Chapter 5

Increasing and Decreasing Functions

Derivatives are used to describe the shapes of graphs of functions.

5.1 Increasing and Decreasing Functions

For any two values x_1 and x_2 in an interval,

$$\begin{aligned} f(x) \text{ is increasing if} f(x_1) < f(x_2) & \quad \text{if} \quad x_1 < x_2, \\ f(x) \text{ is decreasing if} f(x_1) > f(x_2) & \quad \text{if} \quad x_1. \end{aligned}$$

Derivatives can be used to determine whether a function is increasing, decreasing or constant on an interval:

$$\begin{aligned} f(x) \text{ is increasing if} & \quad \text{derivative } f'(x) > 0, \\ f(x) \text{ is decreasing if} & \quad \text{derivative } f'(x) < 0, \\ f(x) \text{ is constant if} & \quad \text{derivative } f'(x) = 0. \end{aligned}$$

A *critical number*, c, is one where $f'(c) = 0$ or $f'(c)$ does not exist; a *critical point* is $(c, f(c))$. After locating the critical number(s), choose test values in each interval between these critical numbers, then calculate the derivatives at the test values to decide whether the function is increasing or decreasing in each given interval. (In general, identify values of the function which are *dis*continuous, so, in addition to critical numbers, also watch for values of the function which are not defined, at vertical asymptotes or singularities ("holes").)

Exercise 10.1 (Increasing and Decreasing Functions)

1. *Application: throwing a ball.*

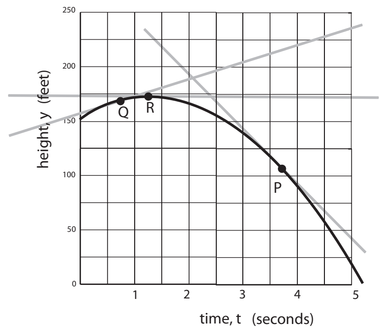


Figure 5.1 (Graph of function of throwing a ball)

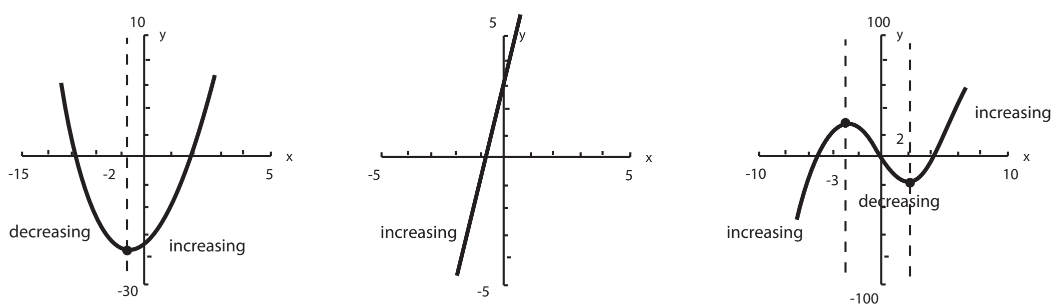
Identify time(s) where height increases and time(s) where height decreases.

(a) *Critical time.* Change in height stops at
$c =$ (i) **0.75** (ii) **1.25** (iii) **3.75**

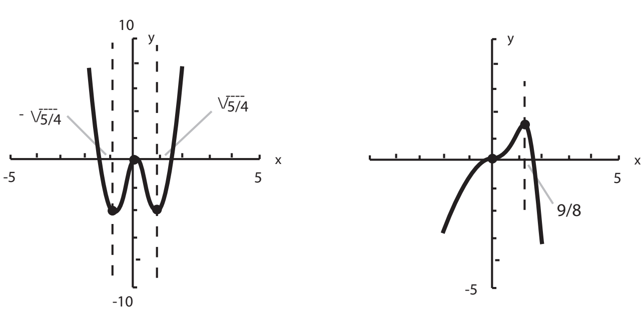
(b) *Intervals*
So there are *two* intervals to investigate
(i) $\mathbf{(-\infty, 1.25)}$ (ii) $\mathbf{(0, 1.25)}$ (iii) $\mathbf{(1.25, 5)}$

(c) *Times for increasing and decreasing heights.*
Height y (i) **increases** (ii) **decreases** over $(0, 1.25)$ interval
and (i) **increases** (ii) **decreases** over $(1.25, 5)$ interval

2. *Functions.*



(a) $f(x) = x^2 + 4x - 21$ (b) $f(x) = 5x + 4$ (c) $f(x) = 2x^3 + 3x^2 - 36x$



(d) $f(x) = 2x^4 - 5x^2$ (e) $f(x) = 3x^3 - 2x^4$

Figure 5.2 (Some functions)

(Type all functions into your calculator using Y= and set WINDOW dimensions to suit each function.

(a) *Figure (a):* $f(x) = x^2 + 4x - 21$.
Since
$$f'(x) = 2x + 4 = 0,$$
there is a *critical number* at
$c = -\frac{4}{2} =$ (i) $\mathbf{-2}$ (ii) $\mathbf{2}$ (iii) $\mathbf{0}$
and so there are *two* intervals to investigate
(i) $\mathbf{(-\infty, -2)}$ (ii) $\mathbf{(-2, 2)}$ (iii) $\mathbf{(-2, \infty)}$
with *two* possible *test values* (in each interval) to check, say:
$x =$ (i) $\mathbf{-3}$ (ii) $\mathbf{-2}$ (iii) $\mathbf{0}$

since $f'(-3) = 2(-3) + 4 = -2$ is negative,
function $f(x)$ (i) **increases** (ii) **decreases** over $(-\infty, -2)$ interval

and $f'(0) = 2(0) + 4 = 4$ is positive,
function $f(x)$ (i) **increases** (ii) **decreases** over $(-2, \infty)$ interval

(b) *Figure (b):* $f(x) = 5x + 4$.
Since
$$f'(x) = 5,$$
there are *no* critical numbers
and so there is only *one* interval to investigate
(i) $\mathbf{(-\infty, \infty)}$ (ii) $\mathbf{(-2, 2)}$ (iii) $\mathbf{(-2, \infty)}$
where since $f'(x) = 5$ is positive for *all* x,
function $f(x)$ (i) **increases** (ii) **decreases** over $(-\infty, \infty)$ interval

(c) *Figure (c):* $f(x) = 2x^3 + 3x^2 - 36x$.
Since
$$f'(x) = 6x^2 + 6x - 36 = 6(x^2 + x - 6) = 6(x + 3)(x - 2) = 0,$$
there are *two* critical numbers at
$c =$ (i) $\mathbf{-3}$ (ii) $\mathbf{2}$ (iii) $\mathbf{6}$
and so there are *three* intervals to investigate
(i) $\mathbf{(-\infty, -3)}$ (ii) $\mathbf{(-\infty, 2)}$ (iii) $\mathbf{(-2, 3)}$ (iv) $\mathbf{(-3, 2)}$ (v) $\mathbf{(2, \infty)}$
with *three* possible test values to check, say:
$x =$ (i) $\mathbf{-4}$ (ii) $\mathbf{-3}$ (iii) $\mathbf{0}$ (iv) $\mathbf{2}$ (v) $\mathbf{3}$

since $f'(-4) = 6(-4 + 3)(-4 - 2)$ is (i) **positive** (ii) **negative**
function $f(x)$ (i) **increases** (ii) **decreases** over $(-\infty, -3)$ interval

since $f'(0) = 6(0+3)(0-2)$ is (i) **positive** (ii) **negative**
function $f(x)$ (i) **increases** (ii) **decreases** over $(-3,-2)$ interval

since $f'(3) = 6(3+3)(3-2)$ is (i) **positive** (ii) **negative**
function $f(x)$ (i) **increases** (ii) **decreases** over $(3,\infty)$ interval

(d) *Figure (d):* $f(x) = 2x^4 - 5x^2$.
Since

$$f'(x) = 8x^3 + 10x = 8x\left(x^2 - \frac{5}{4}\right) = 8x\left(x - \sqrt{\frac{5}{4}}\right)\left(x + \sqrt{\frac{5}{4}}\right) = 0,$$

there are *three* critical numbers at
$c =$ (i) $\mathbf{-\frac{5}{4}}$ (ii) $\mathbf{-\sqrt{\frac{5}{4}}}$ (iii) $\mathbf{0}$ (iv) $\mathbf{\sqrt{\frac{5}{4}}}$ (v) $\mathbf{\frac{5}{4}}$
and so there are *four* intervals to investigate
(i) $\mathbf{\left(-\infty, -\sqrt{\frac{5}{4}}\right)}$ (ii) $\mathbf{\left(-\sqrt{\frac{5}{4}}, 0\right)}$ (iii) $\mathbf{\left(0, \sqrt{\frac{5}{4}}\right)}$
(iv) $\mathbf{\left(0, \sqrt{\frac{5}{4}}\right)}$ (v) $\mathbf{\left(\sqrt{\frac{5}{4}}, \infty\right)}$
with *four* possible test values to check, say:
$x =$ (i) $\mathbf{-2}$ (ii) $\mathbf{-1}$ (iii) $\mathbf{0}$ (iv) $\mathbf{1}$ (v) $\mathbf{2}$

since $f'(-2) = 8(-2)\left(-2 - \sqrt{\frac{5}{4}}\right)\left(-2 + \sqrt{\frac{5}{4}}\right)$ (i) **positive** (ii) **negative**
Notice $\sqrt{\frac{5}{4}} \approx 1.12$, so since $8(-2) < 0$, $\left(-2 - \sqrt{\frac{5}{4}}\right) < 0$ and $\left(-2 + \sqrt{\frac{5}{4}}\right) < 0$, $f'(-2) < 0$.
function $f(x)$ (i) **increases** (ii) **decreases** over $\left(-\infty, -\sqrt{\frac{5}{4}}\right)$ interval

since $f'(-1) = 8(-1)\left(-1 - \sqrt{\frac{5}{4}}\right)\left(-1 + \sqrt{\frac{5}{4}}\right)$ (i) **positive** (ii) **negative**
Notice $\sqrt{\frac{5}{4}} \approx 1.12$, so since $8(-1) < 0$, $\left(-1 - \sqrt{\frac{5}{4}}\right) < 0$ and $\left(-1 + \sqrt{\frac{5}{4}}\right) > 0$, $f'(-2) > 0$.
function $f(x)$ (i) **increases** (ii) **decreases** over $\left(-\sqrt{\frac{5}{4}}, 0\right)$ interval

since $f'(1) = 8(1)\left(1 - \sqrt{\frac{5}{4}}\right)\left(1 + \sqrt{\frac{5}{4}}\right)$ (i) **positive** (ii) **negative**
Notice $\sqrt{\frac{5}{4}} \approx 1.12$, so since $8(1) > 0$, $\left(1 - \sqrt{\frac{5}{4}}\right) < 0$ and $\left(1 + \sqrt{\frac{5}{4}}\right) > 0$, $f'(-2) < 0$.
function $f(x)$ (i) **increases** (ii) **decreases** over $\left(0, \sqrt{\frac{5}{4}}\right)$ interval

since $f'(2) = 8(2)\left(2 - \sqrt{\frac{5}{4}}\right)\left(2 + \sqrt{\frac{5}{4}}\right)$ (i) **positive** (ii) **negative**
Notice $\sqrt{\frac{5}{4}} \approx 1.12$, so since $8(2) > 0$, $\left(2 - \sqrt{\frac{5}{4}}\right) > 0$ and $\left(2 + \sqrt{\frac{5}{4}}\right) > 0$, $f'(-2) > 0$.
function $f(x)$ (i) **increases** (ii) **decreases** over $\left(\sqrt{\frac{5}{4}}, \infty\right)$ interval

(e) *Figure (e):* $f(x) = 3x^3 - 2x^4$.

Since

$$f'(x) = 9x^2 - 8x^3 = 9x^2\left(1 - \frac{8}{9}x\right) = 0,$$

there are *two* critical numbers at
$c =$ (i) $\mathbf{-\frac{9}{8}}$ (ii) $\mathbf{0}$ (iii) $\mathbf{\frac{9}{8}}$
and so there are *three* intervals to investigate
(i) $\mathbf{(-\infty, -3)}$ (ii) $\mathbf{(-\infty, 0)}$ (iii) $\mathbf{\left(0, \frac{9}{8}\right)}$ (iv) $\mathbf{\left(\frac{9}{8}, \infty\right)}$ (v) $\mathbf{(0, \infty)}$
with *three* possible test values to check, say:
$x =$ (i) $\mathbf{-1}$ (ii) $\mathbf{0}$ (iii) $\mathbf{1}$ (iv) $\mathbf{\frac{9}{8}}$ (v) $\mathbf{2}$

since $f'(-1) = 9(-1)^2\left(1 - \frac{8}{9}(-1)\right)$ is (i) **positive** (ii) **negative**
function $f(x)$ (i) **increases** (ii) **decreases** over $(-\infty, 0)$ interval

since $f'(1) = 9(1)^2\left(1 - \frac{8}{9}(1)\right)$ is (i) **positive** (ii) **negative**
function $f(x)$ (i) **increases** (ii) **decreases** over $\left(0, \frac{9}{8}\right)$ interval

since $f'(2) = 9(2)^2\left(1 - \frac{8}{9}(2)\right)$ is (i) **positive** (ii) **negative**
function $f(x)$ (i) **increases** (ii) **decreases** over $\left(\frac{9}{8}, \infty\right)$ interval

3. *Functions whose derivatives do not exist at some values.*

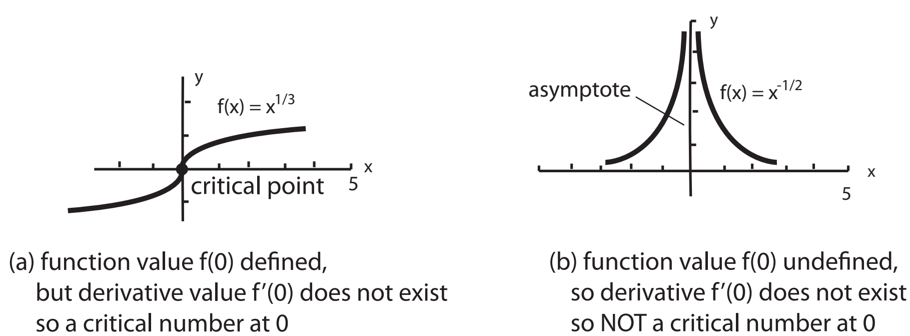


Figure 5.3 (Functions whose derivatives do not exist at some values)

(Type all functions into your calculator using Y= and set WINDOW -5 5 1 -5 5 1 1.

(a) *Figure (a):* $f(x) = x^{\frac{1}{3}}$.
Since

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{\left(\sqrt[3]{x}\right)^2},$$

as $x \to 0$, slope of tangent, f', becomes vertical (does not exist),
$\lim_{x\to 0} f'(x) =$ (i) $\mathbf{-\infty}$ (ii) $\mathbf{0}$ (iii) $\mathbf{\infty}$

and so there is a *critical number* at
$c =$ (i) $\mathbf{-\infty}$ (ii) **0** (iii) $\mathbf{\infty}$
and so there are *two* intervals to investigate
(i) $\mathbf{(-\infty, 0)}$ (ii) $\mathbf{(-0, 0)}$ (iii) $\mathbf{(0, \infty)}$
with *two* possible *test values* (in each interval) to check, say:
$x =$ (i) **0** (ii) **−1** (iii) **1**

since $f'(-1) = \frac{1}{3}(-1)^{-\frac{2}{3}}$ is (i) **positive** (ii) **negative**
function $f(x)$ (i) **increases** (ii) **decreases** over $(-\infty, 0)$ interval

and $f'(1) = \frac{1}{3}(1)^{-\frac{2}{3}}$ is (i) **positive** (ii) **negative**
function $f(x)$ (i) **increases** (ii) **decreases** over $(0, \infty)$ interval

(b) *Figure (b):* $f(x) = \frac{1}{x^2} = x^{-2}$.
Since *function* is *undefined* at $x = 0$, so also derivative $f'(x)$

$$f'(x) = -2x^{-3} = -\frac{2}{x^3}$$

cannot exist at $x = 0$,
so there is a discontinuity (technically, *not* a critical number) at
(i) $\mathbf{-\infty}$ (ii) **0** (iii) $\mathbf{\infty}$
and so there are *two* intervals to investigate
(i) $\mathbf{(-\infty, 0)}$ (ii) $\mathbf{(-0, 0)}$ (iii) $\mathbf{(0, \infty)}$
with *two* possible *test values* (in each interval) to check, say:
$x =$ (i) **0** (ii) **−1** (iii) **1**

since $f'(-1) = -2(-1)^{-3}$ is (i) **positive** (ii) **negative**
function $f(x)$ (i) **increases** (ii) **decreases** over $(-\infty, 0)$ interval

and $f'(1) = -2(1)^{-3}$ is (i) **positive** (ii) **negative**
function $f(x)$ (i) **increases** (ii) **decreases** over $(0, \infty)$ interval

5.2 Relative Extrema

A relative (local) extremum (plural: extrema) ia defined as follows:

$f(c)$ is *relative (local) maximum*	if $f(x) \leq f(c)$, for all x in (a,b)
$f(c)$ is *relative (local) minimum*	if $f(x) \geq f(c)$, for all x in (a,b)
$f(c)$ is *relative (local) extrema*	if c is either a relative minimum or maximum at c.

If function f has a relative extremum at c, then c is either a critical number or an endpoint. *First derivative test* for locating relative extrema in (a, b):

$f(c)$ is *relative maximum* if $f'(x)$ positive in (a, c), negative in (c, b)
$f(c)$ is *relative minimum* if $f'(x)$ negative in (a, c), positive in (c, b)

Although treated in a similar manner, allowances are made for identifying relative extrema for functions with discontinuities.

Exercise 10.2 (Relative Extrema)

1. *Critical points, endpoints and relative extrema.*

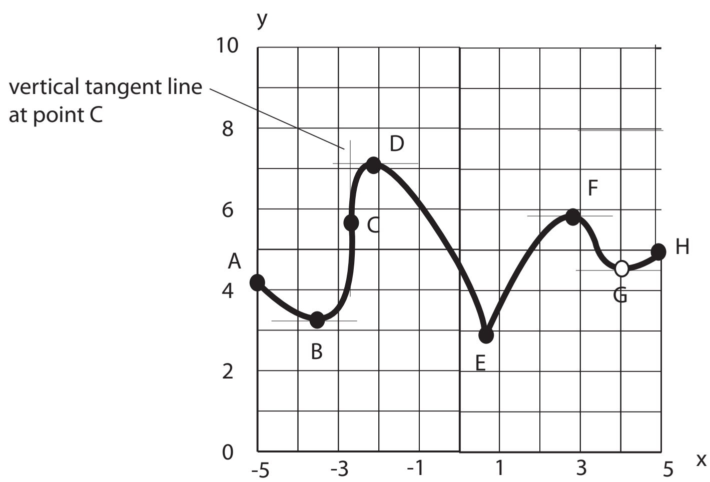


Figure 5.4 (Critical points, endpoints and extrema)

(a) Point A where $x = -5$ is
(i) **a critical point**
(ii) **an endpoint**
(iii) **neither a critical point nor endpoint**
which is
(i) **a relative minimum**
(ii) **a relative maximum**
(iii) **not a relative extremum**
because, as suggested by the text, function heads down after point A;
however, some would dispute this because *if* the function was *previously* heading down to A, then A would neither be a maximum nor minimum; in other words, it really is not clear what A is because there are no points on "both sides" of A to be able to "really" decide if it is a maximum or minimum

(b) Point B where $x = -3.5$ is
(i) **a critical point**
(ii) **an endpoint**

(iii) **neither a critical point nor endpoint**

because tangent line at B is zero, $f'(B) = 0$

which is

(i) **a relative minimum**

(ii) **a relative maximum**

(iii) **not a relative extremum**

(c) Point C where $x = -2.7$ is

(i) **a critical point**

(ii) **an endpoint**

(iii) **neither a critical point nor endpoint**

because although $f(C)$ is defined, $f'(C)$ does not exist

which is

(i) **a relative minimum**

(ii) **a relative maximum**

(iii) **not a relative extremum**

because slope (derivative) $f'(x)$ is positive both before and after critical point C

(d) Point D where $x = -2$ is

(i) **a critical point**

(ii) **an endpoint**

(iii) **neither a critical point nor endpoint**

because tangent line at D is zero, $f'(D) = 0$

which is

(i) **a relative minimum**

(ii) **a relative maximum**

(iii) **not a relative extremum**

(e) Point E where $x = 0.5$ is

(i) **a critical point**

(ii) **an endpoint**

(iii) **neither a critical point nor endpoint**

because although $f(C)$ is defined, $f'(C)$ does not exist

which is

(i) **a relative minimum**

(ii) **a relative maximum**

(iii) **not a relative extremum**

(f) Point F where $x = 3$ is

(i) **a critical point**

(ii) **an endpoint**

(iii) **neither a critical point nor endpoint**

because tangent line at D is zero, $f'(D) = 0$

which is

(i) **a relative minimum**

(ii) **a relative maximum**
(iii) **not a relative extremum**

(g) Point G where $x = 4$ is
(i) **a critical point**
(ii) **an endpoint**
(iii) **neither a critical point nor endpoint**
because removable discontinuity point G is *not* a critical point (since function and so derivative do not exist at this point)
which is
(i) **a relative minimum**
(ii) **a relative maximum**
(iii) **not a relative extremum**
because point G is not a critical point

(h) Point H where $x = 5$ is
(i) **a critical point**
(ii) **an endpoint**
(iii) **neither a critical point nor endpoint**
which is
(i) **a relative minimum**
(ii) **a relative maximum**
(iii) **not a relative extremum**
because function heads up to endpoint H

2. *More critical points, endpoints and extrema.*

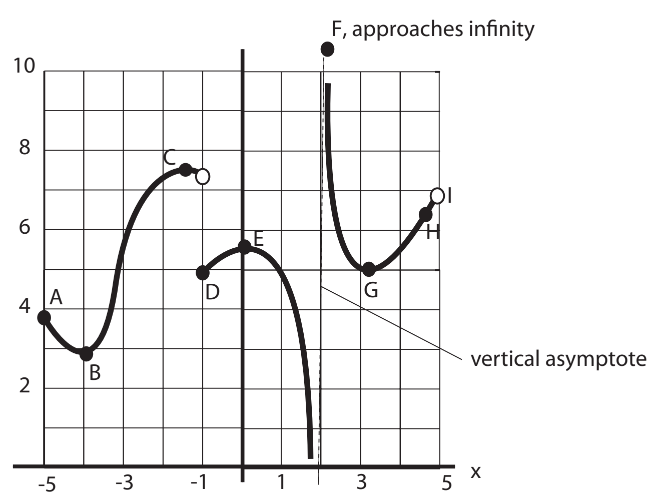


Figure 5.5 (More critical points, endpoints and extrema)

(a) Point A where $x = -5$ is
(i) **a critical point**

(ii) **an endpoint**
(iii) **neither a critical point nor endpoint**
which is
(i) **a relative minimum**
(ii) **a relative maximum**
(iii) **not a relative extremum**
because function heads down after point A

(b) Point B where $x = -4$ is
(i) **a critical point**
(ii) **an endpoint**
(iii) **neither a critical point nor endpoint**
because tangent line at B is zero, $f'(B) = 0$
which is
(i) **a relative minimum**
(ii) **a relative maximum**
(iii) **not a relative extremum**
because function decreases down to B then increases afterwards

(c) Point C where $x = -1.5$ is
(i) **a critical point**
(ii) **an endpoint**
(iii) **neither a critical point nor endpoint**
because tangent line at C is zero, $f'(C) = 0$
which is
(i) **a relative minimum**
(ii) **a relative maximum**
(iii) **not a relative extremum**
because function increases up to C then decreases afterwards

(d) Point D where $x = -1$ is
(i) **a critical point**
(ii) **an endpoint**
(iii) **neither a critical point nor endpoint**
because limit at $x = -1$ does not exist since $\lim_{x \to -1^-} f(x) = 6.8 \neq \lim_{x \to -1^+} f(x) = 5$
which is
(i) **a relative minimum**
(ii) **a relative maximum**
(iii) **not a relative extremum**
because function decreases down to D then increases afterwards

(e) Point E where $x = 0$ is
(i) **a critical point**
(ii) **an endpoint**
(iii) **neither a critical point nor endpoint**

because tangent line at E is zero, $f'(E) = 0$
which is
(i) **a relative minimum**
(ii) **a relative maximum**
(iii) **not a relative extremum**
because function increases up to C then decreases afterwards

(f) Point F where $x = 3$ is
(i) **a critical point**
(ii) **an endpoint**
(iii) **neither a critical point nor endpoint**
because function approaches infinity
which is
(i) **a relative minimum**
(ii) **a relative maximum**
(iii) **not a relative extremum**
because function approaches infinity, not any particular large number

(g) Point G where $x = 3.4$ is
(i) **a critical point**
(ii) **an endpoint**
(iii) **neither a critical point nor endpoint**
because tangent line at G is zero, $f'(G) = 0$
(i) **a relative minimum**
(ii) **a relative maximum**
(iii) **not a relative extremum**
because function decreases down to B then increases afterwards

(h) Point H where $x = 4.2$ is
(i) **a critical point**
(ii) **an endpoint**
(iii) **neither a critical point nor endpoint**
which is
(i) **a relative minimum**
(ii) **a relative maximum**
(iii) **not a relative extremum**
because function increases both before and after H

(i) Point I where $x = 5$ is
(i) **a critical point**
(ii) **an endpoint**
(iii) **neither a critical point nor endpoint**
because the function is defined on an interval which is open at this end
which is
(i) **a relative minimum**

(ii) **a relative maximum**
(iii) **not a relative extremum**
because the function is *not* defined at this point

3. *First derivative test:* $f(x) = x^2 + 4x - 21$ *revisited*

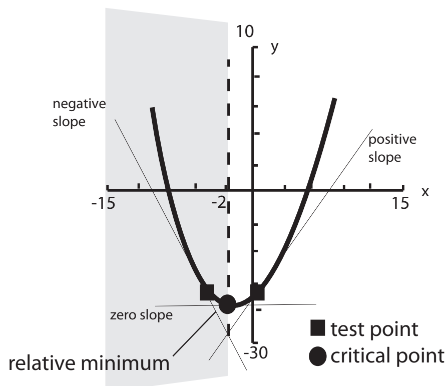


Figure 5.6 (First derivative test: $f(x) = x^2 + 4x - 21$)

GRAPH using $Y_2 = x^2 + 4x - 21$, with WINDOW -15 15 1 -30 10 1 1

(a) *Critical numbers and intervals.*
Recall, since

$$f'(x) = 2x + 4 = 0,$$

there is a *critical number* at
$c = -\frac{4}{2} =$ (i) $\mathbf{-2}$ (ii) $\mathbf{2}$ (iii) $\mathbf{0}$
and so there are *two* intervals to investigate
(i) $\mathbf{(-\infty, -2)}$ (ii) $\mathbf{(-2, 2)}$ (iii) $\mathbf{(-2, \infty)}$
with *two* possible *test values* (in each interval) to check, say:
$x =$ (i) $\mathbf{-3}$ (ii) $\mathbf{-2}$ (iii) $\mathbf{0}$

and since $f'(-3) = 2(-3) + 4 = -2$ is negative,
function $f(x)$ (i) **increases** (ii) **decreases** over $(-\infty, -2)$ interval

and $f'(0) = 2(0) + 4 = 4$ is positive,
function $f(x)$ (i) **increases** (ii) **decreases** over $(-2, \infty)$ interval
so summarizing:

interval	$(-\infty, -2)$	$(-2, \infty)$
test value	$x = -3$	$x = -1$
$f'(x) = 2x + 4$	$f'(-3) = -2$	$f'(-1) = 2$
sign of $f'(x)$	negative	positive

(b) *First derivative test.*
At critical number $c = -2$, sign of derivative $f'(x)$ goes from
(i) **negative to positive**
(ii) **positive to negative**
(iii) **negative to negative**
(iv) **positive to positive**

and so, according to first derivative test, there is
(i) **a relative minimum**
(ii) **a relative maximum**
(iii) **not a relative extremum**
at critical number $c = -2$,
and since $f(-2) = (-2)^2 + 4(-2) - 21 = -25$,
at critical *point* $(c, f(c)) = (-2, -25)$.

4. *First derivative test:* $f(x) = 2x^3 + 3x^2 - 36x$ *revisited*

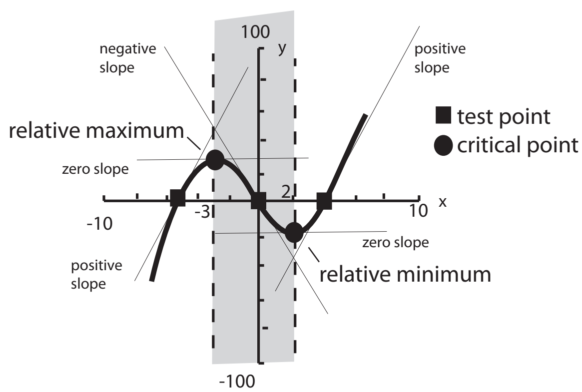


Figure 5.7 (First derivative test: $f(x) = 2x^3 + 3x^2 - 36x$)

GRAPH using $\text{Y}_3 = 2x^3 + 3x^2 - 36x$, with WINDOW -10 10 1 -100 100 1 1

(a) *Critical numbers and intervals.*
Recall, since

$$f'(x) = 6x^2 + 6x - 36 = 6(x^2 + x - 6) = 6(x + 3)(x - 2) = 0,$$

there are *two* critical numbers at
$c =$ (i) $\mathbf{-3}$ (ii) **2** (iii) **6**
and so there are *three* intervals to investigate
(i) $\mathbf{(-\infty, -3)}$ (ii) $\mathbf{(-\infty, 2)}$ (iii) $\mathbf{(-2, 3)}$ (iv) $\mathbf{(-3, 2)}$ (v) $\mathbf{(2, \infty)}$
with *three* possible test values to check, say:
$x =$ (i) $\mathbf{-4}$ (ii) $\mathbf{-3}$ (iii) **0** (iv) **2** (v) **3**

since $f'(-4) = 6(-4+3)(-4-2)$ is (i) **positive** (ii) **negative**
function $f(x)$ (i) **increases** (ii) **decreases** over $(-\infty, -3)$ interval

since $f'(0) = 6(0+3)(0-2)$ is (i) **positive** (ii) **negative**
function $f(x)$ (i) **increases** (ii) **decreases** over $(-3, -2)$ interval

since $f'(3) = 6(3+3)(3-2)$ is (i) **positive** (ii) **negative**
function $f(x)$ (i) **increases** (ii) **decreases** over $(3, \infty)$ interval
so summarizing:

interval	$(-\infty, -3)$	$(-3, 2)$	$(2, \infty)$
test value	$x = -5$	$x = 0$	$x = 3$
$f'(x) = 6x^2 + 6x - 36$	$f'(-5) = 84$	$f'(0) = -36$	$f'(3) = 36$
sign of $f'(x)$	positive	negative	positive

(b) *First derivative test.*
At critical number $c = -3$, sign of derivative $f'(x)$ goes from
(i) **negative to positive**
(ii) **positive to negative**
(iii) **negative to negative**
(iv) **positive to positive**
and so, according to first derivative test, there is
(i) **a relative minimum**
(ii) **a relative maximum**
(iii) **not a relative extremum**
at critical number $c = -3$,
and since $f(-3) = 2(-3)^3 + 3(-3)^2 - 36(-3) = 81$,
at critical *point* $(c, f(c)) = (-3, 81)$.

At critical number $c = 2$, sign of derivative $f'(x)$ goes from
(i) **negative to positive**
(ii) **positive to negative**
(iii) **negative to negative**
(iv) **positive to positive**
and so, according to first derivative test, there is
(i) **a relative minimum**
(ii) **a relative maximum**
(iii) **not a relative extremum**
at critical number $c = 2$,
and since $f(2) = 2(2)^3 + 3(2)^2 - 36(2) = -44$,
at critical *point* $(c, f(c)) = (2, -44)$.

5. *First derivative test:* $f(x) = 3x^3 - 2x^4$ *revisited*

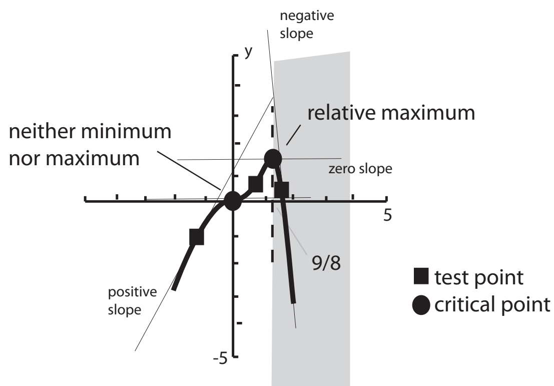


Figure 5.8 (First derivative test: $f(x) = 3x^3 - 2x^4$)

GRAPH using $Y_4 = 3x^3 - 2x^4$, with WINDOW -3 3 1 -3 2 1 1

(a) *Critical numbers and intervals.*
Since

$$f'(x) = 9x^2 - 8x^3 = 9x^2\left(1 - \frac{8}{9}x\right) = 0,$$

there are *two* critical numbers at
$c =$ (i) $\mathbf{-\frac{9}{8}}$ (ii) **0** (iii) $\mathbf{\frac{9}{8}}$
and so there are *three* intervals to investigate
(i) $\mathbf{(-\infty, -3)}$ (ii) $\mathbf{(-\infty, 0)}$ (iii) $\mathbf{\left(0, \frac{9}{8}\right)}$ (iv) $\mathbf{\left(\frac{9}{8}, \infty\right)}$ (v) $\mathbf{(0, \infty)}$
with *three* possible test values to check, say:
$x =$ (i) $\mathbf{-1}$ (ii) **0** (iii) **1** (iv) $\mathbf{\frac{9}{8}}$ (v) **2**

since $f'(-1) = 9(-1)^2\left(1 - \frac{8}{9}(-1)\right)$ is (i) **positive** (ii) **negative**
function $f(x)$ (i) **increases** (ii) **decreases** over $(-\infty, 0)$ interval

since $f'(1) = 9(1)^2\left(1 - \frac{8}{9}(1)\right)$ is (i) **positive** (ii) **negative**
function $f(x)$ (i) **increases** (ii) **decreases** over $\left(0, \frac{9}{8}\right)$ interval

since $f'(2) = 9(2)^2\left(1 - \frac{8}{9}(2)\right)$ is (i) **positive** (ii) **negative**
function $f(x)$ (i) **increases** (ii) **decreases** over $\left(\frac{9}{8}, \infty\right)$ interval
so summarizing:

interval	$(-\infty, 0)$	$\left(0, \frac{9}{8}\right)$	$\left(\frac{9}{8}, \infty\right)$
test value	$x = -1$	$x = 1$	$x = 2$
$f'(x) = 9x^2 - 8x^3$	$f'(-1) = 17$	$f'(1) = 1$	$f'(2) = -28$
sign of $f'(x)$	positive	positive	negative

(b) *First derivative test.*
At critical number $c = 0$, sign of derivative $f'(x)$ goes from
(i) **negative to positive**

(ii) **positive to negative**
(iii) **negative to negative**
(iv) **positive to positive**
and so, according to first derivative test, there is
(i) **a relative minimum**
(ii) **a relative maximum**
(iii) **not a relative extremum**
at critical number $c = 0$,
and since $f(0) = 3(0)^3 - 2(0)^4 = 0$,
at critical *point* $(c, f(c)) = (0, 0)$.

At critical number $c = \frac{9}{8}$, sign of derivative $f'(x)$ goes from
(i) **negative to positive**
(ii) **positive to negative**
(iii) **negative to negative**
(iv) **positive to positive**
and so, according to first derivative test, there is
(i) **a relative minimum**
(ii) **a relative maximum**
(iii) **not a relative extremum**
at critical number $c = \frac{9}{8}$,
and since $f\left(\frac{9}{8}\right) = 3\left(\frac{9}{8}\right)^3 - 2\left(\frac{9}{8}\right)^4 = \frac{2187}{2048}$,
at critical *point* $(c, f(c)) = \left(\frac{9}{8}, \frac{2187}{2048}\right)$.

6. *First derivative test:* $f(x) = x^3(5.5)^x$ *revisited*
Type Y=, then $\text{Y}_1 = x^3(5.5)^x$, WINDOW -2.5 1 1 -0.5 0.5 1 1, then GRAPH

(a) *Derivative.*

Let $u(x) = x^3$ and $v(x) = 5.5^x$
then $u'(x) =$ (i) $\mathbf{3x^2}$ (ii) $\mathbf{3x}$ (iii) $\mathbf{e^{2x}}$
and $v'(x) =$ (i) $\mathbf{5.5^x}$ (ii) $\mathbf{(\ln 5.5)}$ (iii) $\mathbf{(\ln 5.5)\, 5.5^x}$

and so $v(x)u'(x) =$
(i) $\mathbf{5.5^x}$
(ii) $\mathbf{3x^3}$
(iii) $\mathbf{(5.5^x)\,(3x^2)}$

and $u(x)v'(x) =$
(i) $\mathbf{(x^3)\,(\ln 5.5)\, 5.5^x}$
(ii) $\mathbf{(\ln 5.5)\, 5.5^x}$
(iii) $\mathbf{(3x^3)\,(\ln 5.5)}$

and so $f'(x) = v(x) \cdot u'(x) + u(x) \cdot v'(x) =$
(i) $\mathbf{(5.5^x)\,(3x^2) + (3x^3)\,(\ln 5.5)}$
(ii) $\mathbf{(5.5^x) + (3x^3)\,(\ln 5.5)\,5.5^x}$
(iii) $\mathbf{(5.5^x)\,(3x^2) + (x^3)\,(\ln 5.5)\,5.5^x}$
which simplifies to

$$f'(x) = \left(x^2\right)(5.5^x)(3 + x \ln 5.5)$$

Type Y=, then $Y_2 = \left(x^2\right)(5.5^x)(3 + x\ln 5.5)$, WINDOW -2.5 1 1 -0.5 0.5 1 1, then GRAPH

(b) *Critical numbers and intervals.*
Since

$$f'(x) = \left(x^2\right)(5.5^x)(3 + x \ln 5.5) = 0,$$

there are *two* critical numbers at
$c =$ (i) $\mathbf{-\frac{3}{\ln 5.5} \approx -1.76}$ (ii) $\mathbf{0}$ (iii) $\mathbf{\frac{3}{\ln 5.5}}$
(Since $3 + x\ln 5.5 = 0, x = -\frac{3}{\ln 5.5} \approx -1.76$)
and so there are *three* intervals to investigate
(i) $\mathbf{\left(-\infty, -\frac{3}{\ln 5.5}\right)}$ (ii) $\mathbf{\left(-\frac{3}{\ln 5.5}, 0\right)}$ (iii) $\mathbf{(0, 3)}$
(iv) $\mathbf{(3, 5.5)}$ (v) $\mathbf{(0, \infty)}$
with *three* possible test values to check, say:
$x =$ (i) $\mathbf{-2}$ (ii) $\mathbf{-\frac{3}{\ln 5.5}}$ (iii) $\mathbf{-1}$ (iv) $\mathbf{0}$ (v) $\mathbf{1}$

since $f'(-2) = ((-1)^2)(5.5^{-2})(3 - 2\ln 5.5)$ is
(i) **positive** (ii) **negative**
VARS Y-VARS ENTER Y_2 ENTER (-2) ENTER gives approximately -0.054, which is negative
function $f(x)$ (i) **increases** (ii) **decreases** over $\left(-\infty, -\frac{3}{\ln 5.5}\right)$ interval

since $f'(-1) = ((-1)^2)(5.5^{-1})(3 - \ln 5.5)$ is
(i) **positive** (ii) **negative**
VARS Y-VARS ENTER Y_2 ENTER (-1) ENTER gives approximately 0.236, which is positive
function $f(x)$ (i) **increases** (ii) **decreases** over $\left(-\frac{3}{\ln 5.5}, 0\right)$ interval

since $f'(1) = ((1)^2)(5.5^1)(3 + \ln 5.5)$ is
(i) **positive** (ii) **negative**
VARS Y-VARS ENTER Y_2 ENTER (1) ENTER gives approximately 25.876, which is positive
function $f(x)$ (i) **increases** (ii) **decreases** over $(0, \infty)$ interval
so summarizing:

interval	$\left(-\infty, -\frac{3}{\ln 5.5}\right)$	$\left(-\frac{3}{\ln 5.5}, 0\right)$	$(0, \infty)$
test value	$x = -2$	$x = -1$	$x = 1$
$f'(x) = (x^2)(5.5^x)(3 + x\ln 5.5)$	$f'(-2) \approx -0.054$	$f'(-1) \approx 0.236$	$f'(1) \approx 25.877$
sign of $f'(x)$	negative	positive	positive

(c) *First derivative test.*
At critical number $c = -\frac{3}{\ln 5.5}$, sign of derivative $f'(x)$ goes from
(i) **negative to positive**
(ii) **positive to negative**
(iii) **negative to negative**
(iv) **positive to positive**
and so, according to first derivative test, there is
(i) **a relative minimum**
(ii) **a relative maximum**
(iii) **not a relative extremum**
at critical number $c = -\frac{3}{\ln 5.5}$,
and since $f\left(-\frac{3}{\ln 5.5}\right) = \left(-\frac{3}{\ln 5.5}\right)^3 (5.5)^{-\frac{3}{\ln 5.5}} \approx -0.27$,
at critical *point* $(c, f(c)) \approx (-1.76, -0.27)$.

At critical number $c = 0$, sign of derivative $f'(x)$ goes from
(i) **negative to positive**
(ii) **positive to negative**
(iii) **negative to negative**
(iv) **positive to positive**
and so, according to first derivative test, there is
(i) **a relative minimum**
(ii) **a relative maximum**
(iii) **not a relative extremum**
at critical number $c = 0$,
and since $f(0) = (0)^3(5.5)^0 = 0$,
at critical *point* $(c, f(c)) = (0, 0)$.

7. *Another question.* Consider the following table.

interval	$(-\infty, -1)$	$(-1, 1)$	$(1, 5)$	$(5, \infty)$
test value	$x = -2$	$x = 0$	$x = 3$	$x = 6$
$f'(x)$	$f'(-2) = -10$	$f'(0) = -28$	$f'(3) = 18$	$f'(6) = -3$
sign of $f'(x)$	negative	negative	positive	negative

(a) At critical number $c = -1$, there is a
(i) **a relative minimum**
(ii) **a relative maximum**
(iii) **not a relative extremum**

(b) At critical number $c = 1$, there is a
(i) **a relative minimum**
(ii) **a relative maximum**
(iii) **not a relative extremum**

(c) At critical number $c = 5$, there is a
(i) **a relative minimum**
(ii) **a relative maximum**
(iii) **not a relative extremum**

8. *Application: throwing a ball.* A ball is thrown upwards with an initial velocity of 32 feet per second and from an initial height of 150 feet. A function relating height, $f(t)$, to time, f, when throwing this ball is:

$$f(t) = -12t^2 + 32t + 150$$

Find maximum height, $f(t)$, and time, t, ball reaches maximum height.
GRAPH using $Y_5 = -12x^2 + 32x + 150$, with WINDOW 0 6 1 0 250 1 1

(a) *Derivative*

$$f'(t) = -12(2)t^{2-1} + 32(1)t^{1-1} = 0 =$$

(i) $\mathbf{-24t + 32}$ (ii) $\mathbf{24t - 32}$ (iii) $\mathbf{24t^2 + 32}$

(b) *Critical numbers and intervals.*
Since

$$f'(x) = -24t + 32 = 0,$$

there is a *critical number* at
$c = -\frac{32}{-24} =$ (i) $\mathbf{\frac{4}{3}}$ (ii) $\mathbf{-\frac{4}{3}}$ (iii) $\mathbf{\frac{3}{4}}$
and so there are *two* intervals to investigate
(i) $\mathbf{\left(-\infty, \frac{4}{3}\right)}$ (ii) $\mathbf{\left(\frac{4}{3}, \infty\right)}$ (iii) $\mathbf{(-2, \infty)}$
with *two* possible *test values* (in each interval) to check, say:
$x =$ (i) **0** (ii) **1** (iii) **2**

and since $f'(0) = -24(0) + 32 = 32$ is positive,
function $f(x)$ (i) **increases** (ii) **decreases** over $\left(-\infty, \frac{4}{3}\right)$ interval

and $f'(2) = -24(2) + 32 = -16$ is negative,
function $f(x)$ (i) **increases** (ii) **decreases** over $\left(\frac{4}{3}, \infty\right)$ interval
so summarizing:

interval	$\left(-\infty, \frac{4}{3}\right)$	$\left(\frac{4}{3}, \infty\right)$
test value	$x = 0$	$x = 2$
$f'(t) = -24t + 32$	$f'(0) = 32$	$f'(2) = -16$
sign of $f'(x)$	positive	negative

(c) *First derivative test.*
At critical number $c = \frac{4}{3}$, sign of derivative $f'(t)$ goes from
(i) **negative to positive**

(ii) **positive to negative**
(iii) **negative to negative**
(iv) **positive to positive**

and so, according to first derivative test, there is
(i) **a relative minimum**
(ii) **a relative maximum**
(iii) **not a relative extremum**
at critical number $c = \frac{4}{3}$,
and since $f\left(\frac{4}{3}\right) = -12\left(\frac{4}{3}\right)^2 + 32\left(\frac{4}{3}\right) + 150 = \frac{514}{3}$,
at critical *point* $(c, f(c)) = \left(\frac{4}{3}, \frac{514}{3}\right)$.

(d) *Results*
In other words, ball reaches maximum height of (i) $\mathbf{\frac{4}{3}}$ (ii) $\mathbf{\frac{3}{514}}$ (iii) $\mathbf{\frac{514}{3}}$
at time (i) $\mathbf{\frac{4}{3}}$ (ii) $\mathbf{-\frac{4}{3}}$ (iii) $\mathbf{\frac{3}{4}}$