Workbook For Mathematics 224

Introductory Analysis II

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by

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Preface

The point of this course is to acquaint a student with some of the ideas, definitions and concepts of calculus and its applications, particularly those connected to integration. This workbook is a necessary component for a student to successfully complete this course. Without the workbook, a student will not be able to participate in the course.

- This attendance workbook is *based* on the text.

- Although the material covered in each is very similar, the *presentation* of the material in the workbook is quite different from the presentation given in the text. The text consists essentially of definitions, formulas, worked out examples and exercises; this workbook, on the other hand, consists *solely* of exercises to be worked out by the student.

- A student is to use this workbook to follow along with during a lecture.

- There are different kinds of exercises, including multiple choice, true/false, matching and fill–in–the–blank.

- Each week, I recommend you read the text, answer the questions given here in the lecture notes workbook and then do the online homework and quiz or test assignments, in that order.

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SR.1 Special Review Section on Differentiation

Differentiation is quickly reviewed in this section. The derivative is the slope of a tangent to a curve. We look at the power, product, quotient and chain rules.

- *Constant rule.* Derivative of a constant function, $f(x) = k$, k real, is zero:

$$f'(x) = 0.$$

- *Power rule.* Derivative of $f(x) = x^n$, n real, is

$$f'(x) = nx^{n-1}.$$

- *Constant times function rule.* Derivative of $f(x) = k \cdot g(x)$, k real, $g'(x)$ exists:

$$f'(x) = kg'(x).$$

- *Sum or difference rule.* Derivative of $f(x) = u(x) \pm v(x)$, and $u'(x), v'(x)$ exist:

$$f'(x) = u'(x) \pm v'(x).$$

- *Product rule:* If $f(x) = u(x) \cdot v(x)$, $u'(x)$ and $v'(x)$ exist, then

$$f'(x) = v(x) \cdot u'(x) + u(x) \cdot v'(x).$$

- *Quotient rule:* If $y = \frac{u(x)}{v(x)}$, $u'(x)$ and $v'(x)$ exist, and $v(x) \neq 0$, then

$$f'(x) = \frac{v(x) \cdot u'(x) - u(x) \cdot v'(x)}{[v(x)]^2}.$$

- The *chain rule* is used to find the derivative of the composed function $y = g[f(x)]$, where $y = f(u)$ and $u = g(x)$, and is given by,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = f'[g(x)] \cdot g'(x)$$

Exercise SR.1 (Special Review Section on Differentiation)

1. *Power Rule and Other Stuff.*

   (a) If $f(x) = x^2$ (equivalently, $y = x^2$), then derivative:

       i. $f'(x) = 2x^{2-1} =$ (i) **2**   (ii) **$2x$**   (iii) **$2x^3$**

       ii. $\frac{dy}{dx} =$ (i) **x**   (ii) **$2x$**   (iii) **$2x^3$**

    iii. $\frac{d}{dx}[f(x)] =$ (i) $\boldsymbol{x}$ (ii) $\boldsymbol{2x}$ (iii) $\boldsymbol{2x^3}$

    iv. $D_x[f(x)] =$ (i) $\boldsymbol{x}$ (ii) $\boldsymbol{2x}$ (iii) $\boldsymbol{2x^3}$

(b) If $f(x) = \sqrt{x} = x^{\frac{1}{2}}$ then $D_x[f(x)] =$ (i) $\boldsymbol{\frac{3}{2}x^{-\frac{1}{2}}}$ (ii) $\boldsymbol{-\frac{1}{2}x^{-\frac{1}{2}}}$ (iii) $\boldsymbol{\frac{1}{2}x^{-\frac{1}{2}}}$

(c) If $f(x) = \sqrt{x}$ then $D_x[f(x)] =$ (i) $\boldsymbol{\frac{3}{2\sqrt{x}}}$ (ii) $\boldsymbol{-\frac{1}{2\sqrt{x}}}$ (iii) $\boldsymbol{\frac{1}{2\sqrt{x}}}$

(d) If $f(x) = x^{\frac{5}{3}}$ then $f'(x) =$ (i) $\boldsymbol{-\frac{5}{3}x^{-\frac{2}{3}}}$ (ii) $\boldsymbol{-\frac{1}{3}x^{-\frac{1}{3}}}$ (iii) $\boldsymbol{\frac{5}{3}x^{\frac{2}{3}}}$

(e) If $f(x) = x^2$ (or $y = x^2$), so $f'(x) = 2x^{2-1} = 2x$, then at $x = 7$,

    i. $f'(7) = 2(7) =$ (i) $\boldsymbol{14}$ (ii) $\boldsymbol{2x}$ (iii) $\boldsymbol{7}$

    ii. $\frac{dy}{dx}\big|_{x=7} =$ (i) $\boldsymbol{14}$ (ii) $\boldsymbol{2x}$ (iii) $\boldsymbol{7}$

    iii. $\frac{d}{dx}[f(7)] =$ (i) $\boldsymbol{14}$ (ii) $\boldsymbol{2x}$ (iii) $\boldsymbol{7}$

    iv. $D_x[f(7)] =$ (i) $\boldsymbol{14}$ (ii) $\boldsymbol{2x}$ (iii) $\boldsymbol{7}$

(f) If $y = 8p^4 + 4\sqrt{p} - 5 = 8p^4 + 4p^{\frac{1}{2}} - 5$ then $\frac{dy}{dp} = 8\left(4p^{4-1}\right) + 4\left(\frac{1}{2}p^{\frac{1}{2}-1}\right) + 0 =$

    (i) $\boldsymbol{32p^3 + \frac{2}{p}}$ (ii) $\boldsymbol{32p^3 + \frac{2}{\sqrt{p}} + 5}$ (iii) $\boldsymbol{32p^3 + \frac{2}{\sqrt{p}}}$

(g) If $f(x) = \frac{7 + 4\sqrt{x}}{x} = \frac{7}{x} + \frac{4x^{\frac{1}{2}}}{x} = 7x^{-1} + 4x^{-\frac{1}{2}}$

    then $D_x[f(x)] = 7\left(-x^{-1-1}\right) + 4\left(-\frac{1}{2}x^{-\frac{1}{2}-1}\right) =$

    (i) $\boldsymbol{\frac{7}{x^2} + \frac{2}{\sqrt{x}}}$ (ii) $\boldsymbol{-\frac{7}{x^2} + \frac{2}{\sqrt{x}}}$ (iii) $\boldsymbol{-\frac{7}{x^2} - \frac{2}{(\sqrt{x})^3}}$

2. *Product and Quotient Rules*

    (a) *Product rule.* Consider $f(x) = (4x - 1)(5x + 4)$.

    let $u(x) = 4x - 1$ and $v(x) = 5x + 4$.

    then, $u'(x) =$ (i) $\boldsymbol{4x^2}$ (ii) $\boldsymbol{4}$ (iii) $\boldsymbol{-1}$

    and $v'(x) =$ (i) $\boldsymbol{5x^2}$ (ii) $\boldsymbol{5}$ (iii) $\boldsymbol{3 + 4x}$

    and $v(x)u'(x) =$ (i) $\boldsymbol{(5x + 4)(4)}$ (ii) $\boldsymbol{(5x + 4)(4x^2)}$ (iii) $\boldsymbol{(5x^2 + 4)(4)}$

    and $u(x)v'(x) =$ (i) $\boldsymbol{(4x^2 - 1)(5)}$ (ii) $\boldsymbol{(4x - 1)(5)}$ (iii) $\boldsymbol{(4x - 1)(5x)}$

    and so $f'(x) = v(x) \cdot u'(x) + u(x) \cdot v'(x) =$

    (i) $\boldsymbol{(5x + 4)(4) + (4x - 1)(5)}$

    (ii) $\boldsymbol{(5x - 4)(4) + (4x - 1)(5)}$

    (iii) $\boldsymbol{(5x + 4)(4) - (4x - 1)(5)}$

    which equals (i) $\boldsymbol{40x - 11}$ (ii) $\boldsymbol{40x + 11}$ (iii) $\boldsymbol{39 + 11}$

(b) *Quotient rule.* Consider $f(x) = \frac{4x-1}{5x+4}$.

let $u(x) = 4x - 1$ and $v(x) = 5x + 4$.

then, $u'(x) =$ (i) $4x^2$ (ii) 4 (iii) -1

and $v'(x) =$ (i) $5x^2$ (ii) 5 (iii) $3 + 4x$

and $v(x)u'(x) =$ (i) $(5x+4)(4)$ (ii) $(5x+4)(4x^2)$ (iii) $(5x^2+4)(4)$

and $u(x)v'(x) =$ (i) $(4x^2-1)(5)$ (ii) $(4x-1)(5)$ (iii) $(4x-1)(5x)$

and so $f'(x) = \frac{v(x)\cdot u'(x) - u(x)\cdot v'(x)}{[v(x)]^2} =$

(i) $\frac{(5x+4)(4)-(4x-1)(5)}{(5x+4)^2}$ (ii) $\frac{(5x+4)(4)+(4x-1)(5)}{(5x+4)^2}$ (iii) $\frac{(5x+4)(4)+(4x-1)(5)}{(4x-1)^2}$

which equals

(i) $\frac{21}{(5x+4)^2}$ (ii) $\frac{40x+11}{(5x+4)^2}$ (iii) $\frac{20}{(5x+4)^2}$

3. *Chain Rule*

(a) $y = (3x + x^2)^2$

let $f[g(x)] = (3x + x^2)^2$
with "inner" function $g(x) =$ (i) $3x + x^2$ (ii) $3x^2 - 2$ (iii) $3x$
and "outer" function $f(x) =$ (i) x^3 (ii) x^4 (iii) x^2

with derivative $g'(x) =$ (i) $6x - 2$ (ii) 3 (iii) $3 + 2x$
and derivative $f'(x) =$ (i) $3x^2$ (ii) $4x^3$ (iii) $2x$

and so by chain rule

$$
\begin{aligned}
f'[g(x)] \cdot g'(x) &= f'[3x + x^2] \cdot (3 + 2x) \\
&= 2\left[3x + x^2\right](3 + 2x) \\
&= 2\left(9x + 3x^2 + 6x^2 + 2x^3\right) \\
&= 2\left(9x + 9x^2 + 2x^3\right) =
\end{aligned}
$$

(i) $9x + 9x^2 + 2x^3$
(ii) $18x + 18x^2 + 4x^3$
(iii) $9 + 9x + 2x^2$

(b) $y = \sqrt{3x + x^2}$

let $f[g(x)] = (3x + x^2)^{\frac{1}{2}}$
with "inner" function $g(x) = $ (i) $\mathbf{3x^2 - 2}$　(ii) $\mathbf{3x}$　(iii) $\mathbf{3x + x^2}$
and "outer" function $f(x) = $ (i) $\mathbf{x^{\frac{1}{2}}}$　(ii) $\mathbf{x^{\frac{3}{2}}}$　(iii) $\mathbf{x^{\frac{5}{2}}}$

with derivative $g'(x) = $ (i) $\mathbf{3 + 2x}$　(ii) $\mathbf{6x - 2}$　(iii) $\mathbf{3}$
and derivative $f'(x) = $ (i) $\frac{3}{2}\mathbf{x^{\frac{1}{2}}}$　(ii) $\frac{3}{2}\mathbf{x^{\frac{3}{2}}}$　(iii) $\frac{1}{2}\mathbf{x^{-\frac{1}{2}}} = \frac{1}{2\sqrt{x}}$

and so by chain rule

$$
\begin{aligned}
f'[g(x)] \cdot g'(x) &= f'[3x + x^2] \cdot (3 + 2x) \\
&= \frac{1}{2\sqrt{3x + x^2}}(3 + 2x) =
\end{aligned}
$$

(i) $\frac{3+2x}{\sqrt{3x+x^2}}$　(ii) $\frac{3+2x}{2\sqrt{3x+x^2}}$　(iii) $\frac{3+2x}{3\sqrt{3x+x^2}}$

(c) *Natural exponential and chain rule.* $y = 5e^{2x}$.

Let $f[g(x)] = 5e^{2x}$
with inner function $g(x) = 2x$ and outer function $f(x) = 5e^x$
and $g'(x) = $ (i) $\mathbf{2}$　(ii) $\mathbf{2x^2}$　(iii) $\mathbf{4x}$
and $f'(x) = $ (i) $\mathbf{x}$　(ii) $\mathbf{5e^x}$　(iii) $\mathbf{e^{2x}}$
and so by chain rule

$$
f'[g(x)] \cdot g'(x) = f'[2x] \cdot (2) = 5e^{2x}(2) =
$$

(i) $\mathbf{5xe^{2x}}$　(ii) $\mathbf{10e^{2x}}$　(iii) $\mathbf{2xe^{2x}}$

(d) *Natural logarithmic function and chain rule.* $y = 4\ln(7\sqrt{x})$

Let $f[g(x)] = 4\ln(7\sqrt{x})$ and $g(x) = 7\sqrt{x} = 7x^{\frac{1}{2}}$ and $f(x) = 4\ln x$

and $g'(x) = $ (i) $\frac{5}{2}\mathbf{x^{-\frac{1}{2}}}$　(ii) $\frac{7}{2}\mathbf{x^{-\frac{1}{2}}}$　(iii) $\frac{3}{2}\mathbf{x^{-\frac{1}{2}}}$
and $f'(x) = $ (i) $\frac{4}{x^2}$　(ii) $\frac{1}{4x}$　(iii) $\frac{4}{x}$
and so by chain rule

$$
f'[g(x)] \cdot g'(x) = f'\left[7x^{\frac{1}{2}}\right] \cdot \left(\frac{7}{2}x^{-\frac{1}{2}}\right) = \frac{4}{7x^{\frac{1}{2}}}\left(\frac{7}{2}x^{-\frac{1}{2}}\right) =
$$

(i) $\frac{2}{x}$　(ii) $-\frac{7}{2}x^{-3/2}$　(iii) $\frac{7}{2}x^{-\frac{1}{2}}$

6.4 Implicit Differentiation

Thus far, we have looked at *explicit* differentiation where we first express y in terms of x differentiate, calculate $\frac{dy}{dx}$, where y is the *dependent* variable and x is the *independent* variable. In this section, we look at the *implicit differentiation* method which involves finding the derivative without first expressing y in terms of x. Some implicit equations can be expressed as an explicit function of x, but other implicit equations cannot be expressed as explicit functions of x. In general, although explicit solutions can be found for linear, quadratic, cubic and quartic *algebraic functions* (polynomial equations whose coefficients are themselves polynomials) in y, this not true in general for quintic or higher degree equations. The implicit differentiation method can be applied as long as the left of equation of the implicit equation $R(x, y) = 0$ is differentiable (and other mild conditions). Steps of implicit differentiation include

1. differentiate both sides of equation

2. place all terms of $\frac{dy}{dx}$ on one side of equation; all other terms on the other side

3. factor out $\frac{dy}{dx}$, solve for $\frac{dy}{dx}$

Implicit differentiation could also be thought of as an application of the chain rule of differentiation. In the next section, we apply implicit differentiation to *related rates* problems.

Exercise 6.4 (Implicit Differentiation)

1. *Implicit and explicit functions.*

    (a) implicit function $R(x, y) = y - 3x - 5 = 0$
       (i) **can**   (ii) **cannot** be solved as explicit function

$$
\begin{aligned}
y - 3x - 5 &= 0 \\
y &= 3x + 5
\end{aligned}
$$

    which (i) **is**   (ii) **is not** differentiable
    because $\frac{dy}{dx} = 3(1)x^{1-1} = 3$

    (b) implicit function $R(x, y) = y - 3xy - 5 = 0$
       (i) **can**   (ii) **cannot** be solved as explicit function

$$
\begin{aligned}
y - 3xy - 5 &= 0 \\
y(1 - 3x) &= 5 \\
y &= \frac{5}{1 - 3x}
\end{aligned}
$$

which (i) **is**   (ii) **is not** differentiable

because, using quotient rule, $\frac{dy}{dx} = \frac{vu'-uv'}{v^2} = \frac{(1-3x)(0)-(5)(-3)}{(-3)^2} = \frac{15}{9}$

(c) implicit function $R(x,y) = yx^2 + 5y - 2x^2 - 2 = 0$
   (i) **can**   (ii) **cannot** be solved as explicit function

$$\begin{aligned} yx^2 + 5y - 2x^2 - 2 &= 0 \\ yx^2 + 5y &= 2x^2 + 2 \\ y(x^2 + 5) &= 2x^2 + 2 \end{aligned}$$

or (i) $\boldsymbol{y = \frac{2x^2+2}{x^2+5}}$ (ii) $\boldsymbol{y = \frac{2x^2}{x^2+5}}$ (iii) $\boldsymbol{y = \frac{2x^2+2}{x^2-5}}$

which (i) **is**   (ii) **is not** differentiable

because, using quotient rule, $\frac{dy}{dx} = \frac{16x}{x^4+10x^2+25}$

(d) implicit equation $R(x,y) = y^2 + (3x)y + 5 = 0$, where $a = 1, b = 3x, c = 5$,
   (i) **can**   (ii) **cannot** be solved as explicit equation

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3x \pm \sqrt{(3x)^2 - 4(1)(5)}}{2(1)} = \frac{-3x \pm \sqrt{9x^2 - 20}}{2}$$

which is an *equation* which could be thought of as consisting of two *functions*, as given in figure (a);

GRAPH using $Y_1 = -3X + \sqrt{9X^2 - 20}$, and $Y_2 = -3X - \sqrt{9X^2 - 20}$
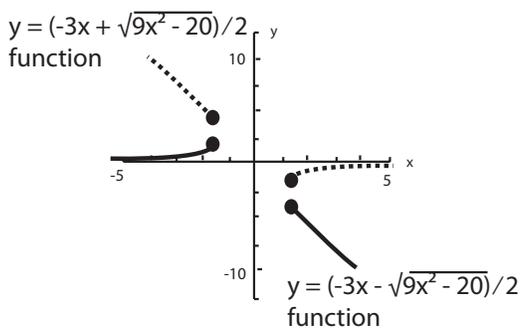
with WINDOW -15 15 1 -30 10 1 1

which (i) **is**   (ii) **is not** differentiable

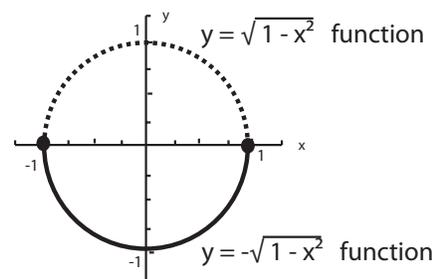because, although $R(x,y) = 0$ is an *equation*, not function,

each "half" of this equation consists of a differentiable function;

upper half $y = \frac{-3x+\sqrt{9x^2-20}}{2}$ has derivative $\frac{dy}{dx} = -3 + \frac{9x}{\sqrt{9x^2-20}}$

and lower half $y = \frac{-3x-\sqrt{9x^2-20}}{2}$ has derivative $\frac{dy}{dx} = -3 - \frac{9x}{\sqrt{9x^2-20}}$

(a) $R(x,y) = y^2 + 3xy + 5 = 0$ equation   (b) $R(x,y) = y^2 + x^2 - 1 = 0$ equation

Figure 6.12 (Implicit equations)

(e) implicit equation $R(x, y) = y^2 + x^2 - 1 = y^2 + 0y + (x^2 - 1) = 0$,
   where $a = 1, b = 0, c = x^2 - 1$,
   (i) **can**   (ii) **cannot** be solved as explicit equation

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{0 \pm \sqrt{(0)^2 - 4(1)(x^2 - 1)}}{2(1)}$$

or (i) $\boldsymbol{y = \pm\sqrt{1 - x^2}}$ (ii) $\boldsymbol{y = \sqrt{1 - x^2}}$ (iii) $\boldsymbol{y = -\sqrt{1 - x^2}}$ which is

an *equation* which could be thought of as consisting of two *functions*, as given in figure (b);

GRAPH using $Y_3 = \sqrt{1 - X^2}$, and $Y_4 = -\sqrt{1 - X^2}$

with WINDOW -3 3 1 -2 2 1 1

which (i) **is**   (ii) **is not** differentiable

because, although $R(x, y) = 0$ is an *equation*, not function,

each "half" of this equation consists of a differentiable function;

upper half $y = \sqrt{1 - x^2}$ has derivative $\frac{dy}{dx} = -\frac{x}{\sqrt{1-x^2}}$

and lower half $y = -\sqrt{1 - x^2}$ has derivative $\frac{dy}{dx} = \frac{x}{\sqrt{1-x^2}}$

(f) implicit function $\ln y - 2x = 0$
   (i) **can**   (ii) **cannot** be solved as explicit function

$$\begin{aligned} \ln y - 2x &= 0 \\ \ln y &= 2x \end{aligned}$$

or (i) $\boldsymbol{y = 2e^x}$ (ii) $\boldsymbol{y = 3e^x}$ (iii) $\boldsymbol{y = 2e^{2x}}$

which (i) **is**   (ii) **is not** differentiable

because, using the chain rule,

where $f[g'(x)] = e^{2x}$ where $f = e^x$, $g = 2x$ then $\frac{dy}{dx} = f'[g(x)] \cdot g'(x) = f'[2x] \cdot (2) = 2e^{2x}$

(g) implicit equation $R(x, y) = y^5 + 2y^4 - 7y^3 + 3y^2 - 6y - x = 0$
   (i) **can**   (ii) **cannot** be solved as an explicit equation of y in terms of x

in general, *usually* can*not* convert implicit to explicit equation if algebraic equation quintic degree or

higher; that is, contains y^5, y^6 or y^7, and so on

but which (i) **is**   (ii) **is not** differentiable

because *implicit* differentiation will allow us to differentiate the function "part" of the relation $R(x, y)$

(h) implicit equation $R(x, y) = e^{x^2 + y^2} - xe^{3x} + ye^x = 0$
   (i) **can**   (ii) **cannot** be solved as an explicit equation of y in terms of x

which (i) **is**   (ii) **is not** (differentiable

because *implicit* differentiation will allow us to differentiate the function "part" of the relation $R(x, y)$

2. *Implicit differentiation.*

    (a) *Find derivative of $R(x, y) = y - 3x - 5 = 0$.*

        derivative of 0 is $\frac{d}{dx}(0) =$ (i) $\frac{dy}{dx}$ (ii) **0**   (iii) $\mathbf{5y}$

        derivative of 5 is $\frac{d}{dx}(5) =$ (i) **0**   (ii) $\frac{dy}{dx}$ (iii) $\mathbf{5y}$

        derivative of $3x$ is $\frac{d}{dx}(3x) = 3x^{1-1} =$ (i) $\frac{dy}{dx}$ (ii) **3**   (iii) $\mathbf{3y}$

        derivative of y is $\frac{d}{dx}(y) =$ (i) $\mathbf{2y\frac{dy}{dx}}$ (ii) $\mathbf{3y}$ (iii) $\frac{dy}{dx}$

        then, together,

$$\frac{d}{dx}(y) - \frac{d}{dx}(3x) - \frac{d}{dx}(5) = \frac{d}{dx}(0)$$
$$\frac{dy}{dx} - 3 - 0 = 0$$

        or (i) $\frac{dy}{dx} = \mathbf{3}$ (ii) $\frac{dy}{dx} = \mathbf{0}$ (iii) $\frac{dy}{dx} = y$

    (b) *Find derivative of $R(x, y) = y - e^{2x} - 5 = 0$.*

        derivative of 0 is $\frac{d}{dx}(0) =$ (i) **0**   (ii) $\frac{dy}{dx}$ (iii) $\mathbf{5y}$

        derivative of 5 is $\frac{d}{dx}(5) =$ (i) **0**   (ii) $\frac{dy}{dx}$ (iii) $\mathbf{5y}$

        derivative of e^{2x}: let $f[g(x)] = e^{2x}$
        with inner function $g(x) = 2x$ and outer function $f(x) = e^x$
        and $g'(x) =$ (i) **2**   (ii) $\mathbf{2x^2}$ (iii) $\mathbf{4x}$
        and $f'(x) =$ (i) x (ii) $\mathbf{e^x}$ (iii) $\mathbf{e^{2x}}$
        and so by chain rule

$$\frac{d}{dx}\left(e^{2x}\right) = f'[g(x)] \cdot g'(x) = f'[2x] \cdot (2) = e^{2x}(2) =$$

        (i) **3**   (ii) $\frac{dy}{dx}$ (iii) $\mathbf{2e^{2x}}$

        derivative of y is $\frac{d}{dx}(y) =$ (i) $\frac{dy}{dx}$ (ii) $\mathbf{2y\frac{dy}{dx}}$ (iii) $\mathbf{3y}$

        then, together,

$$\frac{d}{dx}(y) - \frac{d}{dx}\left(e^{2x}\right) - \frac{d}{dx}(5) = \frac{d}{dx}(0)$$
$$\frac{dy}{dx} - 2e^{2x} - 0 = 0$$

or (i) $\frac{dy}{dx} = 2e^{2x}$ (ii) $\frac{dy}{dx} = 0$ (iii) $\frac{dy}{dx} = y$

(c) *Find derivative of* $R(x, y) = e^{2y} - 3x - 5 = 0$.

derivative of 0 is $\frac{d}{dx}(0) =$ (i) $\frac{dy}{dx}$ (ii) **0**   (iii) $5y$

derivative of 5 is $\frac{d}{dx}(5) =$ (i) $\frac{dy}{dx}$ (ii) **0**   (iii) $5y$

derivative of $3x$ is $\frac{d}{dx}(3x) = 3x^{1-1} =$ (i) $\frac{dy}{dx}$ (ii) **3**   (iii) $3y$

derivative of e^{2y}: let $f[g(x)] = e^{2y}$
with inner function $g(x) = 2y$ and outer function $f(x) = e^x$
and $g'(x) =$ (i) $2y^2$ (ii) $2\frac{dy}{dx}$ (iii) $4y$
and $f'(x) =$ (i) x (ii) e^x (iii) e^{2x}
and so by chain rule

$$\frac{d}{dx}\left(e^{2y}\right) = f'[g(x)] \cdot g'(x) = f'[2y] \cdot \left(2\frac{dy}{dx}\right) = e^{2y}\left(2\frac{dy}{dx}\right) =$$

(i) **3**   (ii) $\frac{dy}{dx}$ (iii) $2e^{2y}\frac{dy}{dx}$

then, together,

$$\frac{d}{dx}\left(e^{2y}\right) - \frac{d}{dx}(3x) - \frac{d}{dx}(5) = \frac{d}{dx}(0)$$
$$2e^{2y}\frac{dy}{dx} - 3 - 0 = 0$$

or (i) $\frac{dy}{dx} = \frac{3}{2e^{2y}}$ (ii) $\frac{dy}{dx} = 0$ (iii) $\frac{dy}{dx} = y$

(d) *Find derivative of* $R(x, y) = y^2 - x = 0$.

derivative of 0 is $\frac{d}{dx}(0) =$ (i) $\frac{dy}{dx}$ (ii) $5y$ (iii) **0**

derivative of x is $\frac{d}{dx}(x) = x^{1-1} =$ (i) **1**   (ii) $\frac{dy}{dx}$ (iii) y

derivative of y^2: let $f[g(x)] = y^2$
with inner function $g(x) = y$ and outer function $f(x) = x^2$
and $g'(x) =$ (i) $\frac{dy}{dx}$ (ii) $2y^2$ (iii) $4y$
and $f'(x) =$ (i) $2x^{2-1} = 2x$ (ii) e^x (iii) e^{2x}
and so by chain rule

$$\frac{d}{dx}\left(y^2\right) = f'[g(x)] \cdot g'(x) = f'[y] \cdot \left(\frac{dy}{dx}\right) = 2y\left(\frac{dy}{dx}\right) =$$

(i) **3**   (ii) $\frac{dy}{dx}$ (iii) $2y\frac{dy}{dx}$

then, together,

$$\frac{d}{dx}\left(y^2\right) - \frac{d}{dx}(x) = \frac{d}{dx}(0)$$
$$2y\frac{dy}{dx} - 1 = 0$$

or (i) $\frac{dy}{dx} = \frac{1}{2y}$ (ii) $\frac{dy}{dx} = \mathbf{0}$ (iii) $\frac{dy}{dx} = \boldsymbol{y}$

(e) *Find derivative of* $y^2 = x$.

derivative of x is $\frac{d}{dx}(x) = x^{1-1} =$ (i) **1**   (ii) $\frac{dy}{dx}$ (iii) $\boldsymbol{y}$

derivative of y^2: let $f[g(x)] = y^2$
with inner function $g(x) = y$ and outer function $f(x) = x^2$
and $g'(x) =$ (i) $\mathbf{2y^2}$ (ii) $\mathbf{4y}$ (iii) $\frac{dy}{dx}$
and $f'(x) =$ (i) $\mathbf{2x^{2-1} = 2x}$ (ii) e^x (iii) e^{2x}
and so by chain rule

$$\frac{d}{dx}\left(y^2\right) = f'[g(x)] \cdot g'(x) = f'[y] \cdot \left(\frac{dy}{dx}\right) = 2y\left(\frac{dy}{dx}\right) =$$

(i) **3**   (ii) $\frac{dy}{dx}$ (iii) $2y\frac{dy}{dx}$

then, together,

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(x)$$
$$2y\frac{dy}{dx} = 1$$

or (i) $\frac{dy}{dx} = \frac{1}{2y}$ (ii) $\frac{dy}{dx} = \mathbf{0}$ (iii) $\frac{dy}{dx} = \boldsymbol{y}$

(f) *Find derivative of* $y^3 + y^2 = x$.

$$\frac{d}{dx}\left(y^3\right) + \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(x)$$
$$3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} = 1$$
$$\frac{dy}{dx}(3y^2 + 2y) = 1$$

or (i) $\frac{dy}{dx} = \frac{1}{3y^2+2y}$ (ii) $\frac{dy}{dx} = \mathbf{0}$ (iii) $\frac{dy}{dx} = \boldsymbol{y}$

(g) *Find derivative of* $y^5 + 2y^4 - 7y^3 + 3y^2 - 6y = x.$

$$\frac{d}{dx}\left(y^5\right) + \frac{d}{dx}\left(2y^4\right) - \frac{d}{dx}\left(7y^3\right) + \frac{d}{dx}\left(3y^2\right) - \frac{d}{dx}(6y) = \frac{d}{dx}(x)$$

$$5y^{5-1}\frac{dy}{dx} + 2(4)y^{4-1}\frac{dy}{dx} - 7(3)y^{3-1}\frac{dy}{dx} + 3(2)y^{2-1}\frac{dy}{dx} - 6y^{1-1}\frac{dy}{dx} = 1$$

$$\frac{dy}{dx}(5y^4 + 8y^3 - 21y^2 + 6y + 6) = 1$$

or (i) $\frac{dy}{dx} = \mathbf{0}$ (ii) $\frac{dy}{dx} = \frac{1}{5y^4+8y^3-21y^2+6y+6}$ (iii) $\frac{dy}{dx} = \mathbf{y}$

(h) *Find derivative of* $y^2 + 3xy = -5.$

derivative of -5 is $\frac{d}{dx}(-5) = $ (i) $\mathbf{0}$ (ii) $\frac{dy}{dx}$ (iii) $\mathbf{y}$

derivative of y^2 is $\frac{d}{dx}(y^2) = $ (i) $\mathbf{0}$ (ii) $\mathbf{2y\frac{dy}{dx}}$ (iii) $\mathbf{y}$

derivative of $3xy$:
let $u(x) = 3x$ and $v(x) = y$
and $u'(x) = $ (i) $\mathbf{3}$ (ii) $\mathbf{2y^2}$ (iii) $\mathbf{4y}$
and $v'(x) = $ (i) $\frac{dy}{dx}$ (ii) $\mathbf{e^x}$ (iii) $\mathbf{e^{2x}}$
and so by product rule

$$\frac{d}{dx}(xy) = v(x)u'(x) + u(x)v'(x) = y(3) + (3x)\left(\frac{dy}{dx}\right) =$$

(i) $\frac{dy}{dx}$ (ii) $\mathbf{3y + 3x\frac{dy}{dx}}$ (iii) $\mathbf{2y\frac{dy}{dx}}$

then, together,

$$\frac{d}{dx}\left(y^2\right) + \frac{d}{dx}(3xy) = \frac{d}{dx}(-5)$$

$$2y\frac{dy}{dx} + \left(3y + 3x\frac{dy}{dx}\right) = 0$$

$$(2y + 3x)\frac{dy}{dx} + 3y = 0$$

or (i) $\frac{dy}{dx} = -\frac{3y}{3x-2y}$ (ii) $\frac{dy}{dx} = -\frac{3y}{2y+3x}$ (iii) $\frac{dy}{dx} = \mathbf{y}$

(i) *Find equation of tangent line to* $y^2 + 3xy = -5$ *at* $(-3, 8.4).$

Since derivative is $\frac{dy}{dx} = -\frac{3y}{3x+2y}$, derivative (slope) at $(x, y) = (-3, 8.4)$ is

$f' = -\frac{3y}{3x+2y} = -\frac{3(8.4)}{3(-3)+2(8.4)} \approx$ (i) -3.2 (ii) -4.2 (iii) -5.3

so equation of tangent line at $(x_1, f(x_1)) = (-3, 8.4)$ is

$$\begin{aligned} y - f(x_1) &= f'(x_1)(x - x_1) \\ y - (8.4) &= (-3.2)\,(x - (-3)) \\ y - 8.4 &= (-3.2x - 9.6) \end{aligned}$$

or (i) $y = x - 5.4$ (ii) $y = -x - 5.4$ (iii) $y = -3.2x - 1.2$

(j) *Application.* Adult body surface area, S, (in cm^2) related to weight, w, is

$$S(w) = w^w.$$

Determine change in surface area, $S'(w)$, at $w = 2$ kilograms.

first, take logarithms both sides:

$$\begin{aligned} \ln y &= \ln w^w \\ &= w \ln w \end{aligned}$$

derivative of $\ln y$ is $\frac{d}{dw}(\ln y) =$ (i) $\frac{dy}{dx}$ (ii) y (iii) $\frac{1}{y}\frac{dy}{dw}$

derivative of $w \ln w$:
let $u(w) = w$ and $v(w) = \ln w$
and $u'(w) =$ (i) $2w^2$ (ii) $4y$ (iii) 1
and $v'(w) =$ (i) $\frac{1}{w}$ (ii) e^w (iii) e^{2w}
and so by product rule

$$\frac{d}{dw}(w \ln w) = v(w)u'(w) + u(w)v'(w) = \ln w(1) + (w)\left(\frac{1}{w}\right) =$$

(i) $\ln w + 1$ (ii) $\frac{dy}{dw}$ (iii) $2y\frac{dy}{dw}$

then, together,

$$\begin{aligned} \frac{d}{dw}(\ln y) &= \frac{d}{dw}(w \ln w) \\ \frac{1}{y}\frac{dy}{dw} &= \ln w + 1 \end{aligned}$$

or $S'(w) = f'(w) =$
(i) $\frac{dy}{dw} = y(\ln w + 1) = w^w(\ln w + 1)$
(ii) $\frac{dy}{dw} = -\frac{3y}{3x-2y}$
(iii) $\frac{dy}{dw} = y$

and so $S'(2) = (2)^2(\ln 2 + 1) \approx$ (i) 4.77 (ii) 5.77 (iii) 6.77

6.5   Related Rates

We apply implicit differentiation to *related rates* problems, where all variables depend on time, t. A characteristic of related rates problems is derivatives of different variables; for example, $\frac{dx}{dt}$ and $\frac{dy}{dt}$, are related to one another because they appear in one equation, say,

$$y^2 \frac{dx}{dt} + 2y \frac{dy}{dt} = 0.$$

Steps for solving related rates problems include

- Identify *variables* and relationship between variables, the *equation*.

  draw a picture to make things clear if possible

  variables are *always* dependent on time, t

- Perform *implicit differentiation* on equation.

  differentiation is *always* with respect to time, t

- Solve for unknown rate of change (derivative).

  use given known quantities and derivatives in solution

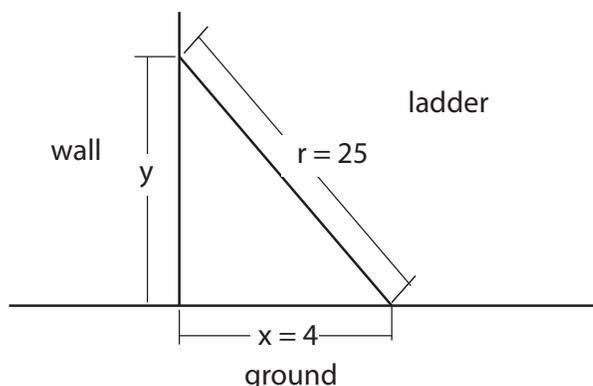**Exercise 6.5 (Related Rates)**

1. *Ladder.*

Figure 6.13 (Ladder)

A 25 foot ladder leans against a wall. The base of the ladder is 4 feet from the wall and is slipping to the right at a rate of 2 feet per minute. Determine the *rate* at which the top of the ladder is slipping down the wall. Also determine the rate at which the area between ladder, wall and ground changes.

(a) *Variables and equations*
Equations relating variables are

$$x^2 + y^2 = r^2, \qquad A = \frac{1}{2}xy$$

where x is distance between base of ladder and wall, y is distance between top of ladder and ground, r is length of ladder (a constant), and A is area between ladder, wall and ground.

(b) *Implicit differentiation, $x^2 + y^2 = r^2$*

derivative of x^2 is $\frac{d}{dt}\left(x^2\right) =$ (i) $\mathbf{2x\frac{dx}{dt}}$ (ii) $\frac{1}{x}\frac{dx}{dt}$ (iii) $\frac{dx}{dt}$

derivative of y^2 is $\frac{d}{dt}\left(y^2\right) =$ (i) $\mathbf{2y\frac{dy}{dt}}$ (ii) $\frac{1}{y}\frac{dy}{dt}$ (iii) $\frac{dy}{dt}$

derivative of r (a constant) is $\frac{d}{dt}\left(r^2\right) =$ (i) $\mathbf{0}$ (ii) $\frac{1}{r}\frac{dr}{dt}$ (iii) $\frac{dr}{dt}$

then, together,

$$\frac{d}{dt}\left(x^2\right) + \frac{d}{dt}\left(y^2\right) = \frac{d}{dt}\left(r^2\right)$$
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

or (i) $\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}$ (ii) $\frac{dy}{dt} = -\frac{y}{x}\frac{dx}{dt}$ (iii) $\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt}$

(c) *Solve for $\frac{dy}{dt}$, $x^2 + y^2 = r^2$*

Since $x =$ (i) $\mathbf{25}$ (ii) $\mathbf{4}$ (iii) $\mathbf{2}$

and $r =$ (i) $\mathbf{25}$ (ii) $\mathbf{4}$ (iii) $\mathbf{2}$

so $y = \sqrt{r^2 - x^2} = \sqrt{25^2 - 4^2} \approx$ (i) $\mathbf{24.68}$ (ii) $\mathbf{25.68}$ (iii) $\mathbf{26.68}$

and $\frac{dx}{dt} =$ (i) $\mathbf{25}$ (ii) $\mathbf{4}$ (iii) $\mathbf{2}$

then, together,

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} \approx -\frac{4}{24.68}(2) \approx$$

(i) $\mathbf{-0.32}$ (ii) $\mathbf{0.18}$ (iii) $\mathbf{0.28}$

(d) *Implicit differentiation, $A = \frac{1}{2}xy$*

derivative of A is $\frac{d}{dt}\left(A\right) =$ (i) $\mathbf{A}$ (ii) $\frac{1}{A}\frac{dA}{dt}$ (iii) $\frac{dA}{dt}$

derivative of $\frac{1}{2}xy$:
let $u(t) = \frac{x}{2}$ and $v(t) = y$
and $u'(t) =$ (i) $\frac{1}{2}$ (ii) $\frac{1}{2} \cdot \frac{dx}{dt}$ (iii) $\frac{dx}{dt}$

and $v'(t) =$ (i) $\frac{dy}{dt}$ (ii) $\mathbf{2y}$ (iii) $\mathbf{2\frac{dy}{dt}}$
and so by product rule

$$\frac{d}{dt}\left(\frac{1}{2}xy\right) = v(t)u'(t) + u(t)v'(t) = y\left(\frac{1}{2} \cdot \frac{dx}{dt}\right) + \left(\frac{x}{2}\right)\left(\frac{dy}{dt}\right) =$$

(i) $\frac{1}{2}\left(\boldsymbol{y} \cdot \frac{d\boldsymbol{x}}{d\boldsymbol{t}} + \boldsymbol{x}\frac{d\boldsymbol{y}}{d\boldsymbol{t}}\right)$ (ii) $\frac{1}{2}\left(\boldsymbol{y} \cdot \frac{d\boldsymbol{x}}{d\boldsymbol{t}}\right)$ (iii) $\frac{1}{2}\boldsymbol{x}\left(\frac{d\boldsymbol{y}}{d\boldsymbol{t}}\right)$

then, together,

$$\begin{aligned}
\frac{d}{dt}(A) &= \frac{d}{dt}\left(\frac{1}{2}xy\right) \\
\frac{dA}{dt} &= \frac{1}{2}\left(y \cdot \frac{dx}{dt} + x\frac{dy}{dt}\right)
\end{aligned}$$

(e) *Solve for* $\frac{dA}{dt}$, $A = \frac{1}{2}xy$

Since $x =$ (i) $\mathbf{25}$ (ii) $\mathbf{4}$ (iii) $\mathbf{2}$

so $y = \sqrt{r^2 - x^2} = \sqrt{25^2 - 4^2} \approx$ (i) $\mathbf{24.68}$ (ii) $\mathbf{25.68}$ (iii) $\mathbf{26.68}$

and $\frac{dx}{dt} =$ (i) $\mathbf{25}$ (ii) $\mathbf{4}$ (iii) $\mathbf{2}$

and from above $\frac{dy}{dt} \approx -0.32$

then, together,

$$\begin{aligned}
\frac{dA}{dt} &= \frac{1}{2}\left(y \cdot \frac{dx}{dt} + x\frac{dy}{dt}\right) \\
&= \frac{1}{2}\left(24.68 \cdot 2 + 4 \cdot (-0.32)\right) \approx
\end{aligned}$$

(i) $\mathbf{24.04}$ (ii) $\mathbf{25.4}$ (iii) $\mathbf{26.04}$

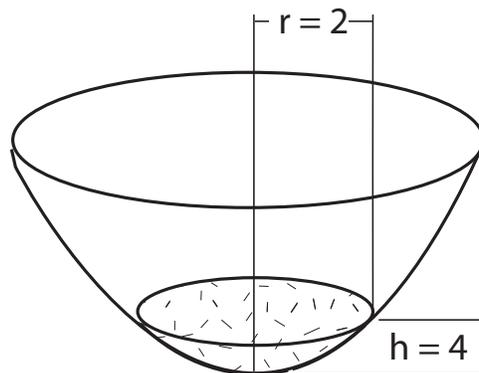2. *Bowl of Chicken Noodle Soup.*

Figure 6.14 (Bowl of Chicken Noodle Soup)

Chicken noodle soup is poured into an (unstable) *paraboloid* bowl at the constant (volume) rate of 200 cubic centimeters (cm^3) per second; furthermore, the radius of soup in bowl is changing at a rate of 5 centimeters per second. Determine the *rate* at which the height of the soup in the bowl is rising (that is, the change in the height (cm) per unit time (seconds)) when the height is 4 cm and the radius is 2 cm.

(a) *Variables and equation*
    Equation relating variables is

$$V = \frac{\pi h r^2}{2}$$

where V is volume, h is height of soup in bowl and r is radius of soup in bowl, and V, h and r and change with time, t; and π is a constant

(b) *Implicit differentiation*

derivative of V is $\frac{d}{dt}(V) =$ (i) $\boldsymbol{V}$ (ii) $\boldsymbol{\frac{1}{V}\frac{dV}{dt}}$ (iii) $\boldsymbol{\frac{dV}{dt}}$

derivative of $\frac{\pi h r^2}{2}$:
let $u(t) = \frac{\pi}{2}h$ and $v(t) = r^2$
and $u'(t) =$ (i) $\boldsymbol{\pi}$ (ii) $\boldsymbol{\frac{\pi}{2} \cdot \frac{dh}{dt}}$ (iii) $\boldsymbol{\frac{dh}{dt}}$
and $v'(t) =$ (i) $\boldsymbol{2r\frac{dr}{dt}}$ (ii) $\boldsymbol{2r}$ (iii) $\boldsymbol{\frac{dr}{dt}}$
and so by product rule

$$\frac{d}{dt}\left(\frac{\pi h r^2}{2}\right) = v(t)u'(t) + u(t)v'(t) = r^2\left(\frac{\pi}{2} \cdot \frac{dh}{dt}\right) + \frac{\pi}{2}h\left(2r\frac{dr}{dt}\right) =$$

(i) $\boldsymbol{\frac{\pi r^2}{2} \cdot \frac{dh}{dt} + \pi h r\frac{dr}{dt}}$ (ii) $\boldsymbol{r^2\left(\pi\frac{dh}{dt}\right)}$ (iii) $\boldsymbol{\pi h\left(2r\frac{dr}{dt}\right)}$

then, together,

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{\pi h r^2}{2}\right)$$

$$\frac{dV}{dt} = \frac{\pi r^2}{2} \cdot \frac{dh}{dt} + \pi h r \frac{dr}{dt}$$

or (i) $\frac{dV}{dt} = \frac{\pi r}{2}\left(r\frac{dh}{dt} + 2h\frac{dr}{dt}\right)$
(ii) $\frac{dV}{dt} = r^2\left(\pi\frac{dh}{dt}\right)$
(iii) $\frac{dV}{dt} = \pi h\left(2r\frac{dr}{dt}\right)$

(c) *Solve for* $\frac{dh}{dt}$

Since $h =$ (i) **4**   (ii) **5**   (iii) **200**   (iv) **2**

and $r =$ (i) **4**   (ii) **5**   (iii) **200**   (iv) **2**

and $\frac{dr}{dt} =$ (i) **4**   (ii) **5**   (iii) **200**   (iv) **2**

and $\frac{dV}{dt} =$ (i) **4**   (ii) **5**   (iii) **200**   (iv) **2**

$$\frac{dV}{dt} = \frac{\pi r}{2}\left(r\frac{dh}{dt} + 2h\frac{dr}{dt}\right)$$

$$200 = \frac{\pi(2)}{2}\left((2)\frac{dh}{dt} + 2(4)(5)\right)$$

$$\frac{dh}{dt} = \frac{100}{\pi} - 20 \approx$$

(i) **11.83**   (ii) **12.83**   (iii) **14.83**

3. *Zip Drives and Profit.* A company estimates that at time t, its future yearly demand, d, for its zip drives, will decrease in the following way,

$$d = \frac{75}{t-2} + 2.$$

Also, its annual profit, P, with respect to demand, d, is given by

$$2P = 2d^3 + 10.$$

Determine the rate of profit (that is, the change in the profit per year) 6 years from now.

(a) *Variables and function*
Equations relating variables are

$$2P = 2d^3 + 10, \quad d = \frac{75}{t-2} + 2$$

where P is profit, d is demand and both P and d change with time, t

(b) *Implicit differentiation,* $2P = 2d^3 + 10$

derivative of P is $\frac{d}{dt}(2P) =$ (i) $\boldsymbol{P}$ (ii) $\frac{1}{P}\frac{dP}{dt}$ (iii) $2\frac{dP}{dt}$

derivative of $2d^3$ is $\frac{d}{dt}(2d^3) =$ (i) $\boldsymbol{6d^2}$ (ii) $\frac{1}{d}\frac{dd}{dt}$ (iii) $\boldsymbol{2(3)d^2\frac{dd}{dt}}$

derivative of 10 is $\frac{d}{dt}(10) =$ (i) $\boldsymbol{0}$ (ii) $\frac{1}{d}\frac{dd}{dt}$ (iii) $\boldsymbol{2(3)d^2\frac{dd}{dt}}$

then, together,

$$\frac{d}{dt}(2P) = \frac{d}{dt}(2d^3) + \frac{d}{dt}(10)$$
$$2\frac{dP}{dt} = 2(3)d^2\frac{dd}{dt} + 0$$

or (i) $\frac{dP}{dt} = 3d^2\frac{dd}{dt}$ (ii) $\frac{dP}{dt} = 6d^2$ (iii) $\frac{dP}{dt} = \frac{dd}{dt}$

(c) *Implicit differentiation,* $d = \frac{75}{t-2} + 2$

derivative of d is $\frac{d}{dt}(d) =$ (i) $\boldsymbol{d}$ (ii) $\frac{1}{d}\frac{dd}{dt}$ (iii) $\frac{dd}{dt}$

derivative of 10 is $\frac{d}{dt}(10) =$ (i) $\boldsymbol{0}$ (ii) $\frac{1}{d}\frac{dd}{dt}$ (iii) $\boldsymbol{2(3)d^2\frac{dd}{dt}}$

derivative of $\frac{75}{t-2}$:
let $u(t) = 75$ and $v(t) = t - 2$
and $u'(t) =$ (i) $\boldsymbol{0}$ (ii) $\boldsymbol{t-2}$ (iii) $\boldsymbol{75}$
and $v'(t) =$ (i) $\boldsymbol{1}$ (ii) $\boldsymbol{2r}$ (iii) $\frac{dr}{dt}$
and so by quotient rule

$$\frac{d}{dt}\left(\frac{75}{t-2}\right) = \frac{v(t)u'(t) - u(t)v'(t)}{[v(t)]^2} = \frac{(t-2)(0) - 75(1)}{[t-2]^2} =$$

(i) $-\frac{75}{(t-2)^2}$ (ii) $\boldsymbol{75}$ (iii) $\frac{75}{t-2}$

then, together,

$$\frac{d}{dt}(d) = \frac{d}{dt}\left(\frac{75}{t-2}\right) + \frac{d}{dt}(2)$$
$$\frac{dd}{dt} = -\frac{75}{(t-2)^2}$$

(d) *Solve for* $\frac{dP}{dt}$

Since

$$\frac{dP}{dt} = 3d^2 \frac{dd}{dt}, \qquad \frac{dd}{dt} = -\frac{75}{(t-2)^2}$$

where $t =$ (i) **4** (ii) **5** (iii) **6**

and so $d = \frac{75}{t-2} + 2 = \frac{75}{6-2} + 2 =$ (i) $\frac{75}{16}$ (ii) $\frac{16}{75}$ (iii) $\frac{83}{4}$

and $\frac{dd}{dt} = \frac{75}{(t-2)^2} = -\frac{75}{(6-2)^2} =$ (i) $-\frac{75}{16}$ (ii) $-\frac{16}{75}$ (iii) $\frac{83}{4}$

so

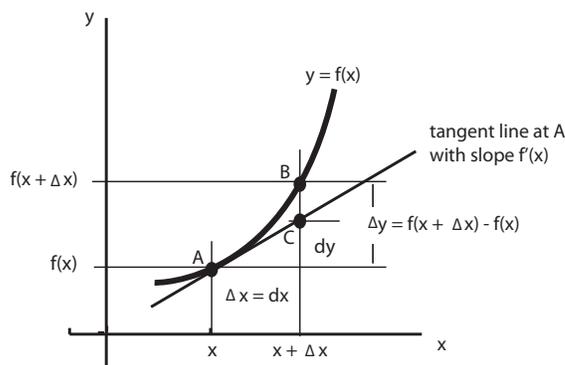$$\frac{dP}{dt} = 3d^2 \frac{dd}{dt} = 3\left(\frac{83}{4}\right)^2 \left(-\frac{75}{16}\right) \approx$$

(i) $-\mathbf{6054.8}$ (ii) $-\mathbf{13109.6}$ (iii) $-\mathbf{84109.6}$

6.6 Differentials: Linear Approximation

For a function $y = f(x)$ whose derivative exists, $f'(x) = \frac{dy}{dx}$, and also given *differential* dx (typically a small number relative to x), the differential of y is given by

$$dy = \frac{dy}{dx}dx = f'(x)dx,$$

where dy is "rise" and dx is "run" of the slope of the tangent to point A in the figure.

Figure 6.15 (Differentials and linear approximation)

Notice the *exact* value of function at point B in figure equals

$$f(x + \Delta x) = f(x) + \Delta y.$$

If differential dx is *small, then* $\Delta y \approx dy$, and so the value of the function at point B could be *(linearly) approximated* by point C,

$$f(x + \Delta x) \approx f(x) + dy = f(x) + f'(x)dx.$$

Exercise 6.6 (Differentials: Linear Approximation)

1. *Approximation $\Delta y \approx dy = f'(x)dx$.*

   (a) $f(x) = x^2 + 3x$, $x = 2$ and $dx = 0.1$.

   Since $f'(x) = 2x^{2-1} + 3(1)x^{1-1} =$
   (i) $\mathbf{2x^2}$ (ii) $\mathbf{2x + 3}$ (iii) $\mathbf{2x^3}$,

   $\Delta y \approx dy = f'(x)dx = (2x + 3)dx =$
   $(2(2) + 3)(0.1) =$ (i) $\mathbf{0.4}$ (ii) $\mathbf{0.5}$ (iii) $\mathbf{0.7}$
   exact $\Delta y = f(2.1) - f(2) = \left[2.1^2 + 3(2.1)\right] - \left[2^2 + 3(2)\right] = 0.71$,
   so approximation minus exact: $0.7 - 0.71 = -0.01$

   (b) $f(x) = x^2 + 3x$, $x = 3$ and $dx = 0.01$.

   Since $f'(x) = 2x^{2-1} + 3(1)x^{1-1} =$
   (i) $\mathbf{2x^2}$ (ii) $\mathbf{2x + 3}$ (iii) $\mathbf{2x^3}$,

   $\Delta y \approx dy = f'(x)dx = (2x + 3)dx =$
   $(2(3) + 3)(0.01) =$ (i) $\mathbf{0.04}$ (ii) $\mathbf{0.05}$ (iii) $\mathbf{0.09}$
   exact $\Delta y = f(3.01) - f(3) = \left[3.01^2 + 3(3.01)\right] - \left[3^2 + 3(3)\right] = 0.0901$,
   so approximation minus exact: $0.09 - 0.0901 = -0.0001$

   (c) $f(x) = 3x^3 + 3x^2 - 4x$, $x = 1$ and $dx = 0.01$.

   Since $f'(x) = 3x^{3-1} + 3(2)x^{2-1} - 4(1)x =$
   (i) $\mathbf{9x^2 + 6x - 4}$ (ii) $\mathbf{2x^2 + 3x}$ (iii) $\mathbf{2x^3 - 4}$,

   $\Delta y \approx dy = f'(x)dx = (9x^2 + 6x - 4)dx =$
   $(9(1)^2 + 6(1) - 4)(0.01) =$ (i) $\mathbf{0.08}$ (ii) $\mathbf{0.09}$ (iii) $\mathbf{0.11}$

   (d) $y = \sqrt{3x + x^2}$, $x = 1$ and $dx = 0.01$

   let $f[g(x)] = (3x + x^2)^{\frac{1}{2}}$
   with "inner" function $g(x) =$ (i) $\mathbf{3x^2 - 2}$ (ii) $\mathbf{3x}$ (iii) $\mathbf{3x + x^2}$
   and "outer" function $f(x) =$ (i) $\mathbf{x^{\frac{1}{2}}}$ (ii) $\mathbf{x^{\frac{3}{2}}}$ (iii) $\mathbf{x^{\frac{5}{2}}}$
   with derivative $g'(x) =$ (i) $\mathbf{3 + 2x}$ (ii) $\mathbf{6x - 2}$ (iii) $\mathbf{3}$
   and derivative $f'(x) =$ (i) $\frac{3}{2}x^{\frac{1}{2}}$ (ii) $\frac{3}{2}x^{\frac{3}{2}}$ (iii) $\frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

   and so by chain rule

   $$f'[g(x)] \cdot g'(x) \quad = \quad f'[3x + x^2] \cdot (3 + 2x)$$

$$= \frac{1}{2\sqrt{3x + x^2}}(3 + 2x) =$$

(i) $\dfrac{3+2x}{\sqrt{3x+x^2}}$ (ii) $\dfrac{3+2x}{2\sqrt{3x+x^2}}$ (iii) $\dfrac{3+2x}{3\sqrt{3x+x^2}}$

so $\Delta y \approx dy = f'(x)dx = \left(\dfrac{3+2x}{2\sqrt{3x+x^2}}\right)dx =$

$\left(\dfrac{3+2(1)}{2\sqrt{3(1)+(1)^2}}\right)(0.01) =$ (i) **0.0125**   (ii) **0.0625**   (iii) **0.0785**

(e) $f(x) = \frac{4x-1}{5x+4}$, $x = 1$ and $dx = 0.01$

let $u(x) = 4x - 1$ and $v(x) = 5x + 4$.
then, $u'(x) =$ (i) $\mathbf{4x^2}$ (ii) $\mathbf{4}$ (iii) $\mathbf{-1}$
and $v'(x) =$ (i) $\mathbf{5x^2}$ (ii) $\mathbf{5}$ (iii) $\mathbf{3 + 4x}$
and $v(x)u'(x) =$ (i) $\mathbf{(5x + 4)(4)}$ (ii) $\mathbf{(5x + 4)(4x^2)}$ (iii) $\mathbf{(5x^2 + 4)(4)}$
and $u(x)v'(x) =$ (i) $\mathbf{(4x^2 - 1)(5)}$ (ii) $\mathbf{(4x - 1)(5)}$ (iii) $\mathbf{(4x - 1)(5x)}$
and so $f'(x) = \frac{v(x)\cdot u'(x) - u(x)\cdot v'(x)}{[v(x)]^2} =$
(i) $\dfrac{(5x+4)(4)-(4x-1)(5)}{(5x+4)^2}$ (ii) $\dfrac{(5x+4)(4)+(4x-1)(5)}{(5x+4)^2}$ (iii) $\dfrac{(5x+4)(4)+(4x-1)(5)}{(4x-1)^2}$
which equals
(i) $\dfrac{21}{(5x+4)^2}$ (ii) $\dfrac{40x+11}{(5x+4)^2}$ (iii) $\dfrac{20}{(5x+4)^2}$

so $\Delta y \approx dy = f'(x)dx = \left(\dfrac{21}{(5x+4)^2}\right)dx =$

$\left(\dfrac{21}{(5(1)+4)^2}\right)(0.01) \approx$ (i) **0.003**   (ii) **0.006**   (iii) **0.009**

exact $\Delta y = f(1.01) - f(1) = \frac{4(1.01)-1}{5(1.01)+4} - \frac{4(1)-1}{5(1)+4} = 0.002578$,

so approximation minus exact: $0.002593 - 0.002578 \approx 0.000015$

2. *Approximation $f(x + \Delta x) \approx f(x) + f'(x)dx$.*

(a) *Approximate $f(65) = \sqrt{65}$.*

let

$$\begin{aligned}
\Delta y &= f(x + \Delta x) - f(x) \\
&= \sqrt{x + \Delta x} - \sqrt{x} \\
&= \sqrt{65} - \sqrt{64} \\
&= \sqrt{65} - \sqrt{8^2} \\
&= \sqrt{65} - 8
\end{aligned}$$

where we take advantage of the fact $\sqrt{8^2} = 8$

implying $f(x) = $ (i) $\sqrt{x} = x^{\frac{1}{2}}$ (ii) x^2 (iii) $\frac{x}{2}$

and so $f'(x) = \frac{1}{2}x^{\frac{1}{2}-1} = $ (i) $\frac{1}{2\sqrt{x}}$ (ii) $\sqrt{x}$ (iii) $\frac{x}{2}$

and $x = $ (i) **1**   (ii) **64**   (iii) **65**

and $\Delta x = dx = 65 - 64 = $ (i) **1**   (ii) **64**   (iii) **65**

so

$$
\begin{aligned}
\sqrt{65} &= \Delta y + 8 \\
&\approx dy + 8 \quad \text{\scriptsize since } dy \approx \Delta y \\
&= f'(x) \cdot dx + 8 \\
&= \frac{1}{2\sqrt{x}} \cdot dx + 8 \\
&= \frac{1}{2\sqrt{64}} \cdot (1) + 8 \approx
\end{aligned}
$$

(i) **8.0625**   (ii) **8.0725**   (iii) **8.0825**

exact $\sqrt{65} \approx 8.0623$, so approximation minus exact: $8.0625 - 8.0623 = 0.0002$

(b) *Approximate* $f(79) = \sqrt{79}.$

let

$$
\begin{aligned}
\Delta y &= f(x + \Delta x) - f(x) \\
&= \sqrt{x + \Delta x} - \sqrt{x} \\
&= \sqrt{79} - \sqrt{81} \\
&= \sqrt{79} - \sqrt{9^2} \\
&= \sqrt{79} - 9
\end{aligned}
$$

where we take advantage of the fact $\sqrt{9^2} = 9$

implying $f(x) = $ (i) $\sqrt{x} = x^{\frac{1}{2}}$ (ii) x^2 (iii) $\frac{x}{2}$

and so $f'(x) = \frac{1}{2}x^{\frac{1}{2}-1} = $ (i) $\frac{1}{2\sqrt{x}}$ (ii) $\sqrt{x}$ (iii) $\frac{x}{2}$

and $x = $ (i) **79**   (ii) **81**   (iii) **2**

and $\Delta x = dx = 79 - 81 = $ (i) **1**   (ii) **−1**   (iii) **−2**

so

$$
\begin{aligned}
\sqrt{79} \; &= \; \Delta y + 9 \\
&\approx \; dy + 9 \quad \text{\scriptsize since } dy \approx \Delta y \\
&= \; f'(x) \cdot dx + 9 \\
&= \; \frac{1}{2\sqrt{x}} \cdot dx + 9 \\
&= \; \frac{1}{2\sqrt{81}} \cdot (-2) + 9 \approx
\end{aligned}
$$

(i) **8.8446**   (ii) **8.8889**   (iii) **8.7448**

exact $\sqrt{79} \approx 8.8882$, so approximation minus exact: $8.8889 - 8.8882 = 0.0007$

(c) *Approximate* $f(0.02) = e^{0.02}$.

let

$$
\begin{aligned}
\Delta y \; &= \; f(x + \Delta x) - f(x) \\
&= \; e^{x + \Delta x} - e^x \\
&= \; e^{0.02} - e^0 \\
&= \; e^{0.02} - 1
\end{aligned}
$$

where we take advantage of the fact $e^0 = 1$

implying $f(x) =$ (i) $\sqrt{x} = x^{\frac{1}{2}}$ (ii) e^x (iii) $\frac{x}{2}$

and so $f'(x) =$ (i) $\frac{1}{2\sqrt{x}}$ (ii) e^x (iii) $\frac{x}{2}$

and $x =$ (i) **0**   (ii) **0.02**   (iii) $e^{0.02}$

and $\Delta x = dx = 0.02 - 0 =$ (i) **0.02**   (ii) **0.01**   (iii) **0**

so

$$
\begin{aligned}
e^{0.02} \; &= \; \Delta y + 1 \\
&\approx \; dy + 1 \quad \text{\scriptsize since } dy \approx \Delta y \\
&= \; f'(x) \cdot dx + 1 \\
&= \; e^x \cdot dx + 1 \\
&= \; e^0 \cdot (0.02) + 1 \approx
\end{aligned}
$$

(i) **1.02**   (ii) **0.98**   (iii) **0.96**

exact $e^{0.02} \approx 1.0202$, so approximation minus exact: $1.02 - 1.0202 = -0.0002$

3. *Application: Cost Function*
   The cost function for a product is given by

   $$C(x) = 0.09x^2 + 1.4x + 95$$

   Use differential $dy = f'(x)dx$ to approximate change in cost when number of units increases from 100 to 101.

   Since $f'(x) = 0.09(2)x^{2-1} + 1.4(1)x^{1-1} + 0 =$
   (i) $\mathbf{0.18x + 1.4}$ (ii) $\mathbf{0.18x^2 + 1.4}$ (iii) $\mathbf{0.9x^2 + 1.4x}$,

   since increase from 100 to 101,
   $dx = 101 - 100 =$ (i) $\mathbf{0.1}$ (ii) $\mathbf{1}$ (iii) $\mathbf{0.01}$

   so $dy = f'(x)dx = (0.18x + 1.4)dx =$
   $(0.18(100) + 1.4)(1) =$ (i) $\mathbf{20.5}$ (ii) $\mathbf{21.7}$ (iii) $\mathbf{19.4}$
   exact $\Delta y = C(101) - C(100) = \left[0.09(101)^2 + 1.4(101) + 95\right] - \left[0.09(100)^2 + 1.4(100) + 95\right] = 19.49$,

   so approximation minus exact: $19.4 - 19.49 = -0.09$