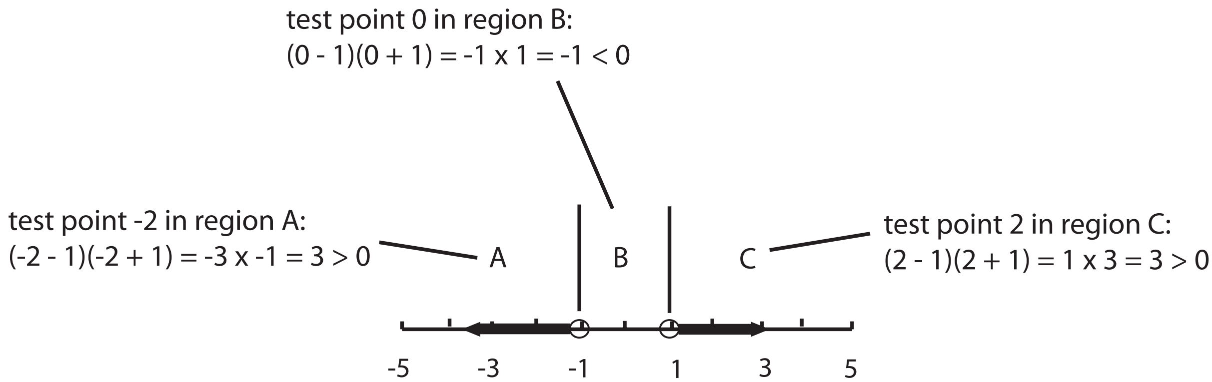

Figure R.4 ($(k-1)(k+1)$ interval)

(b) $(k-1)(k+1) < 0$
(i) $\mathbf{(-\infty, -1) \cup (1, \infty)}$
(ii) $\mathbf{(-\infty, 1) \cap (1, \infty)}$
(iii) $\mathbf{(-1, 1)}$
(Hint: Region B in figure above. $(k-1)(k+1) < 0$ if $-1 < k < 1$.)

(c) $k^2 - 1 < 0$
(i) $\mathbf{(-\infty, -1) \cup (1, \infty)}$
(ii) $\mathbf{(-\infty, 1) \cap (1, \infty)}$
(iii) $\mathbf{(-1, 1)}$
(Hint: $k^2 - 1 = (k-1)(k+1)$ and $(k-1)(k+1) < 0$ if $-1 < k < 1$.)

(d) $(x+1)(x+3) > 0$
(i) $\mathbf{(-3, -1)}$
(ii) $\mathbf{(-\infty, -3) \cup (-1, \infty)}$
(iii) $\mathbf{(-\infty, -3) \cup (-3, -1)}$
(Hint: Regions A and C in figure below.)

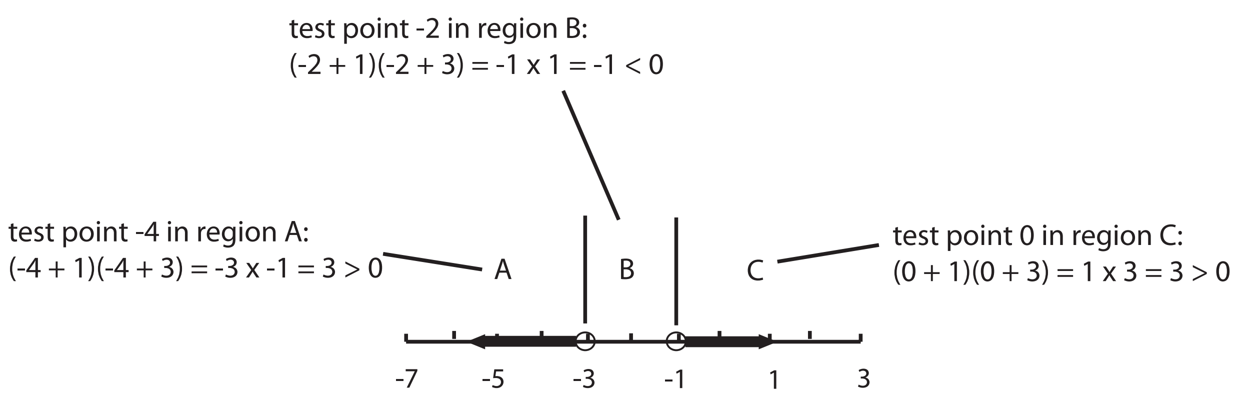

Figure R.5 ($(x+1)(x+8)$ interval)

(e) $(x+1)(x+3) < 0$

(i) $\mathbf{(-3,-1)}$
(ii) $\mathbf{(-\infty,-3)\cup(-1,\infty)}$
(iii) $\mathbf{(-\infty,-3)\cup(-3,-1)}$
(Hint: Region B in figure above. $(x+1)(x+3)<0$ if $-3<x<-1$.)

(f) $x^2+4x+3<0$
(i) $\mathbf{(-\infty,-3)\cup(-1,\infty)}$
(ii) $\mathbf{(-3,-1)}$
(iii) $\mathbf{(-\infty,-3)\cup(-3,-1)}$
(Hint: $x^2+4x+3=(x+1)(x+3)$ and $(x+1)(x+3)<0$ if $-3<x<-1$.)

(g) $x(x+4)<-3$
(i) $\mathbf{(-\infty,-3)\cup(-1,\infty)}$
(ii) $\mathbf{(-\infty,-3)\cup(-3,-1)}$
(iii) $\mathbf{(-3,-1)}$
(Hint: $x(x+4)=-3$ same as $x^2+4x+3=(x+1)(x+3)$ and $(x+1)(x+3)<0$ if $-3<x<-1$.)

(h) $(t-6)t>0$
(i) $\mathbf{(0,6)}$
(ii) $\mathbf{(-\infty,0)\cup(0,6)}$
(iii) $\mathbf{(-\infty,0)\cup(6,\infty)}$
(Hint: $(t-6)t=(t-6)(t+0)=0$ if $t=0$ or $t=6$ and $(t-6)t>0$ if either $t>6$ or $t<0$.)

4. *Solve inequalities with fractions.*

(a) $\frac{k-1}{k+1}>0$
(i) $\mathbf{(-1,1)}$ (ii) $\mathbf{(-\infty,-1)\cup(1,\infty)}$ (iii) $\mathbf{(-\infty,1)\cap(1,\infty)}$
(Hint: Regions A and C in figure below.)

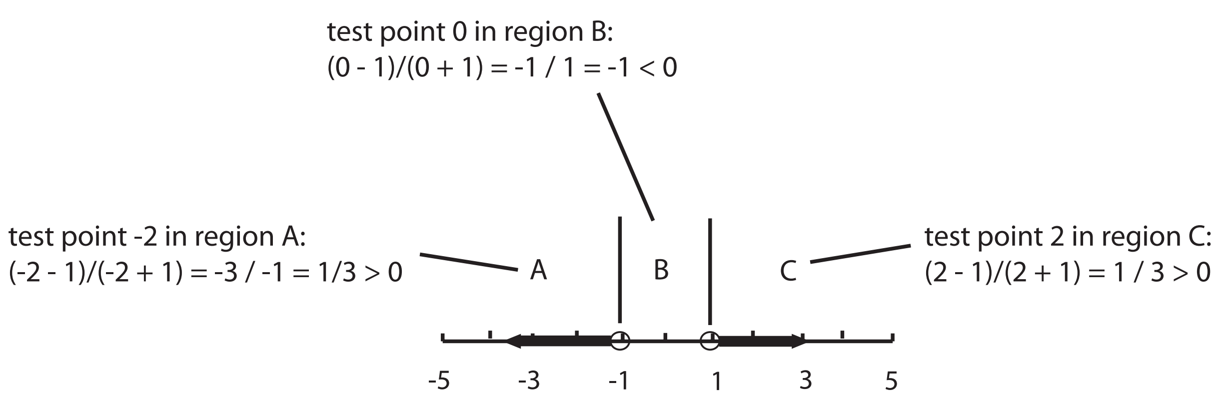

Figure R.6 ($\frac{k-1}{k+1}$ interval)

(b) $\frac{k-1}{k+1}<0$
(i) $\mathbf{(-\infty,-1)\cup(1,\infty)}$ (ii) $\mathbf{(-1,1)}$ (iii) $\mathbf{(-\infty,1)\cap(1,\infty)}$
(Hint: Region B in figure above.)

(c) $\frac{k-1}{k+1} \leq 0$
(i) $\mathbf{[-1, 1)}$ (ii) $\mathbf{[-1, 1]}$ (iii) $\mathbf{(-1, 1]}$
(Hint: Region B in figure above and also $k \neq -1$.)

(d) $\frac{k+1}{k-1} \leq 0$
(i) $\mathbf{(-1, 1]}$ (ii) $\mathbf{[-1, 1]}$ (iii) $\mathbf{[-1, 1)}$
(Hint: Region B in figure above and also $k \neq 1$.)

R.6 Exponents

For natural n,

$$a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n\, factors},$$

where power n is the *exponent* and a is the *base*. Also, $a^0 = 1$ and $a^{-n} = \frac{1}{a^n}$. For real a and b, integers n and m such that following exists,

- $a^m \cdot a^n = a^{m+n}$
- $\frac{a^m}{a^n} = a^{m-n}$
- $(a^m)^n = a^{mn}$
- $(ab)^m = a^m \cdot a^n$
- $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

When a is *real* and n *natural* (a "counting number"), $a^{\frac{1}{n}}$ has n different *roots*[1], but only one or two, possibly none, of these roots, the *principal roots*, are *real* and the others are *complex*. Three cases occur. First, if a is positive and n is *even*, there is one positive real root, one negative real root and the rest complex; for example, square root of 4 has two roots, $4^{\frac{1}{2}} = \pm 2$, whereas the fourth root of 16 has four roots, with two real roots $16^{\frac{1}{4}} = \pm 2$ and two complex roots. Second, if a is positive and n is *odd*, there is one real root with the same sign as a and the rest are complex; for example, $8^{\frac{1}{3}} = 2$ (and two complex roots) and $(-8)^{\frac{1}{3}} = -2$ (and two complex roots). Third, if a is negative, all roots are complex; for example, both roots of $(-2)^{\frac{1}{2}}$ are complex. Also, for all real a and rational $\frac{m}{n}$,

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m.$$

Exercise R.6 (Exponents)

[1]except when $a = 0$, where all n roots are the same: zero(0)

1. *Write without exponents.*

   (a) $\left(\frac{1}{5}\right)^{-3}$ = (i) $\mathbf{5^3}$ (ii) **25** (iii) **125**

   (b) $(5)^{-3}$ = (i) $\mathbf{-5^3}$ (ii) $\mathbf{\frac{1}{125}}$ (iii) $\mathbf{-125}$

   (c) $\left(-\frac{1}{4}\right)^{0}$ = (i) $\mathbf{-\frac{1}{4}}$ (ii) **1** (iii) $\mathbf{-16}$

   (d) $-(-4)^{-3}$ = (i) $\mathbf{\frac{1}{64}}$ (ii) $\mathbf{-\frac{1}{(-4)^3}}$ (iii) $\mathbf{-\frac{1}{64}}$

2. *Simplify to only positive exponents. Assume all variables positive real numbers.*

   (a) $\frac{5^8 \cdot 5^{-9}}{5^4 \cdot 5^2}$ = (i) $\mathbf{\frac{5^{-1}}{5^6}}$ (ii) $\mathbf{\frac{1}{5^7}}$ (iii) $\mathbf{5^{-7}}$

   (b) $\frac{2^{-2}x^2y^3}{x^4y^3}$ = (i) $\boldsymbol{\frac{1}{2^2x^2}}$ (ii) $\boldsymbol{2^{-2}x^{-2}}$ (iii) $\boldsymbol{\frac{1}{4x^2}}$

   (c) $\frac{\left(k^2\right)^{-3}}{7s}$ = (i) $\boldsymbol{\frac{1}{7k^6s}}$ (ii) $\boldsymbol{\frac{k^{-6}}{7s}}$ (iii) $\boldsymbol{7^{-1}k^{-6}s^{-1}}$

   (d) $\left(\frac{k^2}{7s}\right)^{-3}$ = (i) $\boldsymbol{\frac{k^{-6}}{7^{-3}k^{-3}}}$ (ii) $\boldsymbol{\frac{7^3s^3}{k^6}}$ (iii) $\boldsymbol{7^3s^3k^{-6}}$

3. *Write without exponents.*

   (a) $(125)^{\frac{1}{3}}$ = (i) $\mathbf{\frac{1}{5}}$ (ii) **5** (iii) **25**

   (b) $\left(\frac{1}{125}\right)^{\frac{1}{3}}$ = (i) $\mathbf{\frac{1}{5}}$ (ii) **5** (iii) **25**

   (c) $(125)^{\frac{2}{3}}$ = (i) $\mathbf{\frac{1}{25}}$ (ii) $\mathbf{5^2}$ (iii) **25**

   (d) $(125)^{-\frac{2}{3}}$ = (i) $\mathbf{5^{-2}}$ (ii) $\mathbf{\frac{1}{5^2}}$ (iii) $\mathbf{\frac{1}{25}}$

   (e) $\left(\frac{27}{125}\right)^{\frac{1}{3}}$ = (i) $\mathbf{\frac{5}{3}}$ (ii) $\mathbf{\frac{3}{5}}$ (iii) $\mathbf{\frac{1}{3}}$

   (f) $\left(\frac{27}{125}\right)^{\frac{4}{3}}$ = (i) $\mathbf{\frac{3}{625}}$ (ii) $\mathbf{\frac{81}{625}}$ (iii) $\mathbf{\frac{3^4}{5^4}}$

   (g) $\left(\frac{27}{125}\right)^{-\frac{1}{3}}$ = (i) $\mathbf{\frac{5}{3}}$ (ii) $\mathbf{\frac{3}{5}}$ (iii) $\mathbf{\frac{1}{3}}$

4. *Simplify to only positive exponents. Assume all variables positive real numbers.*

   (a) $5^{\frac{1}{3}} \cdot 5^{\frac{5}{3}}$ = (i) $\mathbf{\frac{1}{5}}$ (ii) **25** (iii) **5**

   (b) $\left(\frac{5^8 \cdot 5^{-9}}{5^4 \cdot 5^2}\right)^{\frac{1}{7}}$ = (i) $\mathbf{\frac{5^{-1}}{5^6}}$ (ii) $\mathbf{\frac{1}{5}}$ (iii) $\mathbf{5^{-\frac{7}{7}}}$

   (c) $\left(\frac{2^{-2}x^2y^3}{x^4y^3}\right)^{\frac{1}{2}}$ = (i) $\boldsymbol{\frac{1}{2^2x^2}}$ (ii) $\boldsymbol{2^{-1}x^{-1}}$ (iii) $\boldsymbol{\frac{1}{2x}}$

   (d) $\frac{p^{\frac{5}{3}} \cdot q^{\frac{4}{3}}}{p^{\frac{7}{3}} \cdot q^{\frac{2}{3}}}$ = (i) $\boldsymbol{\frac{p^{-6}}{q}}$ (ii) $\boldsymbol{q^{\frac{2}{3}} \cdot p^{-\frac{2}{3}}}$ (iii) $\boldsymbol{\frac{q^{\frac{2}{3}}}{p^{\frac{2}{3}}}}$

5. *Write a single term without negative exponents.*

(a) $\frac{2p^{-2}+q^{-1}}{p+q^3}$ = (i) $\boldsymbol{\frac{\frac{2}{p^2}+\frac{1}{q}}{p+q^3}}$ (ii) $\boldsymbol{\frac{2q+p^2}{(p^2+q)(p+q^3)}}$ (iii) $\boldsymbol{\frac{\frac{2q+p^2}{p^2+q}}{p+q^3}}$

(b) $(m^{-1}-n^{-1})^{-1}$ = (i) $\boldsymbol{\left(\frac{1}{m}-\frac{1}{n}\right)^{-1}}$ (ii) $\boldsymbol{\left(\frac{n-m}{mn}\right)^{-1}}$ (iii) $\boldsymbol{\frac{mn}{n-m}}$

6. *Factor expression.*

(a) $3x^3(x^2+2x)-6x(x^2+2x)=$
(i) $\boldsymbol{(3x^3-6x)(x^2-2x)}$
(ii) $\boldsymbol{(3x^3-6x)(x^2+2x)}$
(iii) $\boldsymbol{(3x^3+6x)(x^2-2x)}$

(b) $3x^3(x^2+2x)^{-\frac{1}{2}}-6x(x^2+2x)^{\frac{1}{2}}=$
(i) $\boldsymbol{\frac{3x^3}{(x^2+2x)^{\frac{1}{2}}}-\frac{6x(x^2+2x)^{\frac{1}{2}}(x^2+2x)^{\frac{1}{2}}}{(x^2+2x)^{\frac{1}{2}}}}$
(ii) $\boldsymbol{\frac{-3x(x^2+4x)}{(x^2+2x)^{\frac{1}{2}}}}$
(iii) $\boldsymbol{\frac{3x(x^2-2(x^2+2x))}{(x^2+2x)^{\frac{1}{2}}}}$

R.7 Radicals

An alternative notation to the exponential notation of $a^{\frac{1}{n}}$ is the radical notation,

$$a^{\frac{1}{n}} = \sqrt[n]{a},$$

where $\sqrt[n]{a}$ is a *radical*, $\sqrt[n]{\cdot}$ is the *radical sign*, a is the *radicand* and n is the *index* of the radical. Also, $a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$. For real a and b, natural n and m such that $\sqrt[n]{a}$ and $\sqrt[m]{b}$ are real,

- $(\sqrt[n]{a})^n = a$
- $\left(\sqrt[n]{a^n}\right) = \begin{cases} |a| & \text{if } n \text{ is even} \\ a & \text{if } n \text{ is odd} \end{cases}$
- $\sqrt[n]{a}\cdot\sqrt[n]{b} = \sqrt[n]{ab}$
- $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$, $b \neq 0$
- $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

Exercise R.7 (Radicals)

1. *Simplify by removing as many factors from under radical.*

(a) $\sqrt{45}$ = (i) $\boldsymbol{\sqrt{9}\sqrt{5}}$ (ii) $\boldsymbol{\sqrt{3}\sqrt{5}}$ (iii) $\boldsymbol{3\sqrt{5}}$

(b) $\sqrt{1331} =$ (i) $\boldsymbol{\sqrt{121}\sqrt{11}}$ (ii) $\boldsymbol{\sqrt{11^2}\sqrt{11}}$ (iii) $\boldsymbol{11\sqrt{11}}$

(c) $\sqrt{1331} - 7\sqrt{11} =$
(i) $\boldsymbol{4\sqrt{11}}$ (ii) $\boldsymbol{\sqrt{121}\sqrt{11} - 7\sqrt{11}}$ (iii) $\boldsymbol{11\sqrt{11} - 7\sqrt{11}}$

(d) $\sqrt{48k^3} =$ (i) $\boldsymbol{\sqrt{16k^2 \cdot 3k}}$ (ii) $\boldsymbol{4k\sqrt{3k}}$ (iii) $\boldsymbol{4\sqrt{k^3}}$

(e) $\sqrt{325} - \sqrt{192} =$
(i) $\boldsymbol{5\sqrt[3]{3} - 4\sqrt[3]{3}}$ (ii) $\boldsymbol{\sqrt[3]{125}\sqrt[3]{3} - \sqrt[3]{64}\sqrt[3]{3}}$ (iii) $\boldsymbol{5\sqrt{3} - 4\sqrt{3}}$

(f) $\sqrt{m^7n} + \sqrt{m^5n^3} - \sqrt{mn^3} =$
(i) $\boldsymbol{\sqrt{m^6}\sqrt{mn} + \sqrt{m^4n^2}\sqrt{mn} - \sqrt{n^2}\sqrt{mn}}$
(ii) $\boldsymbol{m^3\sqrt{mn} + m^2n\sqrt{mn} - n\sqrt{mn}}$
(iii) $\boldsymbol{(m^3 + m^2n - n)\sqrt{mn}}$

2. *Simplify, if possible.*

(a) $\sqrt{(4y^2 + 20y + 25)} =$
(i) $\boldsymbol{|2y + 5|}$ (ii) $\boldsymbol{\sqrt{(2y + 5)^2}}$ (iii) **cannot simplify**

(b) $\sqrt{(4y^2 + 20y - 25)} =$
(i) $\boldsymbol{\sqrt{(2y + 5)^2}}$ (ii) $\boldsymbol{|2y + 5|}$ (iii) **cannot simplify**

3. *Rationalize denominator.*

(a) $\frac{5}{\sqrt{3}} =$ (i) $\boldsymbol{\frac{5\sqrt{3}}{\sqrt{3}\sqrt{3}}}$ (ii) $\boldsymbol{\frac{\sqrt{3}}{3}}$ (iii) $\boldsymbol{\frac{5\sqrt{3}}{3}}$

(b) $\frac{\sqrt{2}}{\sqrt{3}} =$ (i) $\boldsymbol{\frac{\sqrt{6}}{3}}$ (ii) $\boldsymbol{\frac{\sqrt{2}\sqrt{3}}{\sqrt{3}\sqrt{3}}}$ (iii) $\boldsymbol{\frac{\sqrt{6}}{\sqrt{3^2}}}$

(c) $\frac{6}{2+\sqrt{2}} =$ (i) $\boldsymbol{\frac{6(2-\sqrt{2})}{(2+\sqrt{2})(2-\sqrt{2})}}$ (ii) $\boldsymbol{\frac{\sqrt{6}}{2+\sqrt{2}}}$ (iii) $\boldsymbol{\frac{6(2-\sqrt{2})}{2}}$

(d) $\frac{6k}{2+\sqrt{k^2+1}} =$
(i) $\boldsymbol{\frac{6(2-\sqrt{2})}{(2+\sqrt{2})(2-\sqrt{2})}}$
(ii) **denominator cannot be rationalized**
(iii) $\boldsymbol{\frac{6k^2(2-\sqrt{k^2+1})}{3-k^2}}$

4. *Rationalize numerator.*

(a) $\frac{\sqrt{3}}{5} =$ (i) $\boldsymbol{\frac{3}{5\sqrt{3}}}$ (ii) $\boldsymbol{\frac{\sqrt{3}\sqrt{3}}{5\sqrt{3}}}$ (iii) $\boldsymbol{\frac{\sqrt{3}}{3}}$

(b) $\frac{\sqrt{2}}{\sqrt{3}} =$ (i) $\boldsymbol{\frac{\sqrt{2}\sqrt{2}}{\sqrt{3}\sqrt{2}}}$ (ii) $\boldsymbol{\frac{2}{\sqrt{6}}}$ (iii) $\boldsymbol{\frac{\sqrt{2^2}}{\sqrt{3\cdot 2}}}$

(c) $\frac{2+\sqrt{2}}{6} =$ (i) $\boldsymbol{\frac{2}{6(2-\sqrt{2})}}$ (ii) $\boldsymbol{\frac{(2+\sqrt{2})(2-\sqrt{2})}{6(2-\sqrt{2})}}$ (iii) $\boldsymbol{\frac{\sqrt{6}}{2+\sqrt{2}}}$

(d) $\frac{6k}{2+\sqrt{k^2+1}} =$
(i) $\boldsymbol{\frac{6(2-\sqrt{2})}{(2+\sqrt{2})(2-\sqrt{2})}}$
(ii) **numerator already rationalized**
(iii) $\boldsymbol{\frac{6k^2(2-\sqrt{k^2+1})}{3-k^2}}$