5.8 The Expected Value and Variance of Linear Functions of Random Variables

For random variables $Y_1, Y_2, \ldots, Y_n$ and $X_1, X_2, \ldots, X_m$ with constants $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_m$,

- $E\left(\sum_{i=1}^{n} a_i Y_i\right) = \sum_{i=1}^{n} a_i E(Y_i)$
- $V\left(\sum_{i=1}^{n} a_i Y_i\right) = \sum_{i=1}^{n} a_i^2 \; V(Y_i) + 2 \sum\sum_{i<j} a_i a_j \; \text{Cov}(Y_i, Y_j)$
- $\text{Cov}\left(\sum_{i=1}^{n} a_i Y_i, \sum_{j=1}^{m} b_j X_j\right) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j \; \text{Cov}(Y_i, X_j)$

Exercise 5.8 (The Expected Value and Variance of Linear Functions of Random Variables)

1. *A First Look: Using the Formulas.* For random variables Y_1, Y_2, Y_3, X_1, X_2,

$$\begin{aligned}
&E(Y_1) = -3, \quad E(Y_2) = 2, \quad E(Y_3) = 4, \\
&V(Y_1) = 2, \quad V(Y_2) = 1, \quad V(Y_3) = 4, \\
&\text{Cov}(Y_1, Y_2) = 2, \quad \text{Cov}(Y_1, Y_3) = 0, \quad \text{Cov}(Y_1, Y_2) = -1, \\
&E(X_1) = 1, \;\; E(X_2) = 2, \;\; V(X_1) = 1, \;\; V(X_2) = 3, \;\; \text{Cov}\,(X_1, X_3) = 2, \\
&\text{Cov}(Y_1, X_1) = 2, \quad \text{Cov}(Y_2, X_1) = 0, \quad \text{Cov}(Y_3, X_1) = -1, \\
&\text{Cov}(Y_1, X_2) = 2, \quad \text{Cov}(Y_2, X_2) = 1, \quad \text{Cov}(Y_3, X_2) = 1.
\end{aligned}$$

   (a) Expected value of $U_1 = -5Y_1 + 5Y_2 + 6Y_3$ is

$$\begin{aligned}
E(U_1) &= E(-5Y_1 + 5Y_2 + 6Y_3) \\
&= -5E\,(Y_1) + 5E\,(Y_2) + 6E\,(Y_3) \\
&= -5\,(-3) + 5\,(2) + 6\,(4) \;=
\end{aligned}$$

   (choose one) (i) **48** (ii) **49** (iii) **50** (iv) **51**.

   (b) Variance of $U_1 = -5Y_1 + 5Y_2 + 6Y_3$ is

$$\begin{aligned}
V(U_1) &= V(-5Y_1 + 5Y_2 + 6Y_3) \\
&= (-5)^2 V\,(Y_1) + 5^2\,(Y_2) + 6^2\,(Y_3) \\
&\quad + 2\,[(-5)(5)\text{Cov}(Y_1, Y_2) + (-5)(6)\text{Cov}(Y_1, Y_3) + (5)(6)\text{Cov}(Y_2, Y_3)] \\
&= (25)\,(2) + (25)\,(1) + (36)\,(4) + \; 2\,[(-5)(5)(2) + (-5)(6)(0) + (5)(6)(-1)] \;=
\end{aligned}$$

   (choose one) (i) **59** (ii) **60** (iii) **61** (iv) **62**.

   (c) Expected value of $U_2 = 2X_1 + 2X_2$ is

$$E(U_2) = E(2X_1 + 2X_2) = 2E\,(X_1) + 2E\,(X_2) = 2\,(1) + 2\,(2) \;=$$

   (choose one) (i) **6** (ii) **7** (iii) **8** (iv) **9**.

(d) Variance of $U_2 = 2X_1 + 2X_2$ is

$$\begin{aligned} V(U_2) &= V(2X_1 + 2X_2) \\ &= 2^2 V(X_1) + 2^2(X_2) + 2[(2)(2)\text{Cov}(Y_1, Y_2)] \\ &= (4)(1) + (4)(3) + (6)(2) = \end{aligned}$$

(choose one) (i) **28** (ii) **29** (iii) **30** (iv) **31**.

(e) Covariance of $U_1 = -5Y_1 + 5Y_2 + 6Y_3$ and $U_2 = 2X_1 + 2X_2$ is

$$\begin{aligned} \text{Cov}(U_1, U_2) &= \text{Cov}(-5Y_1 + 5Y_2 + 6Y_3, 2X_1 + 2X_2) \\ &= (-5)(2)\text{Cov}(Y_1, X_1) + (5)(2)\text{Cov}(Y_2, X_1) + (6)(2)\text{Cov}(Y_3, X_1) \\ &\quad + (-5)(2)\text{Cov}(Y_1, X_2) + (5)(2)\text{Cov}(Y_2, X_2) + (6)(2)\text{Cov}(Y_3, X_2) \\ &= (-5)(2)(2) + (5)(2)(0) + (6)(2)(-1) \\ &\quad + (-5)(2)(2) + (5)(2)(1) + (6)(2)(1) = \end{aligned}$$

(choose one) (i) **−31** (ii) **−30** (iii) **−29** (iv) **−28**.

2. *Discrete Distribution: Waiting Times To Catch Fish.* The joint density, $p(y_1, y_2)$, of the number of minutes waiting to catch the *first* fish, y_1, *and* the number of minutes waiting to catch the *second* fish, y_2, is given below.

$y_2 \downarrow y_1 \rightarrow$	1	2	3	total
1	0.01	0.01	0.07	0.09
2	0.02	0.02	0.08	0.12
3	0.08	0.08	0.63	0.79
total	0.11	0.11	0.78	1.00

(a) Since

$$\begin{aligned} E(Y_1) &= \sum_{y_1=1}^{3} y_1 p_1(y_1) = (1)(0.11) + (2)(0.11) + (3)(0.78) = 2.67 \\ E(Y_2) &= \sum_{y_2=1}^{3} y_2 p_2(y_2) = (1)(0.09) + (2)(0.12) + (3)(0.79) = 2.7, \end{aligned}$$

then $E(Y_1 + Y_2) = E(Y_1) + E(Y_2) =$
(choose one) (i) **5.07** (ii) **5.17** (iii) **5.27** (iv) **5.37**

and $E(3Y_1 - 2Y_2) = 3E(Y_1) - 2E(Y_2) =$
(choose one) (i) **2.51** (ii) **2.61** (iii) **2.71** (iv) **2.81**

(b) Since $E[Y_1Y_2] = \sum_{y_1=1}^{3}\sum_{y_2=1}^{3}(y_1y_2)\,p(y_1,y_2)$
$= (1\times 1)(0.01) + \cdots + (3\times 3)(0.63) \;= 7.17$,

then $\text{Cov}(Y_1,Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2) =$
(choose one) (i) $\mathbf{-0.039}$ (ii) **0.039** (iii) **0.139** (iv) **0.239**

and $\text{Cov}(3Y_1,-2Y_2) = (3)(-2)\text{Cov}(Y_1,Y_2) =$
(choose one) (i) **0.034** (ii) **0.134** (iii) **0.234** (iv) **0.334**

(c) Since

$$\begin{aligned}
E(Y_1^2) &= \sum_{y_1=1}^{3} y_1^2 p_1(y_1) = (1^2)(0.11) + (2^2)(0.11) + (3^2)(0.78) = 7.57,\\
V(Y_1) &= E\left[Y_1^2\right] - \left[E[Y_1]\right]^2 = 7.57 - 2.67^2 = 0.4411,\\
E(Y_2^2) &= \sum_{y_2=1}^{3} y_2^2 p_2(y_2) = (1^2)(0.09) + (2^2)(0.12) + (3^2)(0.79) = 2.7,\\
V(Y_2) &= E\left[Y_2^2\right] - \left[E[Y_2]\right]^2 = 7.68 - 2.7^2 = 0.39,
\end{aligned}$$

then $V(Y_1+Y_2) = V(Y_1) + V(Y_2) + 2\text{Cov}(Y_1,Y_2) =$
(choose one) (i) $\mathbf{-0.5531}$ (ii) **0.6531** (iii) **0.7531** (iv) **0.8531**

and $V(3Y_1 - 2Y_2) = 3^2V(Y_1) + (-2)^2V(Y_2) + 2\text{Cov}(3Y_1,-2Y_2) =$
(choose one) (i) **4.9113** (ii) **5.9979** (iii) **6.113** (iv) **7.213**

(d) Since

$$\rho = \rho(Y_1,Y_2) = \frac{\text{Cov}(Y_1,Y_2)}{\sqrt{V(Y_1)V(Y_2)}} = \frac{-0.039}{\sqrt{0.4411\times 0.39}} \approx$$

(choose one) (i) $\mathbf{-0.235}$ (ii) $\mathbf{-0.139}$ (iii) $\mathbf{-0.094}$ (iii) $\mathbf{-0.024}$.

then

$$\rho(3Y_1,-2Y_2) = \frac{\text{Cov}(3Y_1,-2Y_2)}{\sqrt{V(3Y_1)V(-2Y_2)}} = \frac{(3)(-2)\text{Cov}(Y_1,Y_2)}{\sqrt{3^2V(Y_1)(-2)^2V(Y_2)}} \approx$$

(choose one) (i) $\boldsymbol{-2\rho}$ (ii) $\boldsymbol{-\rho}$ (iii) $\boldsymbol{\rho}$ (iv) $\boldsymbol{2\rho}$

(e) Since $\mu_{Y_1+Y_2} = E(Y_1) + E(Y_2) = 5.37$ and $\sigma_{Y_1+Y_2} = \sqrt{V(Y_1+Y_2)} \approx$
(choose one) (i) **0.6678** (ii) **0.7678** (iii) **0.8678** (iv) **0.9678**

then $\mu_{Y_1+Y_2} \pm \sigma_{Y_1+Y_2} =$ (choose one)
(i) $\mathbf{(4.50, 6.24)}$ (ii) $\mathbf{(4.60, 6.24)}$ (iii) $\mathbf{(4.70, 6.24)}$ (iv) $\mathbf{(4.80, 6.24)}$

so a value more than one SD (0.8678) below mean (5.37) is
(choose one or more) (i) **4.45** (ii) **4.55** (iii) **4.65** (iv) **4.75**

3. *Continuous Distribution: Weight and Amount of Salt in Potato Chips.* Bivariate density function for machine A is

$$f(y_1, y_2) = \begin{cases} \frac{1}{12}, & 49 \leq y_1 \leq 51, 2 \leq y_2 \leq 8 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Since

$$\begin{aligned}
f_1(y_1) &= \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_2 = \int_2^8 \frac{1}{12}\, dy_2 = \frac{1}{2}, \\
E(Y_1) &= \int_{49}^{51} y_1 f_1(y_1)\, dy_1 = \int_{49}^{51} y_1 \frac{1}{12}\, dy_1 = 50, \\
f_2(y_2) &= \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_1 = \int_{49}^{51} \frac{1}{12}\, dy_1 = \frac{1}{6}, \\
E(Y_2) &= \int_2^8 y_2 f_2(y_2)\, dy_2 = \int_2^8 y_2 \frac{1}{12}\, dy_2 = 5,
\end{aligned}$$

then $E(Y_1 + Y_2) = E(Y_1) + E(Y_2) =$
(choose one) (i) **40** (ii) **45** (iii) **50** (iv) **55**

and $E(3Y_1 - 2Y_2) = 3E(Y_1) - 2E(Y_2) =$
(choose one) (i) **120** (ii) **130** (iii) **140** (iv) **150**

(b) Since

$$E(Y_1 Y_2) = \int_2^8 \int_{49}^{51} (y_1 y_2)\, f(y_1, y_2)\, dy_1 dy_2 = 250,$$

then $\text{Cov}(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1)E(Y_2) =$
(choose one) (i) $\mathbf{-2}$ (ii) $\mathbf{-1}$ (iii) **0** (iii) **1**

and $\text{Cov}(3Y_1, -2Y_2) = (3)(-2)\text{Cov}(Y_1, Y_2) =$
(choose one) (i) $\mathbf{-2}$ (ii) $\mathbf{-1}$ (iii) **0** (iii) **1**.

(c) Since

$$\begin{aligned}
E(Y_1^2) &= \int_{49}^{51} y_1^2 f_1(y_1)\, dy_1 = \int_{49}^{51} y_1^2 \frac{1}{2}\, dy_1 = \left(\frac{1}{6} y_1^3\right)_{y_1=49}^{y_1=51} = \frac{7501}{3}, \\
V(Y_1) &= E\left[Y_1^2\right] - \left[E(Y_1)\right]^2 = \frac{7501}{3} - (50)^2 = \frac{1}{3}, \\
E(Y_2^2) &= \int_2^8 y_2^2 f_2(y_2)\, dy_2 = \int_2^8 y_2^2 \frac{1}{6}\, dy_2 = \left(\frac{1}{18} y_2^3\right)_{y_2=2}^{y_2=8} = 28, \\
V(Y_2) &= E\left[Y_2^2\right] - \left[E[Y_2]\right]^2 = 28 - (5)^2 = 3,
\end{aligned}$$

then $V(Y_1+Y_2) = V(Y_1)+V(Y_2)+2\text{Cov}(Y_1,Y_2) =$
(choose one) (i) $\mathbf{\frac{8}{3}}$ (ii) $\mathbf{\frac{9}{3}}$ (iii) $\mathbf{\frac{10}{3}}$ (iv) $\mathbf{\frac{11}{3}}$

and $V(3Y_1-2Y_2) = 3^2V(Y_1)+(-2)^2V(Y_2)+2\text{Cov}(3Y_1,-2Y_2) =$
(choose one) (i) **13** (ii) **14** (iii) **15** (iv) **16**

4. *Special Distributions.* Random variable Y_1 with density

$$f(y_1) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^\alpha} y_1^{\alpha-1}e^{-\frac{y_1}{\beta}} = \frac{1}{6}y_1^3e^{-\frac{y_1}{2}}, & y_1 > 0, \\ 0, & \text{otherwise} \end{cases}$$

is a Gamma with parameters $\alpha = 4$ and $\beta = 2$ and random variable Y_2,

$$f(y_2) = \begin{cases} \frac{1}{\beta}e^{-\frac{y_2}{\beta}} = \frac{1}{3}e^{-\frac{y_2}{3}}, & y_2 > 0, \\ 0, & \text{otherwise} \end{cases}$$

is an exponential density with parameter $\beta = 3$.

(a) $E(Y_1+Y_2) = E(Y_1)+E(Y_2) = \alpha\beta+\beta =$
(choose one) (i) **8** (ii) **9** (iii) **10** (iv) **11**

and $E(3Y_1-2Y_2) = 3E(Y_1)-2E(Y_2) =$
(choose one) (i) **16** (ii) **17** (iii) **18** (iv) **19**

(b) If Y_1 and Y_2 are independent,
$V(Y_1+Y_2) = V(Y_1)+V(Y_2)+2\text{Cov}(Y_1,Y_2) = \alpha\beta^2+\beta^2+2(0) =$
(choose one) (i) **24** (ii) **25** (iii) **26** (iv) **27**

and $V(3Y_1-2Y_2) = 3^2V(Y_1)+(-2)^2V(Y_2)+2\text{Cov}(3Y_1,-2Y_2) =$
(choose one) (i) **170** (ii) **180** (iii) **190** (iv) **200**

5. *Mean and Variance of Rolling Dice.* Determine expectation and variance of sum of 15 rolls of a fair die where rolls are independent of one another and X_i is value of outcome of one die.

(a) For ith roll, $Y_i = 1,2,3,4,5,6$,
$E(Y_i) = \sum_{j=1}^{6} y_jp(y_j) = 1(1/6)+2(1/6)+\cdots+6(1/6) =$
(choose one) (i) $\mathbf{\frac{4}{2}}$ (ii) $\mathbf{\frac{5}{2}}$ (iii) $\mathbf{\frac{6}{2}}$ (iv) $\mathbf{\frac{7}{2}}$

(b) So $E\left(\sum_{i=1}^{15} Y_i\right) = \sum_{i=1}^{15} E(Y_i) = 15E(Y_i) = 15\left(\frac{7}{2}\right) =$
(choose one) (i) $\mathbf{\frac{104}{2}}$ (ii) $\mathbf{\frac{105}{2}}$ (iii) $\mathbf{\frac{106}{2}}$ (iv) $\mathbf{\frac{107}{2}}$

(c) Since $E(Y_i^2) = \sum_{j=1}^{6} y_j^2p(y_j) = 1^2(1/6)+2^2(1/6)+\cdots+6^2(1/6) =$
(choose one) (i) $\mathbf{\frac{89}{6}}$ (ii) $\mathbf{\frac{90}{6}}$ (iii) $\mathbf{\frac{91}{6}}$ (iv) $\mathbf{\frac{92}{6}}$

and $V(Y_i) = E[Y_i^2]-(E(Y_i))^2 = \frac{91}{6}-\left(\frac{7}{2}\right)^2 =$
(choose one) (i) $\mathbf{\frac{33}{12}}$ (ii) $\mathbf{\frac{34}{12}}$ (iii) $\mathbf{\frac{35}{12}}$ (iv) $\mathbf{\frac{36}{12}}$

(d) Since Y_i are independent, $V\left(\sum_{i=1}^{15} Y_i\right) = \sum_{i=1}^{15} V(Y_i) = 15V(Y_i) = 15\frac{35}{12} =$ (choose one) (i) $\mathbf{\frac{525}{12}}$ (ii) $\mathbf{\frac{526}{12}}$ (iii) $\mathbf{\frac{527}{12}}$ (iv) $\mathbf{\frac{528}{12}}$

5.9 The Multivariate Probability Distribution

Random variables $Y_1, Y_2, \ldots, Y_k$ have a *multinomial distribution*

$$p(y_1, y_2, \ldots, y_k) = \frac{n!}{n_1!n_2!\cdots n_k!} p_1^{y_1} p_2^{y_2} \cdots p_k^{y_k}$$

where $\sum_{i=1}^{k} p_i = 1$ and $\sum_{i=1}^{k} y_i = n$, $y_i = 0, 1, \ldots, n$ and

$$E(Y_i) = np_i, \quad V(Y_i) = np_i(1-p_i) \quad \mathrm{Cov}(Y_s, Y_t) = -np_sp_t, s \neq t.$$

Exercise 5.9 (The Multivariate Probability Distribution)

1. *Marbles and Jars.* There are 15 different marbles and 3 jars. There is a 20%, 50% and 30% chance of placing a marble in jars 1, 2 and 3, respectively.

(a) Chance 4, 6 and 5 marbles in jars 1, 2 and 3, respectively, is

$$p(4, 6, 5) = \frac{15!}{4!6!5!} 0.2^4 0.5^6 0.3^5 \approx$$

(choose one) (i) **0.023** (ii) **0.038** (iii) **0.045** (iv) **0.051**

(b) $E(Y_1) = np_1$, $E(Y_2) = np_2$ and $E(Y_3) = np_3$ are, respectively (choose one)
(i) $\mathbf{3, 3, 9}$ (ii) $\mathbf{4, 6, 5}$ (iii) $\mathbf{3, 7.5, 4.5}$ (iv) $\mathbf{3.5, 7, 4.5}$

Type 0.2, 0.5 and 0.3 in L_1, define $L_2 = L_1 \times 15$

(c) $V(Y_1)$, $V(Y_2)$ and $V(Y_3)$ are, respectively (choose one)
(i) $\mathbf{2.4, 3.75, 3.15}$ (ii) $\mathbf{4, 6, 5}$ (iii) $\mathbf{3.2, 3.2, 4.15}$ (iv) $\mathbf{2.5, 4.25, 4}$

Type 0.2, 0.5 and 0.3 in L_1, 0.8, 0.5 and 0.7 in L_2, define $L_3 = L_1 \times L_2 \times 15$

(d) $\mathrm{Cov}(Y_1, Y_2)$, $\mathrm{Cov}(Y_1, Y_3)$ and $\mathrm{Cov}(Y_2, Y_3)$ are, respectively (choose one)
(i) $\mathbf{-2, -3, -3.15}$ (ii) $\mathbf{-4, -6.2, -5.1}$
(iii) $\mathbf{-1.25, -3.2, -3.15}$ (iv) $\mathbf{-1.5, -0.9, -2.25}$

Type 0.2, 0.2 and 0.5 in L_1, 0.5, 0.3 and 0.3 in L_2, define $L_3 = L_1 \times L_2 \times -15$

(e) $E(Y_1 + Y_2 + Y_3) = E(Y_1) + E(Y_2) + E(Y_3) =$ (choose one)
(i) **13** (ii) **14** (iii) **15** (iv) **16**

$E(3Y_1 - 2Y_2 + 2Y_3) = 3E(Y_1) - 2E(Y_2) + 2E(Y_3) =$ (choose one)
(i) **0** (ii) **1** (iii) **2** (iv) **3**

(f) $V(Y_1 + Y_2) = V(Y_1) + V(Y_2) + 2\mathrm{Cov}(Y_1, Y_2) = 2.4 + 3.75 + 2(-1.5) =$
(i) **2.75** (ii) **3.15** (iii) **3.65** (iv) **3.95**

and $V(Y_1 + Y_3) = V(Y_1) + V(Y_3) + 2\mathrm{Cov}(Y_1, Y_3) = 2.4 + 3.75 + 2(-0.9) =$
(i) **2.35** (ii) **3.35** (iii) **4.35** (iv) **5.35**

(g) $\text{Cov}(3Y_1, -2Y_2) = (3)(-2)\text{Cov}(Y_1, Y_2) = (3)(-2)(-1.5)$
(choose one) (i) **11** (ii) **10** (iii) **9** (iii) **8**.

and $\text{Cov}(2Y_2, Y_3) = (2)(1)\text{Cov}(Y_2, Y_3) = (2)(1)(-2.25) =$
(choose one) (i) $\mathbf{-4.5}$ (ii) $\mathbf{-5.5}$ (iii) $\mathbf{-6.5}$ (iii) $\mathbf{-7.5}$.

2. *Faculty and Subjects.* There are 9 different faculty members and 3 subjects: mathematics, statistics and physics. There is a 50%, 35% and 15% chance a faculty member teaches mathematics, statistics and physics, respectively.

(a) Chance 4, 3 and 2 faculty members teach mathematics, statistics and physics, respectively, is

$$p(4,3,2) = \frac{9!}{4!3!2!}0.5^4 0.35^3 0.15^2 \approx$$

(choose one) (i) **0.055** (ii) **0.067** (iii) **0.076** (iv) **0.111**

(b) Chance 4, 4 and 1 faculty members teach mathematics, statistics and physics, respectively, is

$$p(4,4,1) = \frac{9!}{4!4!1!}0.5^4 0.35^4 0.15^1 \approx$$

(choose one) (i) **0.089** (ii) **0.098** (iii) **0.108** (iv) **0.131**

(c) $E(Y_1)$, $E(Y_2)$ and $E(Y_3)$ are, respectively (choose one)
(i) $\mathbf{3, 3, 9}$ (ii) $\mathbf{4.5, 3.15, 1.35}$ (iii) $\mathbf{3.1, 7.2, 5.5}$ (iv) $\mathbf{3, 7, 4}$
Type 0.5, 0.35 and 0.15 in L_1, define $L_2 = L_1 \times 9$

(d) $V(Y_1)$, $V(Y_2)$ and $V(Y_3)$ are, respectively (choose one)
(i) $\mathbf{2, 3, 3}$ (ii) $\mathbf{2.25, 2.0475, 1.1475}$ (iii) $\mathbf{3.2, 3.2, 3.1}$ (iv) $\mathbf{2, 4, 4}$
Type 0.5, 0.35 and 0.15 in L_1, 0.5, 0.65 and 0.85 in L_2, define $L_3 = L_1 \times L_2 \times 9$

(e) $\text{Cov}(Y_1, Y_2)$, $\text{Cov}(Y_1, Y_3)$ and $\text{Cov}(Y_2, Y_3)$ are, respectively (choose one)
(i) $\mathbf{-3, -3, -3}$ (ii) $\mathbf{-4.11, -2.2, -3.1}$
(iii) $\mathbf{-1.25, -0.21, -3.15}$ (iv) $\mathbf{-1.575, -0.675, -0.4725}$
Type 0.5, 0.5 and 0.35 in L_1, 0.35, 0.15 and 0.15 in L_2, define $L_3 = -1 \times L_1 \times L_2 \times 9$

(f) If a mathematics, statistics and physics course costs 3 thousand, 2 thousand and 2 thousand dollars, respectively, to teach, expected cost is $E(3Y_1 + 2Y_2 + 2Y_3) = 3E(Y_1) + 2E(Y_2) + 2E(Y_3) =$ (choose one)
(i) **20.5** (ii) **21.5** (iii) **22.5** (iv) **25.2**

(g) Variance in cost of $V(U_1) = V(3Y_1 + 2Y_2 + 2Y_3)$ is

$$\begin{aligned} V(U_1) &= (3)^2 V(Y_1) + 2^2 (Y_2) + 2^2 (Y_3) \\ &\quad + 2\left[(3)(2)\text{Cov}(Y_1, Y_2) + (3)(2)\text{Cov}(Y_1, Y_3) + (2)(2)\text{Cov}(Y_2, Y_3)\right] \\ &\approx (9)(2.25) + (4)(2.0475) + (4)(1.1475) \\ &\quad + 2\left[(3)(2)(-1.575) + (3)(2)(-0.675) + (2)(2)(-0.4725)\right] = \end{aligned}$$

(choose one) (i) **1.52** (ii) **1.64** (iii) **2.25** (iv) **2.56**.

3. *Defective items, large population.* A sample of size n is selected at random from a large number of widgets. A proportion p_1, p_2 and p_3 of all items have one defective, more than one defective and no defectives, respectively. The number of sampled items with one, more than one and no defectives is denoted by Y_1, Y_2 and Y_3, respectively.

   (a) $E(3Y_1 - 2Y_2) =$ (choose one)
   (i) $\boldsymbol{3np_1 - p_2}$ (ii) $\boldsymbol{3np_1 - np_2}$ (iii) $\boldsymbol{3np_1 - 2np_2}$ (iv) $\boldsymbol{3np_1 - 4np_2}$

   (b) $V(3Y_1 - 2Y_2) = 3^2V(Y_1) + (-2)^2V(Y_2) + 2(3)(-2)\text{Cov}(Y_1, Y_2) =$
   (i) $\boldsymbol{-12p_1p_2}$
   (ii) $\boldsymbol{9p_1q_1 - 4p_2q_2 + 12np_1p_2}$
   (iii) $\boldsymbol{9np_1q_1 + 4np_2q_2 + 12np_1p_2}$
   (iv) $\boldsymbol{9np_2q_2 + 4np_3q_3 + 12np_2p_3}$

   (c) $V(3Y_2 - 2Y_3) =$ (choose one)
   (i) $\boldsymbol{-12p_1p_2}$
   (ii) $\boldsymbol{9p_1q_1 - 4p_2q_2 + 12np_1p_2}$
   (iii) $\boldsymbol{9np_1q_1 + 4np_2q_2 + 12np_1p_2}$
   (iv) $\boldsymbol{9np_2q_2 + 4np_3q_3 + 12np_2p_3}$

4. *Defective items, large population.* A sample of size $n = 4$ is selected at random from a large number of widgets. A proportion p_1, p_2 and p_3 of all items have one defective, more than one defective and no defectives, respectively. The number of sampled items with one, more than one and no defectives is denoted by Y_1, Y_2 and Y_3. Furthermore, $p_1 = P(Y_1 \leq 3)$, $p_2 = P(3 < Y_2 \leq 7)$ and $p_3 = P(Y_3 > 7)$ where Y_i are all normal with $\mu = 5$ and $\sigma = 2$.

   (a) $p_1 = P(Y_1 \leq 3) \approx$ (choose one)
   (i) **0.16** (ii) **0.32** (iii) **0.68** (iv) **0.96**
   normalcdf(-E99, 3, 5, 2)

   $p_2 = P(3 < Y_2 \leq 7) \approx$ (choose one)
   (i) **0.16** (ii) **0.32** (iii) **0.68** (iv) **0.96**
   normalcdf(3, 7, 5, 2)

   $p_3 = P(Y_3 > 7) \approx$ (choose one)
   (i) **0.16** (ii) **0.32** (iii) **0.68** (iv) **0.96**
   normalcdf(7, E99, 5, 2)

   (b) $E(3Y_1 - 2Y_2) = 3np_1 - 2np_2 =$ (choose one)
   (i) **−2.56** (ii) **−3.14** (iii) **−3.52** (iv) **−3.86**

   (c) $V(3Y_1 - 2Y_2) = 9np_1q_1 + 4np_2q_2 + 12np_1p_2 =$ (choose one)
   (i) **12.0001** (ii) **12.3455** (iii) **12.9934** (iv) **13.5424**

5.10 The Bivariate Normal Distribution

Not covered.

5.11 Conditional Expectations

We learn about conditional expectation and variance in this section.

- conditional expectation
  $E(g(Y_1)|Y_2 = y_2) = \sum_{y_1} g(y_1)p(y_1|y_2)$
  $E(g(Y_1)|Y_2 = y_2) = \int_{-\infty}^{\infty} g(y_1)f(y_1|y_2)\,dy_1$
- expectation by conditioning
  $E(Y_1) = E(E(Y_1|Y_2)) = \sum_{y_2} E(Y_1|Y_2 = y_2)p(y_2) = \sum_{y_2}\left(\sum_{y_1} y_1 p(y_1|y_2)\right)p(y_2)$
  $E(Y_1) = E(E(Y_1|Y_2)) = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} y_1 f(y_1|y_2)\,dy_1\right)f(y_2)\,dy_2$
- conditional variance
  $V(Y_1|Y_2 = y_2) = E(Y_1^2|Y_2 = y_2) - (E(Y_1|Y_2 = y_2))^2$
- variance by conditioning
  $V(Y_1) = E\left[V(Y_1|Y_2)\right] + V\left[E(Y_1|Y_2)\right]$

Exercise 5.11 (Conditional Expectations)

1. *Discrete Distribution: Waiting Times To Catch Fish.* The joint density, $p(y_1, y_2)$, of the number of minutes waiting to catch the *first* fish, y_1, *and* the number of minutes waiting to catch the *second* fish, y_2, is given below.

$y_2 \downarrow y_1 \rightarrow$	1	2	3	total
1	0.01	0.01	0.07	0.09
2	0.02	0.02	0.08	0.12
3	0.08	0.08	0.63	0.79
total	0.11	0.11	0.78	1.00

   (a) *Compute* $E(Y_1|Y_2 = 1)$.
   $P(Y_1 = 1|Y_2 = 1) = \frac{P(Y_1=1,Y_2=1)}{P(Y_2=1)} = \frac{0.01}{0.09} =$ (i) $\mathbf{\frac{1}{9}}$ (ii) $\mathbf{\frac{2}{9}}$ (iii) $\mathbf{\frac{7}{9}}$ (iv) $\mathbf{\frac{8}{9}}$
   $P(Y_1 = 2|Y_2 = 1) = \frac{P(Y_1=2,Y_2=1)}{P(Y_2=1)} = \frac{0.01}{0.09} =$ (i) $\mathbf{\frac{1}{9}}$ (ii) $\mathbf{\frac{2}{9}}$ (iii) $\mathbf{\frac{7}{9}}$ (iv) $\mathbf{\frac{8}{9}}$
   $P(Y_1 = 3|Y_2 = 1) = \frac{P(Y_1=3,Y_2=1)}{P(Y_2=1)} = \frac{0.07}{0.09} =$ (i) $\mathbf{\frac{1}{9}}$ (ii) $\mathbf{\frac{2}{9}}$ (iii) $\mathbf{\frac{7}{9}}$ (iv) $\mathbf{\frac{8}{9}}$

   $$E(Y_1|Y_2 = 1) = \sum_{y_1=1}^{3} y_1 P(Y_1 = y_1|Y_2 = 1) = (1)\left(\frac{1}{9}\right)+(2)\left(\frac{1}{9}\right)+(3)\left(\frac{7}{9}\right) =$$

   (choose one) (i) $\mathbf{\frac{21}{9}}$ (ii) $\mathbf{\frac{22}{9}}$ (iii) $\mathbf{\frac{23}{9}}$ (iv) $\mathbf{\frac{24}{9}}$

(b) *Compute $E(Y_1|Y_2 = 2)$.*
$P(Y_1 = 1|Y_2 = 2) = \frac{P(Y_1=1,Y_2=2)}{P(Y_2=2)} = \frac{0.02}{0.12} =$ (i) $\mathbf{\frac{1}{12}}$ (ii) $\mathbf{\frac{2}{12}}$ (iii) $\mathbf{\frac{7}{12}}$ (iv) $\mathbf{\frac{8}{12}}$
$P(Y_1 = 2|Y_2 = 2) = \frac{P(Y_1=2,Y_2=2)}{P(Y_2=2)} = \frac{0.02}{0.12} =$ (i) $\mathbf{\frac{1}{12}}$ (ii) $\mathbf{\frac{2}{12}}$ (iii) $\mathbf{\frac{7}{12}}$ (iv) $\mathbf{\frac{8}{12}}$
$P(Y_1 = 3|Y_2 = 2) = \frac{P(Y_1=3,Y_2=2)}{P(Y_2=2)} = \frac{0.08}{0.12} =$ (i) $\mathbf{\frac{1}{12}}$ (ii) $\mathbf{\frac{2}{12}}$ (iii) $\mathbf{\frac{7}{12}}$ (iv) $\mathbf{\frac{8}{12}}$

$$E(Y_1|Y_2 = 2) = \sum_{y_1=1}^{3} y_1 P(Y_1 = y_1|Y_2 = 2) = (1)\left(\frac{2}{12}\right)+(2)\left(\frac{2}{12}\right)+(3)\left(\frac{8}{12}\right) =$$

(choose one) (i) $\mathbf{\frac{25}{12}}$ (ii) $\mathbf{\frac{26}{12}}$ (iii) $\mathbf{\frac{27}{12}}$ (iv) $\mathbf{\frac{30}{12}}$

(c) *Compute $E(Y_1|Y_2 = 3)$.*
$P(Y_1 = 1|Y_2 = 3) = \frac{P(Y_1=1,Y_2=3)}{P(Y_2=3)} = \frac{0.08}{0.79} =$ (i) $\mathbf{\frac{7}{79}}$ (ii) $\mathbf{\frac{8}{79}}$ (iii) $\mathbf{\frac{63}{79}}$ (iv) $\mathbf{\frac{72}{79}}$
$P(Y_1 = 3|Y_2 = 3) = \frac{P(Y_1=3,Y_2=3)}{P(Y_2=3)} = \frac{0.08}{0.79} =$ (i) $\mathbf{\frac{7}{79}}$ (ii) $\mathbf{\frac{8}{79}}$ (iii) $\mathbf{\frac{63}{79}}$ (iv) $\mathbf{\frac{72}{79}}$
$P(Y_1 = 3|Y_2 = 3) = \frac{P(Y_1=3,Y_2=3)}{P(Y_2=3)} = \frac{0.63}{0.79} =$ (i) $\mathbf{\frac{7}{79}}$ (ii) $\mathbf{\frac{8}{79}}$ (iii) $\mathbf{\frac{63}{79}}$ (iv) $\mathbf{\frac{72}{79}}$

$$E(Y_1|Y_2 = 3) = \sum_{y_1=1}^{3} y_1 P(Y_1 = y_1|Y_2 = 3) = (1)\left(\frac{8}{79}\right)+(2)\left(\frac{8}{79}\right)+(3)\left(\frac{63}{79}\right) =$$

(choose one) (i) $\mathbf{\frac{211}{79}}$ (ii) $\mathbf{\frac{212}{79}}$ (iii) $\mathbf{\frac{213}{79}}$ (iv) $\mathbf{\frac{214}{79}}$

(d) *Compute $E(E(Y_1|Y_2))$.*

$$\begin{aligned}
E(E(Y_1|Y_2)) &= \sum_{y_2} E(Y_1|Y_2 = y_2)P(Y_2 = y_2) \\
&= E(Y_1|Y_2 = 1)P(Y_2 = 1) + E(Y_1|Y_2 = 2)P(Y_2 = 2) \\
&\quad + E(Y_1|Y_2 = 3)P(Y_2 = 3) \\
&= \left(\frac{24}{9}\right)(0.09) + \left(\frac{30}{12}\right)(0.12) + \left(\frac{213}{79}\right)(0.79) =
\end{aligned}$$

(choose one) (i) **2.67** (ii) **2.77** (iii) **2.87** (iv) **2.97**

(e) *Compute $E(Y_1)$.*

$$E(Y_1) = \sum_{y_1=1}^{3} y_1 P(Y_1 = y_1) = (1)(0.11) + (2)(0.11) + (3)(0.78) =$$

(choose one) (i) **2.67** (ii) **2.77** (iii) **2.87** (iv) **2.97**

(f) In other words, $E(Y_1)$ (choose one) **does** / **does not** equal $E(E(Y_1|Y_2))$. Here, it is much easier to calculate $E(Y_1)$ than it is to calculate $E(E(Y_1|Y_2))$.

(g) *Compute* $V(Y_1|Y_2 = 1)$.
$E(Y_1^2|Y_2 = 1) = \sum_{y_1} y_1^2 P(Y_1 = y_1|Y_2 = 1) =$
$(1)^2\left(\frac{1}{9}\right) + (2)^2\left(\frac{1}{9}\right) + (3)^2\left(\frac{7}{9}\right) =$ (choose one) (i) $\mathbf{\frac{65}{9}}$ (ii) $\mathbf{\frac{66}{9}}$ (iii) $\mathbf{\frac{67}{9}}$ (iv) $\mathbf{\frac{68}{9}}$

$V(Y_1|Y_2 = 1) = E(Y_1^2|Y_2 = 1) - (E(Y_1|Y_2 = 1))^2 = \frac{68}{9} - \left(\frac{24}{9}\right)^2 =$
(choose one) (i) $\mathbf{\frac{4}{9}}$ (ii) $\mathbf{\frac{5}{9}}$ (iii) $\mathbf{\frac{6}{9}}$ (iv) $\mathbf{\frac{7}{9}}$

(h) *Compute* $V(Y_1|Y_2 = 2)$.
$E(Y_1^2|Y_2 = 2) = \sum_{y_1} y_1^2 P(Y_1 = y_1|Y_2 = 2) =$
$(1)^2\left(\frac{2}{12}\right) + (2)^2\left(\frac{2}{12}\right) + (3)^2\left(\frac{8}{12}\right) =$ (choose one) (i) $\mathbf{\frac{80}{12}}$ (ii) $\mathbf{\frac{81}{12}}$ (iii) $\mathbf{\frac{82}{12}}$ (iv) $\mathbf{\frac{83}{12}}$

$V(Y_1|Y_2 = 2) = E(Y_1^2|Y_2 = 2) - (E(Y_1|Y_2 = 2))^2 = \frac{82}{12} - \left(\frac{30}{12}\right)^2 =$
(choose one) (i) $\mathbf{\frac{5}{12}}$ (ii) $\mathbf{\frac{6}{12}}$ (iii) $\mathbf{\frac{7}{12}}$ (iv) $\mathbf{\frac{8}{12}}$

(i) *Compute* $V(Y_1|Y_2 = 3)$.
$E(Y_1^2|Y_2 = 3) = \sum_{y_1} y_1^2 P(Y_1 = y_1|Y_2 = 3) =$
$(1)^2\left(\frac{8}{79}\right) + (2)^2\left(\frac{8}{79}\right) + (3)^2\left(\frac{63}{79}\right) =$ (i) $\mathbf{\frac{606}{79}}$ (ii) $\mathbf{\frac{607}{79}}$ (iii) $\mathbf{\frac{608}{79}}$ (iv) $\mathbf{\frac{609}{79}}$

$V(Y_1|Y_2 = 3) = E(Y_1^2|Y_2 = 3) - (E(Y_1|Y_2 = 3))^2 = \frac{607}{79} - \left(\frac{213}{79}\right)^2 \approx$
(choose one) (i) $\mathbf{\frac{2583}{6241}}$ (ii) $\mathbf{\frac{2584}{6241}}$ (iii) $\mathbf{\frac{2585}{6241}}$ (iv) $\mathbf{\frac{2586}{6241}}$

(j) *Compute* $E[V(Y_1|Y_2)]$.

$$\begin{aligned}
E[V(Y_1|Y_2)] &= \sum_{y_2} V(Y_1|Y_2 = y_2)P(Y_2 = y_2) \\
&= V(Y_1|Y_2 = 1)P(Y_2 = 1) + V(Y_1|Y_2 = 2)P(Y_2 = 2) \\
&\quad + V(Y_1|Y_2 = 3)P(Y_2 = 3) \\
&= \left(\frac{4}{9}\right)(0.09) + \left(\frac{7}{12}\right)(0.12) + \left(\frac{2584}{6241}\right)(0.79) =
\end{aligned}$$

(choose one) (i) $\mathbf{\frac{3452}{7900}}$ (ii) $\mathbf{\frac{3453}{7900}}$ (iii) $\mathbf{\frac{3454}{7900}}$ (iv) $\mathbf{\frac{3455}{7900}}$

(k) *Compute* $V[E(Y_1|Y_2)]$.

$$\begin{aligned}
E\left\{[E(Y_1|Y_2)]^2\right\} &= \sum_{y_2} [E(Y_1|Y_2 = y_2)]^2 P(Y_2 = y_2) \\
&= [E(Y_1|Y_2 = 1)]^2 P(Y_2 = 1) + [E(Y_1|Y_2 = 2)]^2 P(Y_2 = 2) \\
&\quad + [E(Y_1|Y_2 = 3)]^2 P(Y_2 = 3) \\
&= \left(\frac{24}{9}\right)^2 (0.09) + \left(\frac{30}{12}\right)^2 (0.12) + \left(\frac{213}{79}\right)^2 (0.79) =
\end{aligned}$$

(choose one) (i) $\mathbf{\frac{1125}{158}}$ (ii) $\mathbf{\frac{1126}{158}}$ (iii) $\mathbf{\frac{1127}{158}}$ (iv) $\mathbf{\frac{1128}{158}}$

$\{E(E(Y_1|Y_2))\}^2 = 2.67^2 =$
(choose one) (i) **7.1089** (ii) **7.1189** (iii) **7.1289** (iv) **7.1389**

$$V[E(Y_1|Y_2)] = E\left\{[E(Y_1|Y_2)]^2\right\} - \{E(E(Y_1|Y_2))\}^2 = \frac{1127}{158} - 7.1289 \approx$$

(choose one) (i) **0.004** (ii) **0.005** (iii) **0.006** (iv) **0.007**

(l) *Compute* $E[V(Y_1|Y_2)] + V[E(Y_1|Y_2)]$.

$$V(Y_1) = E[V(Y_1|Y_2)] + V[E(Y_1|Y_2)] = \frac{3453}{7900} + \left(\frac{1127}{158} - 7.1289\right) =$$

(choose one) (i) **0.4111** (ii) **0.4211** (iii) **0.4311** (iv) **0.4411**

(m) *Compute* $V(Y_1)$.

$$E(Y_1^2) = \sum_{y_1=1}^{3} y_1^2 P(Y_1 = y_1) = (1^2)(0.11) + (2^2)(0.11) + (3^2)(0.78) =$$

(choose one) (i) **4.67** (ii) **6.57** (iii) **7.57** (iv) **8.57**

$$V(Y_1) = E(Y_1^2) - (E(Y_1))^2 = 7.57 - (2.67)^2 =$$

(choose one) (i) **0.3411** (ii) **0.4411** (iii) **0.5411** (iv) **0.6411**

(n) $V(Y_1)$ (choose one) **does** / **does not** equal $E[V(Y_1|Y_2)] + V[E(Y_1|Y_2)]$. Here, it is much easier to calculate $V(Y_1)$ than it is to calculate $E[V(Y_1|Y_2)] + V[E(Y_1|Y_2)]$.

(o) *Typical waiting time.* Is it unusual to wait 3 minutes to catch the first fish? Since

$$\frac{y_1 - E(y_1)}{\sqrt{V(y_1)}} = \frac{3 - 2.67}{\sqrt{0.4411}} =$$

(choose one) (i) **0.48** (ii) **0.50** (iii) **0.57** (iv) **0.60**
it is *not* unusual to wait 3 minutes.

2. *Continuous Distribution: Potato Chips.* A machine fills potato chip bags. Although each bag should weigh 50 grams each and contain 5 milligrams of salt, in fact, because of differing machines, weight and amount of salt placed in each bag varies. Bivariate density function for this machine is

$$f(y_1, y_2) = \begin{cases} \frac{1}{12}, & 49 \le y_1 \le 51, 2 \le y_2 \le 8 \\ 0 & \text{elsewhere} \end{cases}$$

(a) *Compute $E(Y_1|Y_2)$.*

$$f(y_2) = \int_{y_1=49}^{51} \left(\frac{1}{12}\right) \, dy_1 = \left(\frac{1}{12}y_1\right)_{y_1=49}^{51} =$$

(choose one) (i) $\mathbf{\frac{0}{12}}$ (ii) $\mathbf{\frac{1}{12}}$ (iii) $\mathbf{\frac{2}{12}}$ (iv) $\mathbf{\frac{3}{12}}$

$$f(y_1|y_2) = \frac{f(y_1, y_2)}{f(y_2)} = \frac{1/12}{2/12} =$$

(choose one) (i) $\mathbf{\frac{1}{4}}$ (ii) $\mathbf{\frac{1}{3}}$ (iii) $\mathbf{\frac{1}{2}}$ (iv) $\mathbf{\frac{1}{1}}$

$$E(Y_1|Y_2) = \int_{y_1=49}^{51} y_1 \left(\frac{1}{2}\right) \, dy_1 = \left(\frac{1}{4}y_1^2\right)_{y_1=49}^{51} =$$

(choose one) (i) $\mathbf{\frac{200}{4}}$ (ii) $\mathbf{\frac{201}{4}}$ (iii) $\mathbf{\frac{202}{4}}$ (iv) $\mathbf{\frac{203}{4}}$

(b) *Compute $E(E(Y_1|Y_2))$.*

$$E(E(Y_1|Y_2)) = \int_{y_2=2}^{8} E(Y_1|Y_2) f(y_2) \, dy_2 = \int_{y_2=2}^{8} \frac{200}{4} \cdot \frac{2}{12} \, dy_2 = \left(\frac{25}{3}y_2\right)_{y_2=2}^{8} =$$

(choose one) (i) **50** (ii) **75** (iii) **100** (iv) **125**

(c) *Compute $E(Y_1)$.*

$$f(y_1) = \int_{y_2=2}^{8} \left(\frac{1}{12}\right) \, dy_2 = \left(\frac{1}{12}y_2\right)_{y_2=2}^{8} =$$

(choose one) (i) $\mathbf{\frac{1}{5}}$ (ii) $\mathbf{\frac{1}{4}}$ (iii) $\mathbf{\frac{1}{3}}$ (iv) $\mathbf{\frac{1}{2}}$

$$E(Y_1) = \int_{y_1=49}^{51} y_1 \left(\frac{1}{2}\right) \, dy_1 = \left(\frac{1}{4}y_1^2\right)_{y_1=49}^{51} =$$

(choose one) (i) **47** (ii) **48** (iii) **49** (iv) **50**

(d) In other words, $E(Y_1)$ (choose one) **does** / **does not** equal $E(E(Y_1|Y_2))$. Here, it is much easier to calculate $E(Y_1)$ than it is to calculate $E(E(Y_1|Y_2))$.

(e) *Compute $V(Y_1|Y_2)$.*

$$E(Y_1^2|Y_2) = \int_{y_1=49}^{51} y_1^2 \left(\frac{1}{2}\right) \, dy_1 = \left(\frac{1}{6}y_1^3\right)_{y_1=49}^{51} =$$

(choose one) (i) $\mathbf{\frac{15000}{6}}$ (ii) $\mathbf{\frac{15001}{6}}$ (iii) $\mathbf{\frac{15002}{6}}$ (iv) $\mathbf{\frac{15003}{6}}$

$$E\left\{[E(Y_1|Y_2)]^2\right\} = \int_{y_1=2}^{8} 50^2 \cdot \frac{2}{12} \, dy_2 = \left(\frac{1250}{3}y_2\right)_{y_2=2}^{8} =$$

(choose one) (i) **2499** (ii) **2500** (iii) **2501** (iv) **2502**

$$V(Y_1|Y_2) = E(Y_1^2|Y_2) - E\left\{[E(Y_1|Y_2)]^2\right\} = \frac{15002}{6} - 2500 =$$

(choose one) (i) $\mathbf{\frac{1}{3}}$ (ii) $\mathbf{\frac{2}{3}}$ (iii) $\mathbf{\frac{3}{3}}$ (iv) $\mathbf{\frac{4}{3}}$

(f) *Compute* $E\left[V(Y_1|Y_2)\right]$.

$$E\left[V(Y_1|Y_2)\right] = \int_{y_1=2}^{8} \frac{1}{3} \cdot \frac{2}{12}\, dy_2 = \left(\frac{1}{18} y_2\right)_{y_2=2}^{8} =$$

(i) $\mathbf{\frac{1}{3}}$ (ii) $\mathbf{\frac{2}{3}}$ (iii) $\mathbf{\frac{3}{3}}$ (iv) $\mathbf{\frac{4}{3}}$

(g) *Compute* $V\left[E(Y_1|Y_2)\right]$.

$$\{E(E(Y_1|Y_2))\}^2 = 50^2 =$$

(choose one) (i) **2499** (ii) **2500** (iii) **2501** (iv) **2502**

$$V\left[E(Y_1|Y_2)\right] = E\left\{[E(Y_1|Y_2)]^2\right\} - \{E(E(Y_1|Y_2))\}^2 = 2500 - 2500 =$$

(choose one) (i) $\mathbf{-2}$ (ii) $\mathbf{-1}$ (iii) $\mathbf{0}$ (iv) $\mathbf{1}$

(h) *Compute* $E\left[V(Y_1|Y_2)\right] + V\left[E(Y_1|Y_2)\right]$.

$$V(Y_1) = E\left[V(Y_1|Y_2)\right] + V\left[E(Y_1|Y_2)\right] = \frac{1}{3} + 0 =$$

(choose one) (i) $\mathbf{\frac{1}{3}}$ (ii) $\mathbf{\frac{2}{3}}$ (iii) $\mathbf{\frac{3}{3}}$ (iv) $\mathbf{\frac{4}{3}}$

(i) *Compute* $V(Y_1)$.

$$E(Y_1^2) = \int_{y_1=49}^{51} y_1^2 \left(\frac{1}{2}\right)\, dy_1 = \left(\frac{1}{6} y_1^3\right)_{y_1=49}^{51} =$$

(choose one) (i) $\mathbf{\frac{15000}{6}}$ (ii) $\mathbf{\frac{15001}{6}}$ (iii) $\mathbf{\frac{15002}{6}}$ (iv) $\mathbf{\frac{15003}{6}}$

$$V(Y_1) = E(Y_1^2) - (E(Y_1))^2 = \frac{15002}{6} - (50)^2 =$$

(choose one) (i) $\mathbf{\frac{1}{3}}$ (ii) $\mathbf{\frac{2}{3}}$ (iii) $\mathbf{\frac{3}{3}}$ (iv) $\mathbf{\frac{4}{3}}$

(j) $V(Y_1)$ (choose one) **does** / **does not** equal $E\left[V(Y_1|Y_2)\right] + V\left[E(Y_1|Y_2)\right]$. Here, it is much easier to calculate $V(Y_1)$ than it is to calculate $E\left[V(Y_1|Y_2)\right] + V\left[E(Y_1|Y_2)\right]$ in this case.

(k) *Typical weight.* Is it unusual for a bag of chips to weigh 55 grams? Since

$$\frac{y_1 - E(y_1)}{\sqrt{V(y_1)}} = \frac{55 - 50}{\sqrt{\frac{1}{3}}} =$$

(choose one) (i) **5.48** (ii) **6.50** (iii) **7.57** (iv) **8.66**
it *is* unusual for a bag to weigh 55 grams.

3. *Bayes hierarchical model: car defects.* Number, Y_1, of car defects from a production line has a Poisson distribution with parameter λ. Furthermore, parameter λ is a random variable with exponential distribution with parameter β.

   (a) *Compute* $E(Y_1|\lambda)$. *If* λ is fixed constant; in other words, given or conditional on a particular value of random variable λ, Y_1 has a discrete Poisson distribution,

   $$p(y_1|\lambda) = \frac{\lambda^{y_1} e^{-\lambda}}{y_1!}, \quad y_1 = 0, 1, 2, \ldots$$

   and so conditional expectation[1] of Y_1 is

   $$E(Y_1|\lambda) = \sum_y y_1 \left(\frac{\lambda^{y_1} e^{-\lambda}}{y_1!} \right) =$$

   (choose one) (i) $\boldsymbol{\lambda}$ (ii) $\boldsymbol{\lambda^2}$ (iii) $\boldsymbol{\lambda^3}$ (iv) $\boldsymbol{\lambda^4}$

   (b) *Compute*[2] $E(E(Y_1|\lambda))$.

   $$E(E(Y_1|\lambda)) = \int_\lambda E(Y_1|\lambda) f(\lambda)\, d\lambda = \int_\lambda \lambda \left(\frac{1}{\beta} e^{-\lambda/\beta} \right) d\lambda =$$

   (choose one) (i) $\boldsymbol{\beta}$ (ii) $\boldsymbol{\beta^2}$ (iii) $\boldsymbol{\beta^3}$ (iv) $\boldsymbol{\beta^4}$

   (c) *Compute* $E(Y_1)$.
   In this case, we know $E(Y_1) = \beta$ because $E(Y_1) = E(E(Y_1|\lambda)) = \beta$. It is much easier to calculate $E(E(Y_1|Y_2))$ than it is to calculate $E(Y_1)$. Calculations for obtaining $E(Y_1)$ directly are typically used in hierarchical Bayesian statistical analyses and involve *joint likelihood* of Y_1 and λ,

   $$L(y_1, \lambda) = f(y_1) \times p(y_1|\lambda) = \left(\frac{1}{\beta} e^{-\lambda/\beta} \right) \times \left(\frac{\lambda^{y_1} e^{-\lambda}}{y_1!} \right).$$

   (d) *Compute* $E\left[V(Y_1|\lambda)\right] + V\left[E(Y_1|\lambda)\right]$. Since[3]

   $$E\left[V(Y_1|\lambda)\right] = E\left[\lambda\right] =$$

   (choose one) (i) $\boldsymbol{\beta}$ (ii) $\boldsymbol{\beta^2}$ (iii) $\boldsymbol{\beta^3}$ (iv) $\boldsymbol{\beta^4}$ and[4]

   $$V\left[E(Y_1|\lambda)\right] = V\left[\lambda\right] =$$

   (choose one) (i) $\boldsymbol{\beta}$ (ii) $\boldsymbol{\beta^2}$ (iii) $\boldsymbol{\beta^3}$ (iv) $\boldsymbol{\beta^4}$ so

   $$E[V(Y_1|\lambda)] + V[E(Y_1|\lambda)] =$$

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[1]Recall, expectation of Poisson Y_1 is λ.
[2]Recall, expectation of exponential λ is β.
[3]Recall, variance of Poisson Y_1 is λ and expectation of of exponential λ is β.
[4]Recall, expectation of Poisson Y_1 is λ and variance of exponential λ is β^2.

(choose one) (i) $\boldsymbol{\beta}$ (ii) $\boldsymbol{\beta+\beta^2}$ (iii) $\mathbf{1}+\boldsymbol{\beta^3}$ (iv) $\boldsymbol{\beta^4-\beta}$

Again, it is much easier to calculate $E\left[V(Y_1|\lambda)\right] + V\left[E(Y_1|\lambda)\right]$ than it is to calculate $V(Y_1)$ directly in this case.

(e) *Typical number of defects.* If $\beta = 1$, is it unusual for 3 defects? Since

$$\frac{y_1 - E(y_1)}{\sqrt{V(y_1)}} = \frac{3-\beta}{\sqrt{\beta+\beta^2}} = \frac{3-1}{\sqrt{1+1^2}} =$$

(choose one) (i) **1.41** (ii) **2.50** (iii) **3.57** (iv) **4.66**
it is *not* unusual for 3 defects.

4. *Bayes hierarchical model: number of replies.* Number, Y_1, of replies from a survey has a binomial distribution with parameters n and p. Furthermore, parameter p is a random variable with gamma distribution with parameters α and β.

(a) *Compute* $E(Y_1)$. Conditional expectation[5] of Y_1 is

$$E(Y_1|p) =$$

(choose one) (i) $\boldsymbol{n}$ (ii) $\boldsymbol{np}$ (iii) $\boldsymbol{np(1-p)}$ (iv) $\boldsymbol{np^2}$
so[6]

$$E(Y_1) = E(E(Y_1|p)) = E(np) = nE(p) =$$

(choose one) (i) $\boldsymbol{n\beta}$ (ii) $\boldsymbol{\alpha\beta^2}$ (iii) $\boldsymbol{n\alpha\beta}$ (iv) $\boldsymbol{n\beta^4}$

(b) *Compute* $V(Y_1)$. Since[7] $V(p) = E\left[p^2\right] - \left[E(p)\right]^2$,

$$E\left[p^2\right] = V(p) + \left[E(p)\right]^2 = \alpha\beta^2 + \left[\alpha\beta\right]^2 = \alpha\beta^2(1+\alpha)$$

then

$$E\left[V(Y_1|p)\right] = E\left[np(1-p)\right] = nE\left[p-p^2\right] = n\left(E(p) - E(p^2)\right) =$$

(choose one) (i) $\boldsymbol{\beta}$ (ii) $\boldsymbol{n\alpha\beta^2}$ (iii) $\boldsymbol{n\alpha\beta(\beta-1)}$ (iv) $\boldsymbol{n\alpha\beta(1-\beta-\alpha\beta)}$
and[8]

$$V\left[E(Y_1|p)\right] = V\left[np\right] = n^2V(p) =$$

(choose one) (i) $\boldsymbol{n\beta}$ (ii) $\boldsymbol{n^2\alpha\beta^2}$ (iii) $\boldsymbol{n^3\alpha^2\beta^3}$ (iv) $\boldsymbol{n\beta^4}$ so

$$V(Y_1) = E[V(Y_1|p)] + V[E(Y_1|p)] =$$

(i) $\boldsymbol{\beta}$ (ii) $\boldsymbol{n(\alpha\beta-\alpha\beta^2)}$ (iii) $\boldsymbol{n\alpha\beta^2+n^2\alpha\beta^2}$ (iv) $\boldsymbol{n\alpha\beta(1-\beta-\alpha\beta+n\beta)}$

---

[5]Recall, expectation of binomial Y_1 is np.
[6]Recall, expectation of gamma p is $\alpha\beta$.
[7]Recall, expectation of gamma p is $\alpha\beta$ and variance of of gamma p is $\alpha\beta^2$.
[8]Recall, expectation of binomial Y_1 is np and variance of gamma p is $\alpha\beta^2$.

5. *Prisoner's Escape and Three Doors.* A prisoner is faced with three doors. First door opens to a tunnel to freedom in 4 hours. Second door opens to a tunnel returning prisoner back to prison in 5 hours. Third door opens to a tunnel returning prisoner back to prison in 10 hours. Assume prisoner is equally likely to choose any door. Let Y_1 represent amount of time until prisoner reaches freedom and let Y_2 represent chosen door (1, 2 or 3). What is expected length of time until prisoner reaches freedom, $E(Y_1)$?

   (a) $E(Y_1|Y_2 = 1) =$ (circle one)
   (circle one) **4** / $\mathbf{5} + \boldsymbol{E(Y_1)}$ / $\mathbf{10} + \boldsymbol{E(Y_1)}$

   (b) and $E(Y_1|Y_2 = 2) =$ (circle one)
   (circle one) **4** / $\mathbf{5} + \boldsymbol{E(Y_1)}$ / $\mathbf{10} + \boldsymbol{E(Y_1)}$

   (c) and $E(Y_1|Y_2 = 3) =$ (circle one)
   (circle one) **4** / $\mathbf{5} + \boldsymbol{E(Y_1)}$ / $\mathbf{10} + \boldsymbol{E(Y_1)}$

   (d) So

$$\begin{aligned}
E(Y_1) &= E[E(Y_1|Y_2)] \\
&= \sum_y E(Y_1|Y_2 = y)P(Y_2 = y) \\
&= E(X|Y_2 = 1)P(Y_2 = 1) + E(Y_1|Y_2 = 2)P(Y_2 = 2) + E(Y_1|Y_2 = 3)P(Y_2 = 3) \\
&= \frac{1}{3}[E(Y_1|Y_2 = 1) + E(Y_1|Y_2 = 2) + E(Y_1|Y_2 = 3)]
\end{aligned}$$

   and so $E(Y_1) =$ (circle one) **16** / **17** / **19**

5.12 Summary