5.3 Higher Derivatives, Concavity, and the Second Derivative Test

Notation for *higher* derivatives of $y = f(x)$ include

$$
\begin{aligned}
\text{second derivative:} \quad & f''(x), \quad , \tfrac{d^2y}{dx^2}, \quad D_x^2[f(x)], \\
\text{third derivative:} \quad & f'''(x), \quad , \tfrac{d^3y}{dx^3}, \quad D_x^3[f(x)], \\
\text{fourth and above, } n\text{th, derivative:} \quad & f^{(n)}(x), \quad , \tfrac{d^{(n)}y}{dx^{(n)}}, \quad D_x^{(n)}[f(x)].
\end{aligned}
$$

Second derivative of function can be used to check for both concavity and points of inflection of the graph of function.

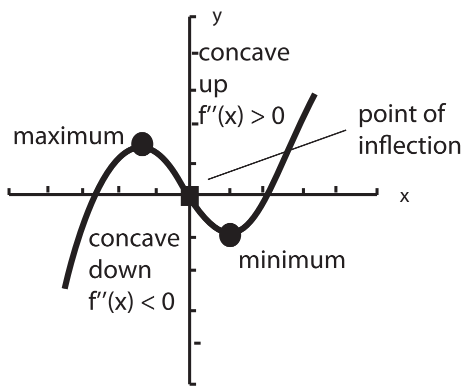

Figure 5.9 (Concave up, concave down and point of inflection)

In particular, if function f has both derivatives f' and f'' for all x in (a, b), the

$$
\begin{aligned}
f(x) \text{ is } \textit{concave up} \quad & \text{if } f''(x) > 0, \quad \text{for all } x \text{ in (a,b)} \\
f(x) \text{ is } \textit{concave down} \quad & \text{if } f''(x) < 0, \text{ for all } x \text{ in (a,b)}
\end{aligned}
$$

At an *inflection point* of function f, either $f''(x) = 0$ or second derivative does not exist (although the reverse is *not* necessarily true). *Second derivative test* is used to check for relative extrema. Let f'' exist on open interval containing c (except maybe c itself) and let $f'(c) = 0$, then

$$
\begin{aligned}
\text{if } f''(c) > 0 \quad & \text{then } f(c) \text{ is } \textit{relative minimum} \\
\text{if } f''(c) < 0 \quad & \text{then } f(c) \text{ is } \textit{relative maximum} \\
\text{if } f''(c) = 0 \text{ or } f''(x) \text{ does not exist} \quad & \text{test gives no information, use first derivative test}
\end{aligned}
$$

Exercise 5.3 (Higher Derivatives, Concavity, and the Second Derivative Test)

1. *Concave up, concave down and inflection points.*

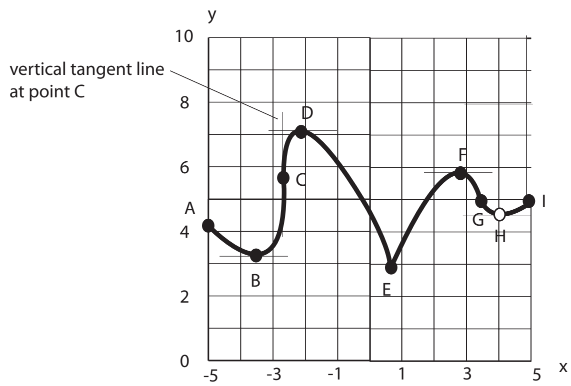

Figure 5.10 (Concave up, concave down and inflection points)

(a) Function is concave *up* between points (choose one or more)
(i) **A to C**
(ii) **C to E**
(iii) **E to F**
(iv) **F to G**
(v) **G to I**
(vi) **none**
because slope *always increases* from left endpoint to right endpoint, *without* "holes"

(b) Function is concave *down* between points (choose one or more)
(i) **A to C**
(ii) **C to E**
(iii) **E to F**
(iv) **F to G**
(v) **G to I**
(vi) **none**
because slope *always decreases* from left endpoint to right endpoint

(c) Inflection point(s) at (choose one or more)
(i) **A** (ii) **B** (iii) **C** (iv) **D** (v) **E**
(vi) **F** (vii) **G** (viii) **H** (ix) **I** (x) **none**
because, at point C, concavity flips from up to down, although f'' does not exist
and, at point G, concavity flips from down to up and $f'' = 0$

2. *More concave up, concave down and inflection points.*

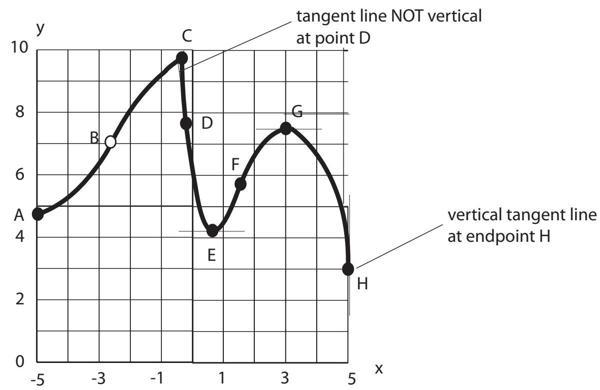

Figure 5.11 (Concave up, concave down and inflection points)

(a) Function is concave *up* between points (choose one or more)
(i) **A to B**
(ii) **B to C**
(iii) **C to F**
(iv) **D to F**
(v) **F to H**
(vi) **none**
because slope *always increases* from left to right endpoints

(b) Function is concave *down* between points (choose one or more)
(i) **A to B**
(ii) **B to C**
(iii) **C to F**
(iv) **D to F**
(v) **F to H**
(vi) **none**
because slope *always decreases* from left endpoint to right endpoint

(c) *One* inflection point at
(i) **A** (ii) **B** (iii) **C** (iv) **D** (v) **E**
(vi) **F** (vii) **G** (viii) **H**
because, at point F, concavity flips from up to down and $f'' = 0$,
but, at point D, concavity *remains* down, so $f'' \neq 0$,
and, at point B, function does not exist, so cannot be an inflection point

3. *Examples of higher derivatives.*

(a) $f(x) = x^2 + 4x - 21$

So $f'(x) = 2x^{2-1} + 4(1)x^{1-1} =$
(i) $\mathbf{2x^2 + 4}$ (ii) $\mathbf{2x - 21}$ (iii) $\mathbf{2x + 4}$

and $f''(x) = 2(1)x^{1-1} =$
(i) $\boldsymbol{2x}$ (ii) $\mathbf{2}$ (iii) $\mathbf{4}$

and so $f''(3) =$
(i) $\mathbf{3}$ (ii) $\mathbf{6}$ (iii) $\mathbf{2}$

and so $f''(-8) =$
(i) $\mathbf{2}$ (ii) $\mathbf{3}$ (iii) $\mathbf{6}$

(b) $f(x) = 2x^3 + 3x^2 - 36x$

So $f'(x) = 2(3)x^{3-1} + 3(2)x^{2-1} - 36(1)x^{1-1} =$
(i) $\boldsymbol{6x+6}$ (ii) $\boldsymbol{6x^3+6x^2-36x}$ (iii) $\boldsymbol{6x^2+6x-36}$

and $f''(x) = 6(2)x^{2-1} + 6(1)x^{1-1} =$
(i) $\boldsymbol{12x+6}$ (ii) $\boldsymbol{12x^2+6x}$ (iii) $\boldsymbol{12+6x}$

and so $f''(3) = 12(3) + 6 =$
(i) $\mathbf{36}$ (ii) $\mathbf{32}$ (iii) $\mathbf{42}$

and so $f''(-8) = 12(-8) + 6 =$
(i) $\mathbf{-45}$ (ii) $\mathbf{-90}$ (iii) $\mathbf{-8}$

and $f'''(x) =$
(i) $\mathbf{12}$ (ii) $\boldsymbol{6x}$ (iii) $\boldsymbol{12x}$

and so $f'''(3) =$
(i) $\mathbf{12}$ (ii) $\mathbf{3}$ (iii) $\mathbf{4}$

and $f^{(4)}(x) =$
(i) $\mathbf{0}$ (ii) $\mathbf{12}$ (iii) $\boldsymbol{12x}$

(c) $f(x) = 3x^2 - 2x^4$

So $f'(x) = 3(2)x^{2-1} - 2(4)x^{4-3} =$
(i) $\boldsymbol{6x-8x^3}$ (ii) $\boldsymbol{6x+2x^2}$ (iii) $\boldsymbol{6x^3+6x^2}$

and $\frac{d^2y}{dx^2} = 6(1)x^{1-1} - 8(3)x^{3-1} =$
(i) $\mathbf{6}$ (ii) $\boldsymbol{6-24x^2}$ (iii) $\boldsymbol{6x}$

and $D_x^3[f(x)] = -24(2)x^{2-1}$
(i) $\boldsymbol{6x}$ (ii) $\mathbf{-12}$ (iii) $\boldsymbol{-48x}$

and $f^{(4)}(x) =$
(i) $\mathbf{12}$ (ii) $\mathbf{48x}$ (iii) $\mathbf{-48}$

(d) $f(x) = \sqrt{3x-3}$

Let $f[g(x)] = (3x-3)^{1/2}$, where $g(x) = 3x-3$, and $f(x) = x^{1/2}$
so $D_x[f(x)] = f'[g(x)]g'(x) = \frac{1}{2}(3x-3)^{\frac{1}{2}-1} \times 3(1)x^{1-1} =$
(i) $\mathbf{\frac{3}{2}(3x-3)^{-\frac{1}{2}}}$ (ii) $\mathbf{\frac{3}{2}(3x-3)^{\frac{1}{2}}}$ (iii) $\mathbf{\frac{3}{2}(3x-3)^{-\frac{3}{2}}}$

Let $f[g(x)] = \frac{3}{2}(3x-3)^{-\frac{1}{2}}$, where $g(x) = 3x-3$, and $f(x) = \frac{3}{2}x^{-\frac{1}{2}}$
so $D_x^2[f(x)] = f'[g(x)]g'(x) = \left(\frac{3}{2}\right)\left(\left(-\frac{1}{2}\right)(3x-3)^{-\frac{1}{2}-1} \times 3 =$
(i) $\mathbf{\frac{9}{4}(3x-3)^{-\frac{3}{2}}}$ (ii) $\mathbf{-\frac{3}{4}(3x-3)^{-\frac{3}{2}}}$ (iii) $\mathbf{-\frac{9}{4}(3x-3)^{-\frac{3}{2}}}$

Let $f[g(x)] = -\frac{9}{4}(3x-3)^{-\frac{3}{2}}$, where $g(x) = 3x-3$, and $f(x) = -\frac{9}{4}x^{-\frac{3}{2}}$
so $D_x^3[f(x)] = f'[g(x)]g'(x) = \left(-\frac{9}{4}\right)\left(\left(-\frac{3}{2}\right)(3x-3)^{-\frac{3}{2}-1} \times 3 =$
(i) $\mathbf{\frac{27}{8}(3x-3)^{-\frac{3}{2}}}$ (ii) $\mathbf{\frac{81}{8}(3x-3)^{-\frac{5}{2}}}$ (iii) $\mathbf{-\frac{81}{8}(3x-3)^{-\frac{5}{2}}}$

(e) $f(x) = 5e^{2x}$

Let $f[g(x)] = 5e^{2x}$, where $g(x) = 2x$, and $f(x) = 5e^{x}$
so $D_x[f(x)] = f'[g(x)]g'(x) = 5e^{2x} \times 2(1)x^{1-1} =$
(i) $\mathbf{5e^{2x}}$ (ii) $\mathbf{5e^{2}}$ (iii) $\mathbf{10e^{2x}}$

Let $f[g(x)] = 10e^{2x}$, where $g(x) = 2x$, and $f(x) = 10e^{x}$
so $D_x^2[f(x)] = f'[g(x)]g'(x) = 10e^{2x} \times 2(1)x^{1-1} =$
(i) $\mathbf{20e^{2x}}$ (ii) $\mathbf{10e^{2x}}$ (iii) $\mathbf{20e^{x}}$

Let $f[g(x)] = 20e^{2x}$, where $g(x) = 2x$, and $f(x) = 20e^{x}$
so $D_x^3[f(x)] = f'[g(x)]g'(x) = 20e^{2x} \times 2(1)x^{1-1} =$
(i) $\mathbf{10e^{2x}}$ (ii) $\mathbf{40e^{2x}}$ (iii) $\mathbf{5e^{2x}}$

(f) $f(x) = \tan 2x$. Recall, derivatives of trigonometric functions include:

$$D_x[\sin x] = \cos x \qquad D_x[\csc x] = -\cot x \csc x$$
$$D_x[\cos x] = -\sin x \qquad D_x[\sec x] = \tan x \sec x$$
$$D_x[\tan x] = \sec^2 x \qquad D_x[\cot x] = -\csc^2 x$$

So, let $f[g(x)] = \tan 2x$, where $g(x) = 2x$, and $f(x) = \tan x$
so $D_x[f(x)] = f'[g(x)]g'(x) = \sec^2 2x \times 2(1)x^{1-1} =$

(i) $\mathbf{2 \sec 2x}$ (ii) $\mathbf{2 \sec^2 2x}$ (iii) $\mathbf{2 \sec x}$

Let $f[g(x)] = 2\sec^2 2x$, where $g(x) = \sec 2x$, and $f(x) = 2x^2$
so $D_x^2[f(x)] = f'[g(x)]g'(x) = 2(2)\sec^{2-1} 2x \times 2\tan 2x \sec 2x =$
(i) $\mathbf{8 \tan 2x \sec^2 2x}$ (ii) $\mathbf{6 \tan 2x \sec^2 2x}$ (iii) $\mathbf{8 \tan 2x \sec 2x}$

4. *Application: throwing a ball.* A ball is thrown upwards with an initial velocity of 32 feet per second and from an initial height of 150 feet. A function relating height, $f(t)$, to time, f, when throwing this ball is:

$$f(t) = -12t^2 + 32t + 150$$

Find velocity, $v(t) = f'(t)$, acceleration, $a(t) = v'(t) = f''(t)$, at $t = 5$ seconds.

So $f'(t) = -12(2)t^{2-1} + 32(1)t^{1-1} =$
(i) $\mathbf{24t + 32}$ (ii) $\mathbf{-24t}$ (iii) $\mathbf{-24t + 32}$

and so $f'(5) = -24(5) + 32 =$
(i) **88** (ii) **5** (iii) **−88** feet per second

and $f''(t) = -24(1)t^{1-1} =$
(i) **−24** (ii) **24** (iii) **−88**

and so $f''(5) =$
(i) **24** (ii) **−24** (iii) **−88** feet per second^2

True / **False**.
Acceleration is first derivative of velocity or second derivative of height.

5. *Application: index of absolute risk aversion.* Index of absolute risk aversion is

$$I(M) = \frac{-U''(M)}{U'(M)},$$

where M is quantity of a commodity owned by a consumer and $U(M)$ is utility (fulfillment) a consumer derives from the quantity M of the commodity. Find $I(M)$ if $U(M) = M^3$.

Since $U'(M) = 3M^{3-1} =$ (i) $\mathbf{M^2}$ (ii) $\mathbf{3M}$ (iii) $\mathbf{3M^2}$

and also $U''(M) = 3(2)M^{2-1} =$ (i) $\mathbf{6M^2}$ (ii) $\mathbf{6M}$ (iii) **6** and so

$$I(M) = \frac{-U''(M)}{U'(M)} = \frac{-6M}{3M^2} =$$

(i) $\boldsymbol{\frac{-1}{M}}$ (ii) $\boldsymbol{\frac{-2}{M}}$ (iii) $\boldsymbol{\frac{2}{M}}$

6. *Second derivative test:* $f(x) = 5x - 4$ *revisited*

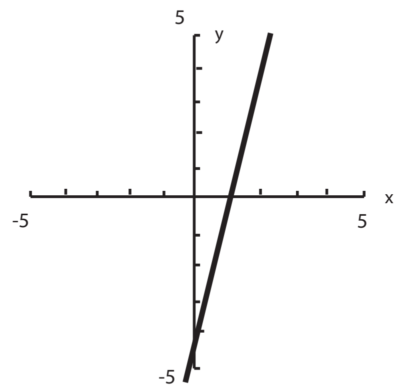

Figure 5.12 (Second derivative test: $f(x) = 5x - 4$)

GRAPH using $Y_1 = 5x - 4$, with WINDOW -5 5 1 -5 5 1 1

(a) *Critical numbers, points of inflection and intervals.*
Recall, since

$$f'(x) = 5(1)x^{1-1} = 5,$$

there (i) **are** (ii) **are no** *critical numbers*,
because $f'(x) = 5$ can *never* equal zero

and also since

$$f''(x) = 0,$$

there (i) **are** (ii) **are no** *points of inflection*,
because a point of inflection is that point on function where concavity is defined to *change*; but $f''(x) = 0$ everywhere with no concave up or concave down sections of function anywhere, so certainly there could not be any points where concavity *changes*

and so there is only *one* interval to investigate
(i) $\boldsymbol{(2, \infty)}$ (ii) $\boldsymbol{(-2, 2)}$ (iii) $\boldsymbol{(-\infty, \infty)}$

so summarizing:

interval	$(-\infty, \infty)$
critical value	none
$f''(x) = 0$	$f''(x) = 0$
sign of $f''(x)$	zero

(b) *Second derivative test.*
It (i) **is** (ii) **is not** possible to perform second derivative test
because there are no critical numbers to test

7. *Second derivative test:* $f(x) = x^2 + 4x - 21$ *revisited*

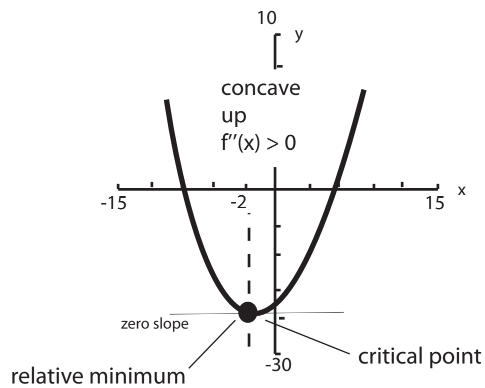

Figure 5.13 (Second derivative test: $f(x) = x^2 + 4x - 21$)

GRAPH using $Y_2 = x^2 + 4x - 21$, with WINDOW -15 15 1 -30 10 1 1

(a) *Critical numbers, points of inflection and intervals.*
Recall, since

$$f'(x) = 2x^{2-1} + 4(1)x^{1-1} = 2x + 4 = 0,$$

there is a critical number at
$c = -\frac{4}{2} =$ (i) $\mathbf{-2}$ (ii) $\mathbf{2}$ (iii) $\mathbf{0}$

and also since

$$f''(x) = 2(1)x^{1-1} = 2 > 0,$$

there (i) **is** (ii) **is no** *point of inflection*,
there is no point of inflection because $f''(x) = 2$ can *never* equal zero

and so there is *one* interval to investigate
(i) $\mathbf{(2, \infty)}$ (ii) $\mathbf{(-2, 2)}$ (iii) $\mathbf{(-\infty, \infty)}$

and at critical number $c = -2$, $f''(-2) = 2$ is
(i) **positive** (ii) **negative** (iii) **zero**
so summarizing:

interval	$(-\infty, \infty)$
critical value	$c = -2$
$f''(x) = 2$	$f''(-2) = 2$
sign of $f''(x)$	positive

(b) *Second derivative test.*
At critical number $c = -2$, sign of derivative $f''(x)$
(i) **positive, so concave up**
(ii) **negative, so concave down**
(iii) **zero**
and so, according to second derivative test, there is
(i) **a relative minimum**
(ii) **a relative maximum**
(iii) **not enough information to decide, use first derivative test**
at critical number $c = -2$,
and since $f(-2) = (-2)^2 + 4(-2) - 21 = -25$,
at critical *point* $(c, f(c)) = (-2, -25)$.

8. *Second derivative test:* $f(x) = 2x^3 + 3x^2 - 36x$ *revisited*

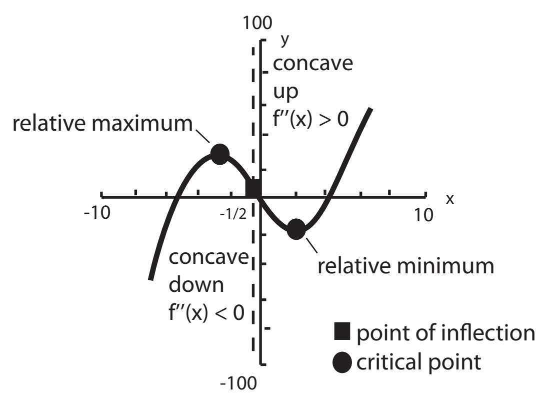

Figure 5.14 (Second derivative test: $f(x) = 2x^3 + 3x^2 - 36x$)

GRAPH using $Y_3 = 2x^3 + 3x^2 - 36x$, with WINDOW -10 10 1 -100 100 1 1

(a) *Critical numbers, points of inflection and intervals.*
Recall, since

$$f'(x) = 2(3)x^{3-1} + 3(2)x^{2-1} - 36(1)x^{1-1} = 6x^2 + 6x + 36 = 6(x+3)(x-2) = 0,$$

there are *two* critical numbers at
$c =$ (i) $\mathbf{-3}$ (ii) **2** (iii) **6**

and also since

$$f''(x) = 6(2)x^{2-1} + 6(1)x^{1-1} = 12x + 6 = 0,$$

there is *one* point of inflection at
$x =$ (i) $-\frac{1}{2}$ (ii) **0** (iii) $\frac{1}{2}$

and so there are *two* intervals to investigate
(i) $\left(-\infty, -\frac{1}{2}\right)$ (ii) $\left(-\infty, \frac{1}{2}\right)$ (iii) $\left(-\frac{1}{2}, \infty\right)$

and at critical number $c = -3$, $f''(-1) = 12(-3) + 6 = -30$ is
(i) **positive** (ii) **negative** (iii) **zero**

and at critical number $c = 2$, $f''(0) = 12(2) + 6 = 30$ is
(i) **positive** (ii) **negative** (iii) **zero**

so summarizing:

interval	$\left(-\infty, -\frac{1}{2}\right)$	$\left(-\frac{1}{2}, \infty\right)$
critical value	$c = -3$	$c = 2$
$f''(x) = 12x + 6$	$f''(-3) = -30$	$f''(2) = 30$
sign of $f''(x)$	negative	positive

Notice interval here for second derivative bounded by *point of inflection* (bounds concavity) and critical point is *inside* this interval, whereas for first derivative test, interval bounded by critical point (bounds in/decreasing function) and so critical point is on "edge" of interval.

(b) *Second derivative test.*
At critical number $c = -3$, sign of derivative $f''(x)$
(i) **positive, so concave up**
(ii) **negative, so concave down**
(iii) **zero**
and so, according to second derivative test, there is
(i) **a relative minimum**
(ii) **a relative maximum**
(iii) **not enough information to decide, use first derivative test**
at critical number $c = -3$,
and since $f(-3) = 2(-3)^3 + 3(-3)^2 - 36(-3) = 81$,
at critical *point* $(c, f(c)) = (-3, 81)$.

At critical number $c = 2$, sign of derivative $f''(x)$
(i) **positive, so concave up**
(ii) **negative, so concave down**
(iii) **zero**
and so, according to second derivative rule, there is
(i) **a relative minimum**
(ii) **a relative maximum**
(iii) **not enough information to decide, use first derivative test**

at critical number $c = 2$,
and since $f(2) = 2(2)^3 + 3(2)^2 - 36(2) = -44$,
at critical *point* $(c, f(c)) = (2, -44)$.

9. *Second derivative test:* $f(x) = 3x^3 - 2x^4$ *revisited*

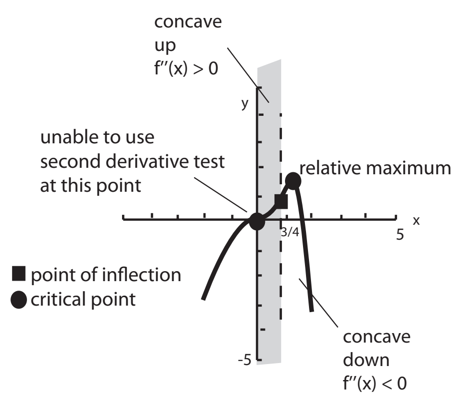

Figure 5.15 (Second derivative test: $f(x) = 3x^3 - 2x^4$)

GRAPH using $Y_4 = 3x^3 - 2x^4$, with WINDOW -3 3 1 -3 2 1 1

(a) *Critical numbers, points of inflection and intervals.*
Recall, since

$$f'(x) = 3(3)x^{3-1} - 2(4)x^{4-1} = 9x^2 - 8x^3 = 9x^2\left(1 - \frac{8}{9}x\right) = 0,$$

there are *two* critical numbers at
$c =$ (i) $\mathbf{-\frac{9}{8}}$ (ii) $\mathbf{0}$ (iii) $\mathbf{\frac{9}{8}}$

and also since

$$f''(x) = 9(2)x^{2-1} - 8(3)x^{3-1} = 18x - 24x^2 = 18x\left(1 - \frac{24}{18}x\right) = 0,$$

there are *two* points of inflection at
$x =$ (i) $\mathbf{-\frac{3}{4}}$ (ii) $\mathbf{0}$ (iii) $\mathbf{\frac{3}{4}}$
because $18x = 0$ when $x = 0$ and $1 - \frac{24}{18}x = 1 - \frac{4}{3}x = 0$ when $x = \frac{3}{4}$

and so there are *three* intervals to investigate
(i) $\mathbf{(-\infty, 0)}$ (ii) $\mathbf{\left(-\infty, \frac{3}{4}\right)}$ (iii) $\mathbf{\left(0, \frac{3}{4}\right)}$ (iv) $\mathbf{\left(\frac{3}{4}, \infty\right)}$

and at critical number $c = 0$, $f''(0) = 18(0) - 24(0)^2 = 0$ is
(i) **positive** (ii) **negative** (iii) **zero**

and at $c = \frac{9}{8}$, $f''\left(\frac{9}{8}\right) = 18\left(\frac{9}{8}\right) - 24\left(\frac{9}{8}\right)^2 = -\frac{27}{4}$ is

(i) **positive** (ii) **negative** (iii) **zero**

so summarizing:

interval	$(-\infty, 0)$	$\left(0, \frac{3}{4}\right)$	$\left(\frac{3}{4}, \infty\right)$
critical value	$c = 0$	$c = 0$	$c = \frac{9}{8}$
$f''(x) = 18x - 24x^2$	$f''(0) = 0$	$f''(0) = 0$	$f''\left(\frac{9}{8}\right) = -\frac{27}{4}$
sign of $f''(x)$	zero	zero	negative

(b) *Second derivative test.*
At critical number $c = 0$, sign of derivative $f''(x)$
(i) **positive, so concave up**
(ii) **negative, so concave down**
(iii) **zero**
and so, according to second derivative test, there is
(i) **a relative minimum**
(ii) **a relative maximum**
(iii) **not enough information to decide, use first derivative test**
at critical number $c = 0$,

At critical number $c = \frac{3}{4}$, sign of derivative $f''(x)$
(i) **positive, so concave up**
(ii) **negative, so concave down**
(iii) **zero**
and so, according to second derivative rule, there is
(i) **a relative minimum**
(ii) **a relative maximum**
(iii) **not enough information to decide, use first derivative test**
at critical number $c = \frac{3}{4}$,
and since $f\left(\frac{3}{4}\right) = 3\left(\frac{3}{4}\right)^3 - 2\left(\frac{3}{4}\right)^4 = \frac{81}{128}$,
at critical *point* $(c, f(c)) = \left(\frac{3}{4}, \frac{81}{128}\right)$.

10. *Application of second derivative test: throwing a ball.* A ball is thrown upwards with an initial velocity of 32 feet per second and from an initial height of 150 feet. A function relating height, $f(t)$, to time, f, when throwing this ball is:

$$f(t) = -12t^2 + 32t + 150$$

Find maximum height, $f(t)$, and time, t, ball reaches maximum height.
GRAPH using $Y_5 = -12x^2 + 32x + 150$, with WINDOW 0 6 1 0 250 1 1

(a) *Critical numbers, points of inflection and intervals.*
Recall, since

$$f'(x) = -12(2)t^{2-1} + 32(1)t^{1-1} = -24t + 32 = 0,$$

there is a *critical number* at
$c = -\frac{32}{-24} =$ (i) $\mathbf{\frac{4}{3}}$ (ii) $\mathbf{-\frac{4}{3}}$ (iii) $\mathbf{\frac{3}{4}}$

and also since

$$f''(x) = -24(1)t^{1-1} = -24 < 0,$$

there (i) **is** (ii) **is no** *point of inflection*,
there is no point of inflection because $f''(x) = -24$ can *never* equal zero

and so there is *one* interval to investigate
(i) $\mathbf{(2, \infty)}$ (ii) $\mathbf{(-2, 2)}$ (iii) $\mathbf{(-\infty, \infty)}$

and at critical number $c = \frac{4}{3}$, $f''\left(\frac{4}{3}\right) = -24$ is
(i) **positive** (ii) **negative** (iii) **zero**
so summarizing:

interval	$(-\infty, \infty)$
critical value	$c = \frac{4}{3}$
$f''(x) = -24$	$f''\left(\frac{4}{3}\right) = -24$
sign of $f''(x)$	negative

(b) *Second derivative test.*
At critical number $c = \frac{4}{3}$, sign of derivative $f''(x)$
(i) **positive, so concave up**
(ii) **negative, so concave down**
(iii) **zero**
and so, according to second derivative test, there is
(i) **a relative minimum**
(ii) **a relative maximum**
(iii) **not enough information to decide, use first derivative test**
at critical number $c = \frac{4}{3}$,
and since $f\left(\frac{4}{3}\right) = -12\left(\frac{4}{3}\right)^2 + 32\left(\frac{4}{3}\right) + 150 = \frac{514}{3}$,
at critical *point* $(c, f(c)) = \left(\frac{4}{3}, \frac{514}{3}\right)$.

(c) *Results*
In other words, ball reaches maximum height of (i) $\mathbf{\frac{4}{3}}$ (ii) $\mathbf{\frac{3}{514}}$ (iii) $\mathbf{\frac{514}{3}}$
at time (i) $\mathbf{\frac{4}{3}}$ (ii) $\mathbf{-\frac{4}{3}}$ (iii) $\mathbf{\frac{3}{4}}$

5.4 Curve Sketching

We combine a number of previous ideas to sketch a graph of a function. First, determine the following properties of the function:

1. *domain*, note restrictions

   cannot divide by 0, or take square root of negative number, or take logarithm of 0 or of a negative number

2. *y-intercept, x-intercept*, if they exist

   y-intercept: let $x = 0$ in $f(x)$, x-intercept: solve $f(x) = 0$ for x

3. *vertical, horizontal, oblique asymptotes*

   vertical asymptote when denominator 0, horizontal asymptote when $x \to \infty$, or $x \to -\infty$

4. *symmetry*

   symmetric about y-axis if $f(-x) - f(x)$; symmetric about origin if $f(-x) = -f(x)$

5. *first derivative test*

   note critical points (when $f'(x) = 0$), relative extrema, in/decreasing sections of function

6. *points of inflection, intervals and concavity*

   note inflection points (when $f''(x) = 0$), concave up/down

Then, plot all points and connect them with a smooth curve, taking into account asymptotes, concavity and in/decreasing sections of function. Check result with a graphing calculator. Commonly recurring shapes are given in the figure; for example, an increasing concave up function is given in upper left corner.

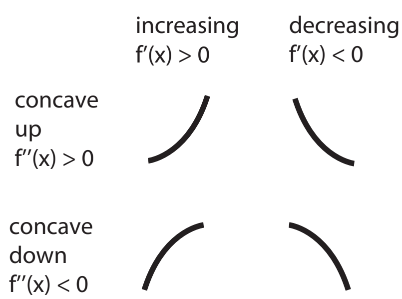

Figure 5.16 (Some important function shapes)

Exercise 5.4 (Curve Sketching)

1. *Increasing, decreasing and concavity together:* $f(x) = 2x^3 + 3x^2 - 36x$ *revisited.*

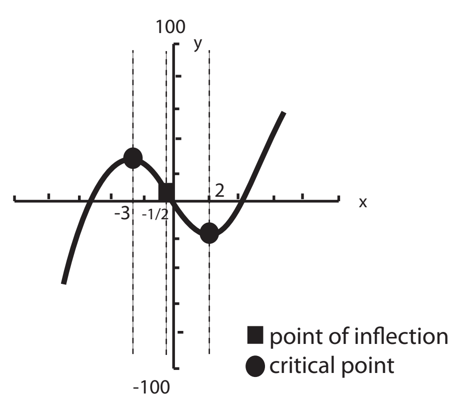

Figure 5.17 (Increasing, decreasing and concavity together)

(a) On interval $(-\infty, -3)$, $f(x) = 2x^3 + 3x^2 - 36x$ is
   i. increasing and concave up
   ii. increasing and concave down
   iii. decreasing and concave up
   iv. decreasing and concave down

(b) On interval $\left(-3, -\frac{1}{2}\right)$, $f(x) = 2x^3 + 3x^2 - 36x$ is
   i. increasing and concave up
   ii. increasing and concave down
   iii. decreasing and concave up
   iv. decreasing and concave down

(c) On interval $\left(-\frac{1}{2}, 2\right)$, $f(x) = 2x^3 + 3x^2 - 36x$ is
   i. increasing and concave up
   ii. increasing and concave down
   iii. decreasing and concave up
   iv. decreasing and concave down

(d) On interval $(2, \infty)$, $f(x) = 2x^3 + 3x^2 - 36x$ is
   i. increasing and concave up
   ii. increasing and concave down
   iii. decreasing and concave up
   iv. decreasing and concave down

2. $f(x) = x^2 + 4x - 21$ *revisited*

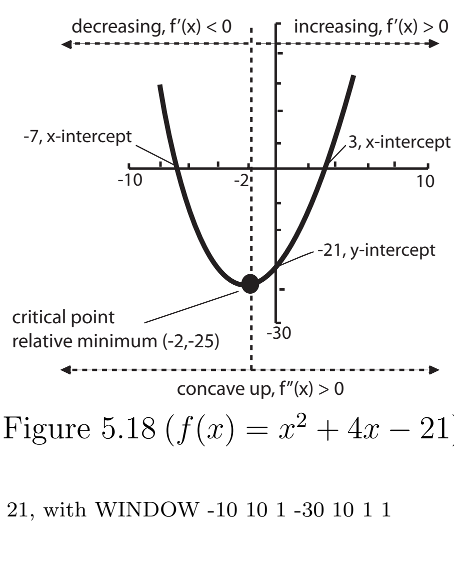

Figure 5.18 ($f(x) = x^2 + 4x - 21$)

GRAPH using $Y_2 = x^2 + 4x - 21$, with WINDOW -10 10 1 -30 10 1 1

(a) *domain*
(i) $(-\infty, \infty)$ (ii) $(-\infty, \infty), x \neq 0$ (iii) $(-\infty, \infty), x \neq -2$

(b) *intercepts*, if they exist

y-intercept
when $x = 0$, $f(0) = (0)^2 + 4(0) - 21 =$
(i) -21 (ii) 21 (iii) -23

x-intercept(s)
$f(x) = x^2 + 4x - 21 = (x-3)(x+7) = 0$ when $x =$
(i) 3 (ii) -3 (iii) -7

(c) *asymptotes*

vertical
$f(x) = x^2 + 4x - 21$ (i) **does** (ii) **does not** have any vertical asymptotes
because $f(x)$, a polynomial, does not have a denominator (and so cannot be divided by 0, causing a vertical asymptote)

horizontal
$f(x) = x^2 + 4x - 21$ (i) **does** (ii) **does not** have horizontal asymptotes
because $\lim_{x \to \infty} (x^2 + 4x - 21) =$ (i) ∞ (ii) $-\infty$ (iii) 0
and $\lim_{x \to -\infty} (x^2 + 4x - 21) =$ (i) ∞ (ii) $-\infty$ (iii) 0

oblique
$f(x) = x^2 + 4x - 21$ (i) **does** (ii) **does not** have any oblique asymptotes
because $f(x)$ cannot be rewritten in the form $g(x) + \frac{1}{ax+b}$

(d) *symmetry*

about y-axis
$f(x)$ (i) **is** (ii) **is not** symmetric about y-axis
because $f(-x) = (-x)^2 + 4(-x) - 21 \neq x^2 + 4x - 21 = f(x)$

about origin
$f(x)$ (i) **is** (ii) **is not** symmetric about origin
because $f(-x) = (-x)^2 + 4(-x) - 21 \neq -x^2 - 4x + 21 = -f(x)$

(e) *first derivative test*

i. *critical numbers and intervals.*
Recall, since

$$f'(x) = 2x + 4 = 0,$$

there is a *critical number* at
$c = -\frac{4}{2} =$ (i) -2 (ii) 2 (iii) 0

and so there are *two* intervals to investigate
(i) $\mathbf{(-\infty, -2)}$ (ii) $\mathbf{(-2, 2)}$ (iii) $\mathbf{(-2, \infty)}$
with *two* possible *test values* (in each interval) to check, say:
$x =$ (i) $\mathbf{-3}$ (ii) $\mathbf{-2}$ (iii) $\mathbf{0}$

and since $f'(-3) = 2(-3) + 4 = -2$ is negative,
function $f(x)$ (i) **increases** (ii) **decreases** over $(-\infty, -2)$ interval

and $f'(0) = 2(0) + 4 = 4$ is positive,
function $f(x)$ (i) **increases** (ii) **decreases** over $(-2, \infty)$ interval
so summarizing:

interval	$(-\infty, -2)$	$(-2, \infty)$
test value	$x = -3$	$x = -1$
$f'(x) = 2x + 4$	$f'(-3) = -2$	$f'(-1) = 2$
sign of $f'(x)$	negative	positive

ii. *first derivative rule.*
At critical number $c = -2$, sign of derivative $f'(x)$ goes from
(i) **negative to positive**
(ii) **positive to negative**
(iii) **negative to negative**
(iv) **positive to positive**
and so, according to first derivative rule, there is
(i) **a relative minimum**
(ii) **a relative maximum**
(iii) **not a relative extremum**
at critical number $c = -2$,
and since $f(-2) = (-2)^2 + 4(-2) - 21 = -25$,
at critical *point* $(c, f(c)) = (-2, -25)$.

(f) *points of inflection, intervals and concavity*
Recall, since

$$f'(x) = 2x^{2-1} + 4(1)x^{1-1} = 2x + 4 = 0,$$

there is a critical number at
$c = -\frac{4}{2} =$ (i) $\mathbf{-2}$ (ii) $\mathbf{2}$ (iii) $\mathbf{0}$

and also since

$$f''(x) = 2(1)x^{1-1} = 2 > 0,$$

there (i) **is** (ii) **is no** *point of inflection*,
there is no point of inflection because $f''(x) = 2$ can *never* equal zero

and so there is *one* interval to investigate
(i) $\mathbf{(2, \infty)}$ (ii) $\mathbf{(-2, 2)}$ (iii) $\mathbf{(-\infty, \infty)}$

and at test number 0, say, $f''(0) = 2$ is positive
so $f(x)$ is concave (i) **up** (ii) **down** over $(-\infty, \infty)$

so summarizing

interval	$(-\infty, \infty)$
test number	0
$f''(x) = 2$	$f''(0) = 2$
sign of $f''(x)$	positive

(g) *Graph summary.* Combining information from above:

interval	$(-\infty, -2)$	$(-2, \infty)$
sign of $f'(x)$	$-$	$+$
sign of $f''(x)$	$+$	$+$
increasing/decreasing?	decreasing	increasing
concave up/down?	upward	upward

3. $f(x) = \frac{2x^2+2}{x^2+5}$ *revisited*

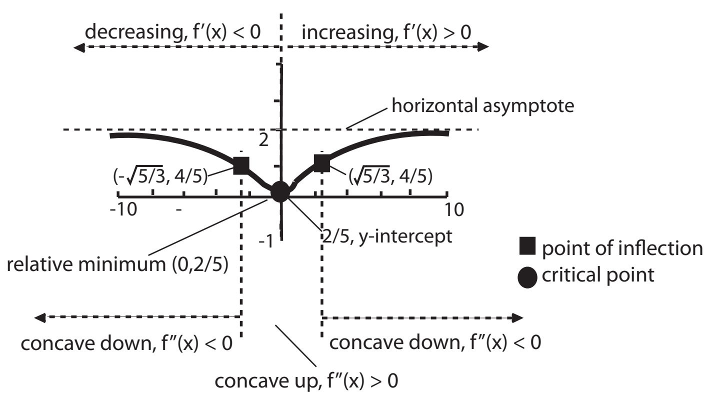

Figure 5.19 ($f(x) = \frac{2x^2+2}{x^2+5}$)

GRAPH using $Y_2 = \frac{2x^2+2}{x^2+5}$, with WINDOW -10 10 1 -1 3 1 1

(a) *domain*
(i) $\boldsymbol{(-\infty, \infty)}$ (ii) $\boldsymbol{(-\infty, \infty), x \neq 0}$ (iii) $\boldsymbol{(-\infty, \infty), x \neq -2}$
because denominator of $f(x) = \frac{2x^2+2}{x^2+5}$; namely, $x^2 + 5$, can never be 0 and so interrupt the domain with a vertical asymptote

(b) *intercepts*, if they exist

y-intercept
when $x = 0$, $f(0) = \frac{2(0)^2+2}{(0)^2+5} =$ (i) $\boldsymbol{\frac{2}{5}}$ (ii) $\boldsymbol{\frac{3}{5}}$ (iii) $\boldsymbol{-\frac{2}{5}}$

x-intercept(s)
$f(x) = \frac{2x^2+2}{x^2+5}$ (i) **does** (ii) **does not** have an x-intercept
because numerator of $f(x)$; specifically, $2x^2+2$ can *never* be zero and so an x-intercept is not possible

(c) *asymptotes*

vertical
$f(x) = \frac{2x^2+2}{x^2+5}$ (i) **does** (ii) **does not** have any vertical asymptotes
because denominator of $f(x)$, x^2+5, can never be 0 and so a vertical asymptote is not possible

horizontal
$f(x) = \frac{2x^2+2}{x^2+5}$ (i) **does** (ii) **does not** have a horizontal asymptote because

i. *limit at positive infinity*

$$\lim_{x\to\infty} \frac{2x^2+2}{x^2+5} = \lim_{x\to\infty} \frac{\frac{2x^2}{x^2}+\frac{2}{x^2}}{\frac{x^2}{x^2}+\frac{5}{x^2}} = \lim_{x\to\infty} \frac{2+\frac{2}{x^2}}{1+\frac{5}{x^2}} =$$

(i) **0** (ii) **1** (iii) **2**.

ii. *limit at negative infinity*

$$\lim_{x\to-\infty} \frac{2x^2+2}{x^2+5} = \lim_{x\to-\infty} \frac{\frac{2x^2}{x^2}+\frac{2}{x^2}}{\frac{x^2}{x^2}+\frac{5}{x^2}} = \lim_{x\to-\infty} \frac{2+\frac{2}{x^2}}{1+\frac{5}{x^2}} =$$

(i) **0** (ii) **1** (iii) **2**.

iii. so horizontal asymptote at (i) $\boldsymbol{y=0}$ (ii) $\boldsymbol{y=1}$ (iii) $\boldsymbol{y=2}$

oblique
$f(x) = \frac{2x^2+2}{x^2+5}$ (i) **does** (ii) **does not** have any oblique asymptotes
because $f(x)$ cannot be rewritten in the form $g(x) + \frac{1}{ax+b}$

(d) *symmetry*

about y-axis
$f(x)$ (i) **is** (ii) **is not** symmetric about y-axis
because $f(-x) = \frac{2(-x)^2+2}{(-x)^2+5} = \frac{2x^2+2}{x^2+5} = f(x)$

about origin
$f(x)$ (i) **is** (ii) **is not** symmetric about origin
because $f(-x) = \frac{2(-x)^2+2}{(-x)^2+5} \neq -\frac{2x^2+2}{x^2+5} = -f(x)$

(e) *first derivative test*

i. *critical numbers and intervals.*
Let $u(x) = 2x^2+2$ and $v(x) = x^2+5$.

then, $u'(x) = 2(2)x^{2-1} = 4x$ and $v'(x) = 2x^{2-1} = 2x$
and so $v(x)u'(x) = (x^2 + 5)(4x)$ and $u(x)v'(x) = (2x^2 + 2)(2x)$
and so since

$$f'(x) = \frac{v(x) \cdot u'(x) - u(x) \cdot v'(x)}{[v(x)]^2} = \frac{(x^2+5)(4x) - (2x^2+2)(2x)}{[x^2+5]^2} = \frac{16x}{x^4 + 10x^2 + 25} = 0$$

there is a *critical number* at
$c =$ (i) $\mathbf{-2}$ (ii) $\mathbf{2}$ (iii) $\mathbf{0}$
because $16x = 0$ when $x = 0$
and so there are *two* intervals to investigate
(i) $\mathbf{(-\infty, 0)}$ (ii) $\mathbf{(-2, 2)}$ (iii) $\mathbf{(0, \infty)}$
with *two* possible *test values* (in each interval) to check, say:
$x =$ (i) $\mathbf{-1}$ (ii) $\mathbf{0}$ (iii) $\mathbf{1}$

and since $f'(-1) = \frac{16(-1)}{(-1)^4+10(-1)^2+25} = -\frac{4}{9}$ is negative,
function $f(x)$ (i) **increases** (ii) **decreases** over $(-\infty, 0)$ interval

and $f'(1) = \frac{16(1)}{(1)^4+10(1)^2+25} = \frac{4}{9}$ is positive,
function $f(x)$ (i) **increases** (ii) **decreases** over $(0, \infty)$ interval
so summarizing:

interval	$(-\infty, 0)$	$(0, \infty)$
test value	$x = -1$	$x = 1$
$f'(x) = \frac{16x}{x^4+10x^2+25}$	$f'(-1) = -\frac{4}{9}$	$f'(1) = \frac{4}{9}$
sign of $f'(x)$	negative	positive

ii. *first derivative test.*
At critical number $c = 0$, sign of derivative $f'(x)$ goes from
(i) **negative to positive**
(ii) **positive to negative**
(iii) **negative to negative**
(iv) **positive to positive**
and so, according to first derivative rule, there is
(i) **a relative minimum**
(ii) **a relative maximum**
(iii) **not a relative extremum**
at critical number $c = 0$,
and since $f(0) = \frac{2(0)^2+2}{(0)^2+5} = \frac{2}{5}$,
at critical *point* $(c, f(c)) = \left(0, \frac{2}{5}\right)$.

(f) *points of inflection, intervals and concavity*
Recall, since

$$f'(x) = \frac{16x}{x^4 + 10x^2 + 25} = 0$$

there is a *critical number* at
$c =$ (i) $\mathbf{-2}$ (ii) $\mathbf{2}$ (iii) $\mathbf{0}$

and also since
Let $u(x) = 16x$ and $v(x) = (x^2+5)^2 = x^4 + 10x^2 + 25$.
then, $u'(x) = 16$ and $v'(x) = 4x^3 + 20x$
so $v(x)u'(x) = (x^4 + 10x^2 + 25)(16)$ and $u(x)v'(x) = (16x)(4x^3 + 20x)$
and so since

$$f''(x) = \frac{v(x) \cdot u'(x) - u(x) \cdot v'(x)}{[v(x)]^2} = \frac{(x^2+5)^2(16) - (16x)(4x^3+20x)}{[x^2+5]^4} = \frac{16(5-3x^2)}{[x^2+5]^3} = 0$$

there are *two* points of inflection at
$x =$ (i) $\mathbf{-\sqrt{\frac{5}{3}}}$ (ii) $\mathbf{0}$ (iii) $\mathbf{\sqrt{\frac{5}{3}}}$
because $5 - 3x^2 = 0$ when $x^2 = \frac{5}{3}$ or $x = \pm\sqrt{\frac{5}{3}}$

and so there are *three* intervals to investigate
(i) $\mathbf{(-\infty, 0)}$ (ii) $\mathbf{\left(-\infty, -\sqrt{\frac{5}{3}}\right)}$ (iii) $\mathbf{\left(-\sqrt{\frac{5}{3}}, \sqrt{\frac{5}{3}}\right)}$ (iv) $\mathbf{\left(\sqrt{\frac{5}{3}}, \infty\right)}$
and *three* tests numbers, say (i) $\mathbf{-2}$ (ii) $\mathbf{-\sqrt{\frac{5}{3}}}$ (iii) $\mathbf{0}$ (iv) $\mathbf{\sqrt{\frac{5}{3}}}$ (v) $\mathbf{2}$

and at test number -1, $f''(-2) = \frac{16(5-3(-2)^2)}{[(-2)^2+5]^3} = -\frac{112}{729}$ is negative,
so $f(x)$ is concave (i) **up** (ii) **down** over $\left(-\infty, -\sqrt{\frac{5}{3}}\right)$

and at test number 0, $f''(0) = \frac{16(5-3(0)^2)}{[(0)^2+5]^3} = \frac{16}{25}$ is positive,
so $f(x)$ is concave (i) **up** (ii) **down** over $\left(-\sqrt{\frac{5}{3}}, \sqrt{\frac{5}{3}}\right)$

and at test number 1, $f''(2) = \frac{16(5-3(2)^2)}{[(2)^2+5]^3} = -\frac{112}{729}$ is negative,
so $f(x)$ is concave (i) **up** (ii) **down** over $\left(\sqrt{\frac{5}{3}}, \infty\right)$

so summarizing:

interval	$\left(-\infty, -\sqrt{\frac{5}{3}}\right)$	$\left(-\sqrt{\frac{5}{3}}, \sqrt{\frac{5}{3}}\right)$	$\left(\sqrt{\frac{5}{3}}, \infty\right)$
test number	-2	0	2
$f''(x) = \frac{16(5-3x^2)}{[x^2+5]^3}$	$f''(-2) = -\frac{112}{729}$	$f''(0) = \frac{16}{25}$	$f''(2) = -\frac{112}{729}$
sign of $f''(x)$	negative	positive	negative

(g) *Graph summary.* Combining information from above:

interval	$\left(-\infty, -\sqrt{\frac{5}{3}}\right)$	$\left(-\sqrt{\frac{5}{3}}, 0\right)$	$\left(0, \sqrt{\frac{5}{3}}\right)$	$\left(\sqrt{\frac{5}{3}}, \infty\right)$
sign of $f'(x)$	$-$	$-$	$+$	$+$
sign of $f''(x)$	$-$	$+$	$+$	$-$
increasing/decreasing?	decreasing	decreasing	increasing	increasing
concave up/down?	down	up	up	down

4. *Sketching functions.* Sketch function on graph which satisfies conditions.

   (a) f is continuous everywhere

   (b) y-intercept at $y = 3$, no x-intercepts

   (c) critical points at $(0, 3)$ and $(3, 6)$

   (d) inflection point at $(-3, 4)$

   (e) $f'(x) < 0$ on $(-5, 0)$ and $(3, 5)$

   (f) $f'(x) > 0$ on $(0, 3)$

   (g) $f''(x) > 0$ on $(-5, -3)$

   (h) $f''(x) < 0$ on $(-3, 5)$

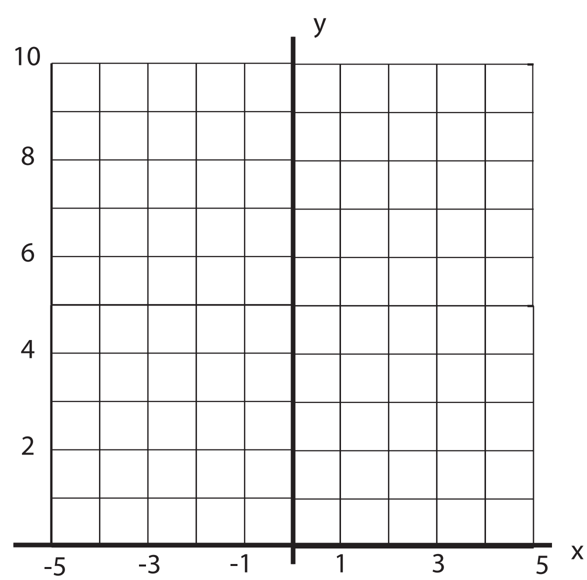

Figure 5.20 (Sketching functions)

5. *More Sketching Functions.* Sketch function on graph which satisfies conditions.

   (a) f is continuous everywhere except at vertical asymptote at $x = 2$
   (b) y-intercept at $y = 5$, no x-intercepts
   (c) critical point at $(3, 5)$
   (d) inflection point at $(-1, 4)$
   (e) $f'(x) < 0$ on $(2, 3)$
   (f) $f'(x) > 0$ on $(-5, 2)$ and $(3, 5)$
   (g) $f''(x) > 0$ on $(-1, 2)$ and $(2, 5)$
   (h) $f''(x) < 0$ on $(-5, -1)$

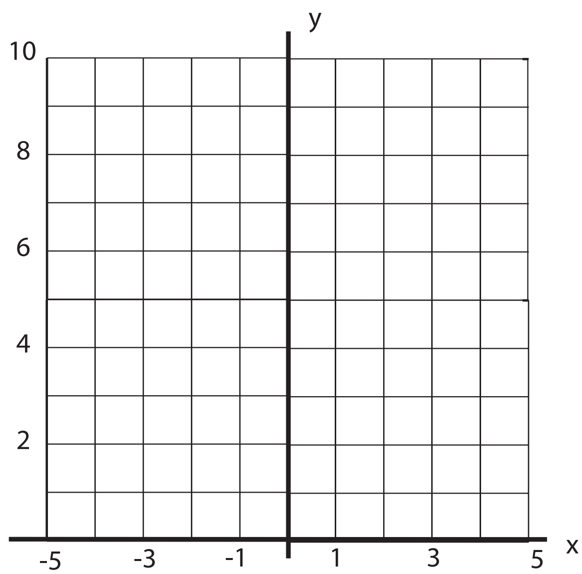

Figure 5.21 (More sketching functions)