10.3 Euler's Method

Difficult–to–solve differential equations can always be *approximated* by numerical methods. We look at one numerical method called Euler's Method. Euler's method uses the readily available slope information to start from the point (x_0, y_0) then move from one point to the next along the polygon approximation of the graph of the particular differential equation to ultimately reach the terminal point, (x_n, y_n). Although interested in determining *all* of the points along the differential equation, it is often the case that the value of y_n at the terminal point is of most interest. More specifically, let $y = f(x)$ be the solution to the differential equation

$$\frac{dy}{dx} = g(x, y), \quad \text{with} \quad y(x_0) = y_0$$

for $x_0 \leq x \leq x_n$ and let $x_{i+1} = x_i + h$, where $h = \frac{x_n - x_0}{n}$ and

$$y_{i+1} = y_i + g(x_i, y_i)h,$$

for $0 \leq i \leq n-1$, then

$$f(x_{i+1}) \approx y_{i+1}.$$

Exercise 10.3 (Euler's Method)

1. *Approximate* $\frac{dy}{dx} = y - 2x$, *start at* $(x, y) = (0, 1)$, $0 \leq x \leq 2$.

   (a) *approximate* $\frac{dy}{dx} = y - 2x$, *when subinterval* $h = 0.4$

   Since
   $$\frac{dy}{dx} = y - 2x, \quad \text{then} \quad g(x, y) =$$
   (i) $\boldsymbol{y - 2x}$ (ii) $\boldsymbol{2x + 2 - e^x}$ (iii) $\boldsymbol{3e^{\frac{1}{2}}}$

   and since $x_0 = 0$, $y_0 = 1$, then
   $$g(x_0, y_0) = y_0 - 2x_0 = 1 - 2(0) =$$
   (i) **0** (ii) **1** (iii) **2**,

   and since $h = 0.4$,
   $$y_1 = y_0 + g(x_0, y_0)h = 1 + 1(0.4) =$$
   (i) **1** (ii) **1.4** (iii) **1.8**,

   but, now, since $x_1 = x_0 + h = 0 + 0.4 = 0.4$ and
   $$g(x_1, y_1) = y_1 - 2x_1 = 1.4 - 2(0.4) =$$

(i) **0.6** (ii) **0.8** (iii) **1.0**,

and since $h = 0.4$,

$$y_2 = y_1 + g(x_1, y_1)h = 1.4 + 0.6(0.4) =$$

(i) **1.64** (ii) **1.84** (iii) **2.04**

Remainder of (x_i, y_i) values given in table.

i	x_i	Euler's Approximation $y_0 = 1$ $y_i = y_{i-1} + (y_{i-1} - 2x_{i-1})(0.4), i \geq 1$	Actual Solution $f(x_i) = 2x_i + 2 - e^{x_i}$	Difference $y_i - f(x_i)$
0	0	1.00	1.00	0.00
1	0.4	1.40	1.31	0.09
2	0.8	1.64	1.37	0.27
3	1.2	1.66	1.08	0.58
4	1.6	1.36	0.25	1.11
5	2.0	0.62	-1.39	2.01

TI-84 calculator: For Euler's approximation, define $Y_1 = Y - 2X$, initialize X and Y with -0.4 and 1, respectively: $-0.4 \to X$, $1 \to Y$; type Euler's approximation: $X + 0.4 \to X : Y + Y_1 \times 0.4 \to Y$ ENTER for 1.4, then ENTER for 1.64, and so on. Recall $\frac{dy}{dx} = y - 2x$ is a first order linear differential equation whose particular solution, where $(x, y) = (0, 1)$, is $y = 2x + 2 - e^x$ as explained in previous section 10.2 of the lecture notes. So, for actual solution, define $Y_2 = 2x + 2 - e^X$, then VARS, Y-VARS ENTER Y_2 ENTER $Y_2(0.4)$ ENTER for 1.31, and so on.

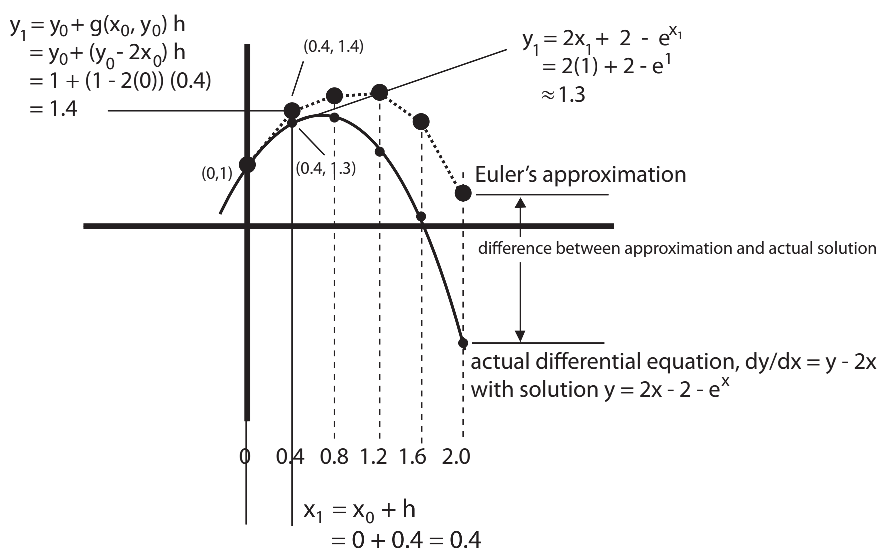

Figure 10.3 (Euler approximation to $\frac{dy}{dx} = y - 2x, h = 0.4$)

(b) *approximate* $\frac{dy}{dx} = y - 2x$, *when subinterval* $h = 0.1$

As before,

$$\frac{dy}{dx} = y - 2x, \quad \text{then} \quad g(x,y) =$$

(i) $\boldsymbol{y - 2x}$ (ii) $\boldsymbol{2x + 2 - e^x}$ (iii) $\boldsymbol{3e^{\frac{1}{2}}}$

and since $x_0 = 0$, $y_0 = 1$, then

$$g(x_0, y_0) = y_0 - 2x_0 = 1 - 2(0) =$$

(i) **0** (ii) **1** (iii) **2**,

and since $h = 0.1$ (instead of $h = 0.4$),

$$y_1 = y_0 + g(x_0, y_0)h = 1 + 1(0.1) =$$

(i) **1.1** (ii) **1.4** (iii) **1.8**,

but, now, since $x_1 = x_0 + h = 0 + 0.1 = 0.1$ and

$$g(x_1, y_1) = y_1 - 2x_1 = 1.1 - 2(0.1) =$$

(i) **0.6** (ii) **0.8** (iii) **0.9**,

and since $h = 0.1$,

$$y_2 = y_1 + g(x_1, y_1)h = 1.1 + 0.9(0.1) =$$

(i) **1.19** (ii) **1.84** (iii) **2.04**

i	x_i	Euler's Approximation $y_0 = 1$ $y_i = y_{i-1} + (y_{i-1} - 2x_{i-1})(0.1), i \geq 1$	Actual Solution $f(x_i) = 2x_i + 2 - e^{x_i}$	Difference $y_i - f(x_i)$
0	0	1.00	1.00	0.00
1	0.1	1.10	1.09	0.01
2	0.2	1.19	1.18	0.01
⋮	⋮	⋮	⋮	⋮
19	1.9	−0.32	−0.89	0.57
20	2.0	−0.73	−1.39	0.66

For Euler's approximation, define $Y_1 = Y - 2X$, initialize X and Y with -0.1 and 1, respectively: $-0.1 \to X$, $1 \to Y$; type Euler's approximation: $X + 0.1 \to X : Y + Y_1 \times 0.1 \to Y$ ENTER for 1.1, then ENTER for 1.19, and so on. For actual solution, define $Y_2 = 2x + 2 - e^X$, then VARS, Y-VARS ENTER Y_2 ENTER $Y_2(0.1)$ ENTER for 1.09, $Y_2(0.2)$ for 1.18 and so on.

Euler's method (i) **improves** (ii) **worse** for *smaller* h subintervals.

2. *Approximate* $\frac{dy}{dx} = xy$, *start* $f(1) = 3$, *over* $[1, 2]$, *using 10 subintervals.*

Since

$$\frac{dy}{dx} = xy, \quad \text{then} \quad g(x, y) =$$

(i) $\boldsymbol{3e^{-\frac{1}{2}}e^{\frac{1}{2}x^2}}$ (ii) $\boldsymbol{2x + 2 - e^x}$ (iii) $\boldsymbol{xy}$

and since $x_0 = 1$, $y_0 = 3$, then

$$g(x_0, y_0) = x_0 y_0 = 1(3) =$$

(i) **0** (ii) **1** (iii) **3**,

and since $[1, 2]$ has 10 subintervals

$$h = \frac{2-1}{10} =$$

(i) **0** (ii) **0.1** (iii) **0.2**,

and so

$$y_1 = y_0 + g(x_0, y_0)h = 3 + 3(0.1) =$$

(i) **3.1** (ii) **3.3** (iii) **3.8**,

but, now, since $x_1 = x_0 + h = 0 + 0.1 = 0.1$ and

$$g(x_1, y_1) = x_1 y_1 = 0.1(1.1) =$$

(i) **0.10** (ii) **0.11** (iii) **0.12**,

and since $h = 0.1$,

$$y_2 = y_1 + g(x_1, y_1)h = 3.3 + 0.11(0.1) \approx$$

(i) **3.11** (ii) **3.31** (iii) **3.51**

Fill in the missing (x_i, y_i) values given in table.

i	x_i	Euler's Approximation $y_0 = 3$ $y_i = y_{i-1} + (x_{i-1}y_{i-1})(0.1), i \geq 1$	Actual Solution $f(x_i) = 3e^{-\frac{1}{2}}e^{\frac{1}{2}x_i^2}$	Difference $y_i - f(x_i)$
0	1	3.00	3.00	0.00
1	1.1	3.30	3.33	-0.03
2	1.2	3.66	3.74	-0.08
3	1.3	4.10	4.24	-0.14
4	1.4	4.64	4.85	-0.21
5	1.5	5.28	5.60	-0.40
6	1.6	______	______	______
7	1.7	______	______	______
8	1.8	______	______	______
9	1.9	______	______	______
10	2.0	______	______	______

TI-84 calculator: For Euler's approximation, define $Y_1 = XY$, initialize X and Y with 0.9 and 3, respectively: $0.9 \to X$, $3 \to Y$; type Euler's approximation: $X + 0.1 \to X : Y + Y_1 \times 0.1 \to Y$ ENTER for 3.3, then ENTER for 3.66, and so on. Recall $\frac{dy}{dx} = xy$ is a separable differential equation whose particular solution, where $(x, y) = (1, 3)$, is $y = 3e^{-\frac{1}{2}}e^{\frac{1}{2}x^2}$ as explained in previous section 10.1 of the lecture notes. So, for actual solution, define $Y_2 = 3e^{-\frac{1}{2}}e^{\frac{1}{2}X^2}$, then VARS, Y-VARS ENTER Y_2 ENTER $Y_2(1.1)$ ENTER for 3.33, ...

3. *Approximate bear population* Assume bear population grows according to following differential equation.

$$\frac{dy}{dt} = 0.02y(y+1)(y+3)$$

Assume initial population at time $t = 0$ is $y = 5$, use Euler's method, where $h = 1$ year, to approximate bear population at time $t = 3$ years.

Since

$$\frac{dy}{dt} = 0.02y(y+1)(y+3), \quad \text{then} \quad g(t, y) =$$

(i) $\boldsymbol{0.02y(y-1)(y+3)}$ (ii) $\boldsymbol{2x + 2 - e^t}$ (iii) $\boldsymbol{ty}$

and since $t_0 = 1$, $y_0 = 5$, then

$$g(t_0, y_0) = 0.02y_0(y_0+1)(y_0+3) = 0.02 \times 5(5+1)(5+3) =$$

(i) **4.7** (ii) **4.8** (iii) **4.9**,

and since $h = 1$ and so

$$y_1 = y_0 + g(t_0, y_0)h = 5 + 4.8(1) =$$

(i) **9.6** (ii) **9.7** (iii) **9.8**,

but, now, since $t_1 = x_0 + h = 0 + 1 = 1$ and

$$g(t_1, y_1) = 0.02y_1(y_1 + 1)(y_1 + 3) = 0.02 \times 9.8(9.8 + 1)(9.8 + 3) \approx$$

(i) **27.0** (ii) **27.1** (iii) **27.2**,

and since $h = 1$,

$$y_2 = y_1 + g(t_1, y_1)h \approx 9.8 + 27.1(1) \approx$$

(i) **36.9** (ii) **37.3** (iii) **40.2**

Fill in the missing (t_i, y_i) value given in table.

i	t_i	Euler's Approximation $y_0 = 5$ $y_i = y_{i-1} + (0.02y_{i-1}(y_{i-1} + 1)(y_{i-1} + 3))(1), i \geq 1$
0	0	9.800
1	1	36.895
2	2	1152.471
3	3	______

For Euler's approximation, define $Y_1 = 0.02Y(Y + 1)(Y + 3)$, initialize X and Y with -1 and 5, respectively: $-1 \to X$, $5 \to Y$; type Euler's approximation: $X + 1 \to X : Y + Y_1 \times 1 \to Y$ ENTER for 9.8, then ENTER for 36.895, and so on. Notice there is no actual solution because, although $\frac{dy}{dt} = 0.02y(y + 1)(y + 3)$ is a separable differential equation where, with the aid of wolfram's integrator web site, gives $\frac{50}{3} \ln y - 25 \ln(y + 1) + \frac{25}{3} \ln(y + 3) = t$, there is no closed analytic solution for y that I am aware of.

10.4 Applications of Differential Equations

We look at a variety of applications of differential equations.

Exercise 10.4 (Applications of Differential Equations)

1. *Application: limited growth rate model,* $\frac{dy}{dt} = k(N - y)$.
   After 10 days, 40% of the 24000 viewers of a local TV station had seen an advertisement on car parts. How long must the advertisement air to reach 80% of the station's viewers?

(a) *General Solution.*
Since $\frac{dy}{dt} = k(N-y)$,

$$\begin{aligned}
\frac{1}{N-y}\,dy &= k\,dt \quad \text{separation of variables} \\
\int \frac{1}{N-y}\,dy &= \int k\,dt \quad \text{integrate both sides} \\
-\ln(N-y) &= k \cdot \frac{1}{0+1} t^{0+1} + C \quad \text{notice } -\ln(N-y) \text{ not } \ln(N-y) \text{ because of } -y \\
\ln(N-y) &= -kt + C \\
e^{\ln(N-y)} &= e^{-kt+C} \\
N - y &= e^{-kt+C}
\end{aligned}$$

so (i) $\boldsymbol{y = Ne^{-kt} + M}$ (ii) $\boldsymbol{y = N + e^{Mt}}$ (iii) $\boldsymbol{y = N - Me^{-kt}}$

(b) *Particular Solution.*
Since no viewers see advertisement before it airs, at $t = 0$, $y = 0$,
so solve $\frac{dy}{dt} = k(N-y)$, given $f(0) = 0$.
Since $y = N - Me^{-kt} = 24000 - Me^{-kt}$,

$$0 = 24000 - Me^{k(0)} \quad \text{since } t = 0, y = 0$$

or $M =$ (i) **24000** (ii) **25000** (iii) **10000**

and so the particular solution is $y = N - Me^{-kt} =$
(i) $\boldsymbol{2500 - 2500e^{-kt}}$ (ii) $\boldsymbol{24000 - 24000e^{-kt}}$ (iii) $\boldsymbol{25000 - 10000e^{-kt}}$

(c) *What is k when $t = 10$, $y = 0.4(24000) = 9600$?*

$$9600 = 24000 - 24000e^{-k(10)}$$

so $e^{-10k} = \frac{14400}{24000}$ or $-10k = \ln 0.6$, so
$k \approx$ (i) **0.05108** (ii) **0.08244** (iii) **0.09232**.

(d) *What is t when $k \approx 0.05108$, $y = 0.8(24000) = 19200$?*

$$19200 = 24000 - 24000e^{-0.05108t}$$

so $e^{-0.05108t} = \frac{4800}{24000}$ or $-0.05108t = \ln 0.2$, so
$t \approx$ (i) **29.5** (ii) **30.5** (iii) **31.5**.

2. *Application: logistic growth rate model,* $\frac{dy}{dt} = k\left(1 - \frac{y}{N}\right)y$.
   After 4 days, an initial butterfly population of 15 grows to 56. If the restricted ecosystem supports 300 butterflies, how many butterflies will there be in 12 days? Assuming the butterfly population grows fastest when there are $\frac{N}{2} = \frac{300}{2} = 150$ butterflies, when does this happen?

   (a) *Particular Solution.*
   Assume particular solution to $\frac{dy}{dt} = k\left(1 - \frac{y}{N}\right)y$, $y(0) = y_0$ is

$$y = \frac{N}{1 + be^{-kt}}, \quad b = \frac{N - y_0}{y_0}$$

   in other words, since

$$b = \frac{N - y_0}{y_0} = \frac{300 - 15}{15} = 19,$$

   so $y = \frac{N}{1+be^{-kt}} =$ (i) $\mathbf{\frac{300}{1+19e^{-kt}}}$ (ii) $\mathbf{\frac{300}{1+20e^{-kt}}}$ (iii) $\mathbf{\frac{300}{1+21e^{-kt}}}$

   (b) *What is k when $t = 4$, $y = 56$?*

$$\begin{aligned} 56 &= \frac{300}{1 + 19e^{-k(4)}} \\ 1 + 19e^{-4k} &= \frac{300}{56} \\ 19e^{-4k} &= \frac{300}{56} - 1 \\ e^{-4k} &= \frac{\frac{300}{56} - 1}{19} \\ -4k &= \ln\left(\frac{\frac{300}{56} - 1}{19}\right) \\ k &= \frac{\ln\left(\frac{\frac{300}{56}-1}{19}\right)}{-4} \approx \end{aligned}$$

   (i) **0.368** (ii) **0.743** (iii) **0.876**.

   (c) *What is y when $k \approx 0.368$, $t = 12$?*

$$y \approx \frac{300}{1 + 19e^{-(0.368)(12)}} \approx$$

   (i) **113** (ii) **214** (iii) **244**.
   round up

(d) *Time of maximum growth rate of butterflies, when $y = 150$?* Since

$$\begin{aligned}
150 &= \frac{300}{1+19e^{-0.368t}} \\
150\left(1+19e^{-0.368t}\right) &= 300 \\
150 + 19(150)e^{-0.368t} &= 300 \\
e^{-0.368t} &= \frac{300-150}{19(150)} \\
\ln\left(e^{-0.368t}\right) &= \ln\frac{1}{19} \\
-0.368t &= \ln\frac{1}{19} \\
t &= -\frac{\ln\frac{1}{19}}{0.368} \approx
\end{aligned}$$

(i) **8** (ii) **9** (iii) **10** days.

3. *Application: predator-prey models.*
Consider the following system of differential equations which describe two competing species such as foxes (x) and rabbits (y), where each species keeps the other species in check, or, as one population increases, the other decreases,

$$\begin{aligned}
\frac{dy}{dt} &= 3y - 3xy \quad \text{prey (rabbit) differential equation} \\
\frac{dx}{dt} &= -2x + xy. \quad \text{predator (fox) differential equation}
\end{aligned}$$

Rabbit differential equation equals rate of increase of rabbits (since no foxes around to eat rabbits) minus rate of rabbit-fox encounters (negative because foxes eat rabbits during encounters), whereas fox differential equation equals negative rate of predators (since no rabbits for foxes to eat) plus rate of rabbit-fox encounters. Determine equation relating x to y, assume $y = 1$ when $x = 1$.

(a) *General Solution.*

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy/dt}{dx/dt} \\
\frac{dy}{dx} &= \frac{3y-3xy}{-2x+xy} \\
\frac{dy}{dx} &= \frac{y(3-3x)}{x(-2+y)} \\
\frac{-2+y}{y}\,dy &= \frac{3-3x}{x}\,dx \quad \text{separation of variables}
\end{aligned}$$

$$\int \left(-\frac{2}{y}+1\right) dy \;=\; \int \left(\frac{3}{x}-3\right) dx$$
$$-2\ln y + \frac{1}{1+0}y^{0+1} \;=\; 3\ln x - \frac{3}{1+0}x^{0+1} + C$$

so (i) $\mathbf{2\ln y + \frac{1}{2}y^2 = 3\ln x - \frac{3}{2}x^3 + C}$
(ii) $\mathbf{-2\ln y + y = 3\ln x - 3x + C}$

(b) *Particular Solution, at* $(x,y) = (1,1)$.
Since $-2\ln y + y = 3\ln x - 3x + C$,

$$-2\ln 1 + 1 = 3\ln 1 - 3(1) + C \quad \text{since } x = 1, y = 1$$

and so
$C =$ (i) **2** (ii) **3** (iii) **4**

and so particular solution is
(i) $\mathbf{-2\ln y + y = 3\ln x - 3x + 2}$
(ii) $\mathbf{-2\ln y + y = 3\ln x - 3x + 3}$
(iii) $\mathbf{-2\ln y + y = 3\ln x - 3x + 4}$
graph of this equation of number of foxes, x, and number of rabbits, y, given in figure

(c) *Equilibrium and particular solution at* $(x,y) = (1,1)$.

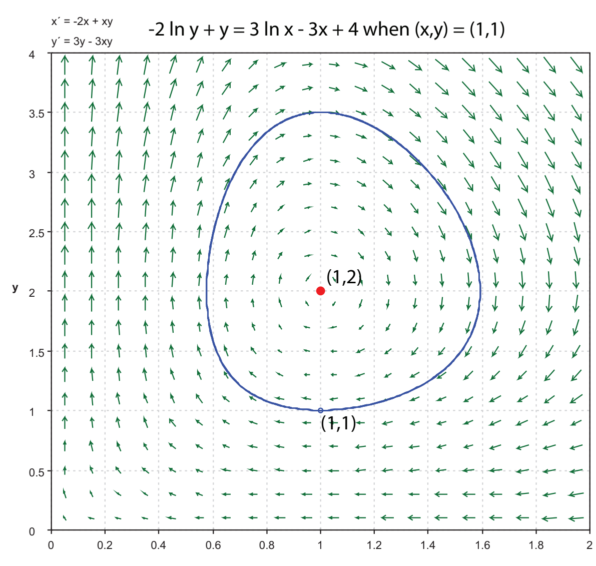

Figure 10.4 (Rabbit-Fox plot, (http://math.rice.edu/ dfield/dfpp.html))

Equilibrium occurs when number of foxes and rabbits does not change; when both differential equations equal zero:

$$\frac{dy}{dt} \;=\; 3y - 3xy \;=\; 0$$

$$\frac{dx}{dt} = -2x + xy = 0,$$

either if both $(x, y) = (0, 0)$ when there are no rabbits or foxes
or $3y - 3xy = 3y(1 - x) = 0$ and $-2x + xy = x(-2 + y) = 0$ so

equilibrium $(x, y) =$ (i) $\mathbf{(1, 2)}$ (ii) $\mathbf{(1, 1)}$ (iii) $\mathbf{(2, 1)}$

In this case, rabbit-fox population starting at $(x, y) = (1, 1)$ spirals indefinitely around equilibrium $(x, y) = (1, 2)$, different starting points give different spirals; different differential equations can give different behaviors such as spiraling outwards from or inwards to equilibrium.

4. *Application: Wine mixture.*
Five grams of crushed pepper is dissolved in 200 liters of wine. Wine is added at a rate of 3 liters per hour and also the solution is drained at 2 liters per hour. Determine the equation describing the mixture at time t. How much crushed pepper is present after 25 hours?

   (a) *Mixture equation.* If $y = f(t)$ is the amount of pepper in wine then change in pepper over time is, since no pepper is being added,

$$\begin{aligned} \frac{dy}{dt} &= \text{(rate of pepper in) - (rate of pepper out)} \\ \frac{dy}{dt} &= 0 - \left(\frac{y}{V} \text{ grams per liter}\right) \text{(2 liters per hour)}, \\ \frac{dy}{dt} &= -\frac{2y}{V}, \end{aligned}$$

   and also the change in volume of wine is

$$\begin{aligned} \frac{dV}{dt} &= \text{(rate of wine in) - (rate of wine out)} \\ \frac{dV}{dt} &= 3 - 2 = 1 \\ dV &= 1\,dt, \quad \text{separation of variables} \\ \int dV &= \int 1\,dt \\ V(t) &= t + C, \\ 200 &= 0 + C \quad \text{since } V = 200, t = 0 \end{aligned}$$

   so $C =$ (i) **0** (ii) **100** (iii) **200**

   and so particular solution is
   (i) $\boldsymbol{V(t) = t}$ (ii) $\boldsymbol{V(t) = t + 100}$ (iii) $\boldsymbol{V(t) = t + 200}$

so combining with first equation

$$\begin{aligned}
\frac{dy}{dt} &= -\frac{2y}{V} \\
\frac{dy}{dt} &= -\frac{2y}{t+200} \\
\frac{1}{y}\,dy &= -\frac{2}{t+200}\,dt \quad \text{separation of variables} \\
\int \left(\frac{1}{y}\right) dy &= \int \left(-\frac{2}{t+200}\right) dx \\
\ln y &= -2\ln(t+200) + C \\
\ln 5 &= -2\ln(0+200) + C \quad \text{since } y=5, t=0 \\
\ln 5 + 2\ln 200 &= C \quad \text{since } y=5, t=0
\end{aligned}$$

so $C =$ (i) $\mathbf{\ln(10 \times 200)}$ (ii) $\mathbf{\ln(5 \times 200)}$ (iii) $\mathbf{\ln(5 \times 200^2)}$

and so particular solution is
(i) $\boldsymbol{\ln y = \ln(t+200) + \ln(5 \times 200^2)}$
(ii) $\boldsymbol{\ln y = -2\ln(t+200) + \ln(5 \times 200^2)}$
(iii) $\boldsymbol{\ln y = -2\ln(t+200) - \ln(5 \times 200^2)}$

or

$$\ln y = -2\ln(t+200) + \ln(5 \times 200^2) = \ln(t+200)^{-2}(5 \times 200^2)$$

and so $y =$ (i) $\boldsymbol{\frac{200^2}{(t+200)^2}}$ (ii) $\boldsymbol{\frac{5\times 200^2}{(t+200)^2}}$ (iii) $\boldsymbol{\frac{5\times 200^2}{(t+200)}}$

(b) *Crushed pepper after 25 hours?*

$$y = \frac{5 \times 200^2}{(t+200)^2} = \frac{5 \times 200^2}{(25+200)^2} \approx$$

(i) **3.85** (ii) **3.95** (iii) **4.05** grams