Chapter 11

Probability and Calculus

Determination of probability for a continuous random variable involves integrating a probability density function for this random variable. We first look at general probability density functions and their expected value and variance, then look at special probability density functions; specifically, the uniform, exponential and normal density functions and their expected values and variances.

11.1 Continuous Probability Models

The *(cumulative) distribution function* for *random variable* X is

$$F(x) = P\left(X \leq x\right), \quad -\infty < x < \infty,$$

and has properties

- $\lim_{x \to -\infty} F(x) = 0$,
- $\lim_{x \to \infty} F(x) = 1$,
- if $x_1 < x_2$, then $F(x_1) \leq F(x_2)$; that is, F is nondecreasing.

Also, $\lim_{n \to \infty} F(x_n) = F(x)$; that is, F must be right continuous (which determines where the solid and empty endpoints are on the graph of a distribution function). A random variable X with distribution function $F(x)$ is *continuous* if $F(x)$ is continuous, and the first derivative of $F(x)$ exists and continuous except, possibly, at a finite number of points in a finite interval. The *(probability) density function*, $f(x)$, is given by

$$f(x) = \frac{dF(x)}{dy} = F'(x), \quad \text{and so, also,} \quad F(x) = \int_{-\infty}^{x} f(t)\, dt$$

Properties of the density function for a continuous random variable are

- $f(x) \geq 0$, for all x, $-\infty < x < \infty$,

- $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

And,

$$P(a \leq X \leq b) = P(X \leq b) - P(X \leq a) = F(b) - F(a) = \int_a^b f(x)\, dy.$$

Exercise 11.1 (Continuous Probability Models)

1. *Discrete function: number of heads when flipping a coin twice.*
   Let the number of heads flipped in two flips of a coin be a random variable X. There are four possible cases, TT ($X = 0$), HT or TH ($X = 1$) and HH ($X = 2$) and so, assuming each of the four cases are equally likely, then the probability density function is

x (number of heads)	0	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

   or, $P(X = 0) = 0.25$, $P(X = 1) = 0.50$ and $P(X = 2) = 0.25$.

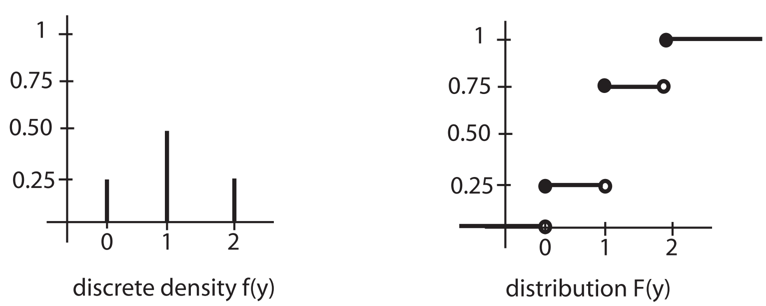

Figure 11.1: Density and distribution function: flipping a coin twice

   (a) $F(0) = P(X \leq 0) = P(X = 0) =$ (choose one)
   (i) **0** (ii) **0.25** (iii) **0.75** (iv) **1**.

   (b) $F(1) = P(X \leq 1) = P(X = 0) + P(X = 1) =$ (choose one)
   (i) **0** (ii) **0.25** (iii) **0.75** (iv) **1**.

   (c) $F(2) = P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) =$ (choose one)
   (i) **0** (ii) **0.25** (iii) **0.75** (iv) **1**.

(d) Also, if $x < 0$, $F(x) = P(X \leq x) =$ (choose one)
(i) **0** (ii) **0.25** (iii) **0.75** (iv) **1**.

(e) And, if $x \geq 2$, $F(x) = P(X \leq x) =$ (choose one)
(i) **0** (ii) **0.25** (iii) **0.75** (iv) **1**.

(f) So, in this case,

$$F(x) = \begin{cases} 0, & x < 0 \\ 0.25, & 0 \leq x < 1 \\ 0.75, & 1 \leq x < 2 \\ 1, & x \geq 2. \end{cases}$$

(i) **True** (ii) **False**

(g) The discontinuous "step function" graph of this $F(x)$ is given in the figure. Notice that $F(x)$ is right continuous, which is indicated by the solid and empty endpoints on the graph of this distribution function.
(i) **True** (ii) **False**

(h) $P(X < 1) = P(X = 0) = 0.25 =$ (choose one)
(i) $\boldsymbol{F(0)}$ (ii) $\boldsymbol{F(1)}$ (iii) $\boldsymbol{F(2)}$ (iv) $\boldsymbol{F(3)}$.

(i) $P(X < 2) = P(X = 0) + P(X = 1) = 0.75 =$ (choose one)
(i) $\boldsymbol{F(0)}$ (ii) $\boldsymbol{F(1)}$ (iii) $\boldsymbol{F(2)}$ (iv) $\boldsymbol{F(3)}$.

(j) $P(X > 1) = 1 - P(X \leq 1) = 1 - F(1) = 1 - 0.75 =$ (choose one)
(i) **0** (ii) **0.25** (iii) **0.75** (iv) **1**.

2. *Another discrete distribution function.*
Let random variable X have distribution (*not* density),

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{3}, & 0 \leq x < 1 \\ \frac{1}{2}, & 1 \leq x < 2 \\ 1, & x \geq 2. \end{cases}$$

(a) $F(1) = P(X \leq 1) = P(X = 0) + P(X = 1) =$ (choose one)
(i) **0** (ii) $\boldsymbol{\frac{1}{6}}$ (iii) $\boldsymbol{\frac{1}{3}}$ (iv) $\boldsymbol{\frac{1}{2}}$.

(b) $P(X < 2) = P(X = 0) + P(X = 1) =$ (choose one)
(i) $\mathbf{0}$ (ii) $\mathbf{\frac{1}{6}}$ (iii) $\mathbf{\frac{1}{3}}$ (iv) $\mathbf{\frac{1}{2}}$.

(c) $P(X \leq 1.5) = P(X = 0) + P(X = 1) =$ (choose one)
(i) $\mathbf{0}$ (ii) $\mathbf{\frac{1}{6}}$ (iii) $\mathbf{\frac{1}{3}}$ (iv) $\mathbf{\frac{1}{2}}$.

(d) $p(1) = P(X = 1) = P(X \leq 1) - P(X \leq 0) = F(1) - F(0) =$ (choose one)
(i) $\mathbf{0}$ (ii) $\mathbf{\frac{1}{6}}$ (iii) $\mathbf{\frac{1}{3}}$ (iv) $\mathbf{\frac{1}{2}}$.

(e) $p(2) = P(X = 2) = P(X \leq 2) - P(X \leq 1) = F(2) - F(1) =$ (choose one)
(i) $\mathbf{0}$ (ii) $\mathbf{\frac{1}{6}}$ (iii) $\mathbf{\frac{1}{3}}$ (iv) $\mathbf{\frac{1}{2}}$.

(f) Random variable X is discrete, not continuous, because the associated $F(x)$ is a discontinuous ("step", in this case) function.
(i) **True** (ii) **False**

(g) Notice that
(1) $\lim_{x\to-\infty} F(x) = 0$,
(2) $\lim_{x\to\infty} F(x) = 1$,
(3) if $x_1 < x_2$, then $F(x_1) \leq F(x_2)$; that is, F is nondecreasing.
(i) **True** (ii) **False**

3. *Continuous distribution function: waiting time.*
Let the time waiting in line, in minutes, be described by the random variable X which has the following probability *density* (not distribution),

$$f(x) = \begin{cases} 0, & x < 2, \\ \frac{1}{2}, & 2 \leq x < 4, \\ 0, & x \geq 4. \end{cases}$$

(a) The chance of waiting *at most* $x = 1$ minute is

$$F(1) = P(X \leq 1) = \int_{-\infty}^{1} f(x)\, dx = \int_{-\infty}^{1} 0\, dx =$$

(choose one) (i) **0** (ii) **0.1** (iii) **0.5** (iv) **1**.

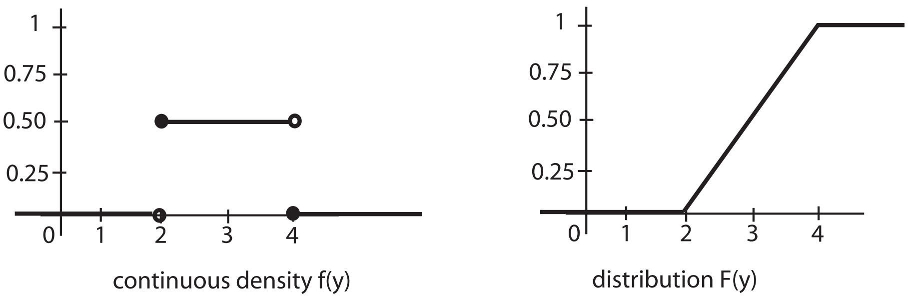

Figure 11.2: Distribution function: waiting time

(b) The chance of waiting any time less than 2 minutes, $x < 2$,

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} 0 \; dx =$$

(i) **0** (ii) **0.1** (iii) **0.5** (iv) **1**.

(c) The chance of waiting at most $x = 3$ minutes is

$$\begin{aligned} F(3) = P(X \leq 3) &= \int_{-\infty}^{3} f(x) \; dx \\ &= \int_{-\infty}^{2} 0 \; dy + \int_{2}^{3} \frac{1}{2} \; dx \\ &= 0 + \left. \frac{x}{2} \right]_{x=2}^{x=3} = \frac{3}{2} - \frac{2}{2} = \end{aligned}$$

(choose one) (i) **0** (ii) **0.1** (iii) **0.5** (iv) **1**.

(d) The chance of waiting at most $x = 5$ minutes is

$$\begin{aligned} F(5) = P(X \leq 5) &= \int_{-\infty}^{5} f(x) \; dx \\ &= \int_{-\infty}^{2} 0 \; dy + \int_{2}^{4} \frac{1}{2} \; dy + \int_{4}^{5} 0 \; dx \\ &= 0 + \left. \frac{x}{2} \right]_{x=2}^{x=4} + 0 = \frac{4}{2} - \frac{2}{2} = \end{aligned}$$

(choose one) (i) **0** (ii) **0.1** (iii) **0.5** (iv) **1**.

Integrals involving zero will not always be explicitly stated as they are here; instead of $\int_{-\infty}^{5} f(x) \; dy = \int_{-\infty}^{2} 0 \; dy + \int_{2}^{4} \frac{1}{2} \; dy + \int_{4}^{5} 0 \; dy$, integral $\int_{-\infty}^{5} f(x) \; dy = \int_{2}^{4} \frac{1}{2} \; dy$ will be used instead.

(e) The chance of waiting any time more than 4 minutes, $x \geq 4$,

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dy = \int_{2}^{4} \frac{1}{2}\,dx =$$

(i) **0** (ii) **0.1** (iii) **0.5** (iv) **1**.

(f) In general, the distribution function is

$$F(x) = \begin{cases} 0, & x < 2, \\ \frac{x}{2} - 1, & 2 \leq x < 4, \\ 1, & x \geq 4. \end{cases}$$

Random variable X is continuous because, as shown in the figure, even though the density, $f(x)$, is a discontinuous function, the associated distribution, $F(x)$, is a continuous function in this case.
(i) **True** (ii) **False**

(g) $F(3) = \frac{1}{2}$ is *both* the (positive) *area* under density $f(x)$ from $-\infty$ up to 3 *and also* the *point* on distribution $F(x)$ at $x = 3$.
(i) **True** (ii) **False**

(h) Since X is a continuous random variable

$$P(X < 3) = \int_{2}^{3} \frac{1}{2}\,dt = \left.\frac{x}{2}\right]_{x=2}^{x=3} = \frac{3}{2} - \frac{2}{2} = \frac{1}{2} =$$

(i) $\boldsymbol{P(X > 3)}$ (ii) $\boldsymbol{P(X < 4)}$ (iii) $\boldsymbol{P(X \leq 3)}$ (iv) $\boldsymbol{P(X \leq 4)}$.
and, consequently, the chance of waiting *exactly* $x = 3$ minutes is *zero*,

$$P(X = 3) = \int_{3}^{3} f(x)\,dx = 0.$$

(i) Since

$$P(2.5 < X < 3) = \int_{2.5}^{3} \frac{1}{2}\,dy = \left.\frac{x}{2}\right]_{x=2.5}^{x=3} = \frac{3}{2} - \frac{2.5}{2} = \frac{0.5}{2} = \frac{1}{4} =$$

(choose one or *more*)
(i) $\boldsymbol{P(2.5 \leq X < 3)}$ (ii) $\boldsymbol{P(2.5 \leq X \leq 3)}$
(iii) $\boldsymbol{P(2.5 < X < 3)}$ (iv) $\boldsymbol{P(2.5 < X \leq 3)}$.
because the chance of waiting *exactly* x minutes is *zero*,

$$P(X = x) = \int_{x}^{x} f(t)\,dt = 0.$$

(j) (i) **True** (ii) **False**
Function $f(x)$ is a *probability* density here because *both* $f(x) \geq 0$ and also $\int f(x)\,dx = 1$ on interval $[2, 4)$.

4. *Continuous distribution function: triangle.*
Let random variable X have the following probability *density*,

$$f(x) = \begin{cases} 0, & x < 2, \\ \frac{1}{6}x, & 2 \leq x < 4, \\ 0, & x \geq 4. \end{cases}$$

In other words, $f(x) = \frac{x}{6}$ on $[2, 4)$, equivalently $[2, 4]$, and zero elsewhere.

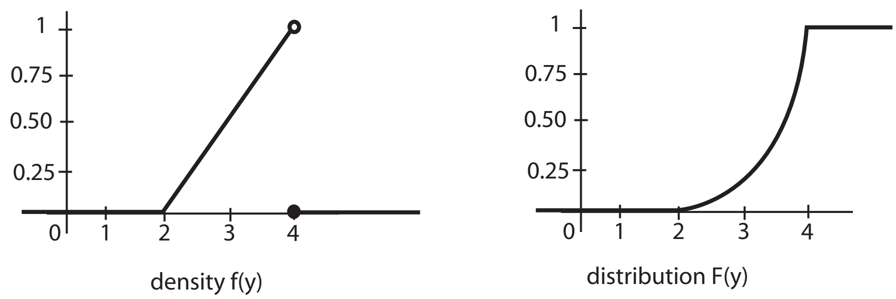

Figure 11.3: Density and distribution function: triangle

(a) (i) **True** (ii) **False** The *distribution function* is

$$F(x) = \begin{cases} \int_{-\infty}^{x} 0\ dt \ = \ 0, & x < 2, \\ \int_{-\infty}^{2} 0\ dt \ + \ \int_{2}^{x} \frac{t}{6}\ dt \ = \ 0 + \frac{t^2}{12}\Big]_{t=2}^{t=x} \ = \ \frac{x^2}{12} - \frac{4}{12}, & 2 \leq x < 4, \\ \int_{-\infty}^{2} 0\ dt \ + \ \int_{t=2}^{t=4} \frac{t}{6}\ dt \ + \ \int_{4}^{\infty} 0\ dt \ = \ 0 + \frac{x^2}{12}\Big]_{t=2}^{t=4} + 0 \ = \ 1, & x \geq 4. \end{cases}$$

In other words, $F(x) = \frac{x^2}{12} - \frac{4}{12}$ on $[2, 4)$ or equivalently $[2, 4]$ and zero elsewhere. Both density and distribution are given in the figure.

(b) $F(1) =$ (choose one) (i) $\mathbf{0}$ (ii) $\mathbf{\frac{5}{12}}$ (iii) $\mathbf{\frac{7}{12}}$ (iv) $\mathbf{1}$.

(c) $F(2) = \frac{2^2}{12} - \frac{4}{12} =$ (choose one) (i) $\mathbf{0}$ (ii) $\mathbf{\frac{5}{12}}$ (iii) $\mathbf{\frac{9}{12}}$ (iv) $\mathbf{1}$.

(d) $F(3) = \frac{3^2}{12} - \frac{4}{12} =$ (choose one) (i) $\mathbf{0}$ (ii) $\mathbf{\frac{5}{12}}$ (iii) $\mathbf{\frac{9}{12}}$ (iv) $\mathbf{1}$.

(e) $F(5) =$ (choose one) (i) $\mathbf{0}$ (ii) $\mathbf{\frac{5}{12}}$ (iii) $\mathbf{\frac{9}{12}}$ (iv) $\mathbf{1}$.

(f) Also,

$$\begin{aligned} P(1 < X < 3) &= P(X < 3) - P(X < 1) \\ &= P(X \leq 3) - P(X \leq 1) \\ &= F(3) - F(1) = \left(\frac{3^2}{12} - \frac{4}{12}\right) - 0 = \end{aligned}$$

(choose one) (i) $\mathbf{0}$ (ii) $\mathbf{\frac{5}{12}}$ (iii) $\mathbf{\frac{9}{12}}$ (iv) $\mathbf{1}$.

(g) $P(2.5 < X < 3.5) = F(3.5) - F(2.5) = \left(\frac{3.5^2}{12} - \frac{4}{12}\right) - \left(\frac{2.5^2}{12} - \frac{4}{12}\right) =$
(choose one) (i) $\mathbf{0}$ (ii) $\mathbf{0.25}$ (iii) $\mathbf{0.50}$ (iv) $\mathbf{1}$.

(h) $P(X > 3.5) = 1 - P(X \leq 3.5) = 1 - F(3.5) = 1 - \left(\frac{3.5^2}{12} - \frac{4}{12}\right) =$
(choose one) (i) $\mathbf{0.3125}$ (ii) $\mathbf{0.4425}$ (iii) $\mathbf{0.7650}$ (iv) $\mathbf{1}$.

(i) (i) **True** (ii) **False**
Function $f(x)$ is a probability density here because
both $f(x) \geq 0$ and also $\int f(x)\,dx = 1$ on interval $[2, 4]$.

5. *Continuous distribution function: triangle with unknown k.*
Let random variable X have the probability density

$$f(x) = kx$$

on the interval $[1, 5]$ and zero elsewhere. What is k? Find $P(2.5 < X < 3.5)$.

(a) The distribution function is

$$F(x) = \int_1^x kt\,dt = k\frac{t^2}{2}\Bigg]_{t=1}^{t=x} = \frac{kx^2}{2} - \frac{k}{2} = \frac{k(x^2-1)}{2}$$

(i) **True** (ii) **False**

(b) *What is k?* Since $1 \leq x < 5$ and the total probability must equal 1,

$$F(5) = \frac{k(5^2-1)}{2} = \frac{24k}{2} = 1,$$

then $k =$ (choose one) (i) $\mathbf{\frac{1}{11}}$ (ii) $\mathbf{\frac{1}{12}}$ (iii) $\mathbf{\frac{1}{13}}$ (iv) $\mathbf{\frac{1}{14}}$.

(c) In other words,

$$f(x) = kx = \frac{1}{12}x,$$

and

$$F(x) = \frac{k(x^2-1)}{2} = \frac{1}{12} \times \frac{x^2-1}{2} = \frac{(x^2-1)}{24}$$

on $[1, 5]$ and zero elsewhere.
(i) **True** (ii) **False**

(d) $P(2.5 < X < 3.5) = F(3.5) - F(2.5) = \left(\frac{3.5^2-1}{24}\right) - \left(\frac{2.5^2-1}{24}\right) =$
(choose one) (i) **0** (ii) **0.25** (iii) **0.50** (iv) **1**.

6. *Another continuous probability function with unknown k.*
Find k such that $f(x) = kx^2$ is a probability density function where x is in the interval $[1, 5]$. Then find $P(X \geq 2)$.

(a) Since the distribution function is

$$F(x) = \int_1^x kt^2\, dt = k\frac{t^3}{3}\Bigg]_{t=1}^{t=x} = \frac{kx^3}{3} - \frac{k}{3} = \frac{k(x^3-1)}{3}$$

and since $1 \leq x < 5$ and the total probability must equal 1,

$$F(5) = \frac{k(5^3-1)}{3} = \frac{124k}{3} = 1,$$

then $k =$ (i) $\mathbf{\frac{3}{124}}$ (ii) $\mathbf{\frac{2}{124}}$ (iii) $\mathbf{\frac{1}{124}}$

(b) And so

$$F(x) = \frac{k(x^3-1)}{3} = \frac{3}{124} \times \frac{x^3-1}{3} =$$

(i) $\boldsymbol{\frac{3}{124}(x^3-1)}$ (ii) $\boldsymbol{\frac{2}{124}(x^3-1)}$ (iii) $\boldsymbol{\frac{1}{124}(x^3-1)}$

(c) And also

$$f(x) = kx^2 = \frac{3}{124}x^2 =$$

(i) $\boldsymbol{\frac{3}{124}x^2}$ (ii) $\boldsymbol{\frac{2}{124}x^2}$ (iii) $\boldsymbol{\frac{1}{124}x^2}$

(d) And so

$$P(X \geq 2) = 1 - P(X < 2) = 1 - F(2) = 1 - \frac{1}{124}((2)^3 - 1) =$$

(i) $\mathbf{\frac{116}{124}}$ (ii) $\mathbf{\frac{117}{124}}$ (iii) $\mathbf{\frac{118}{124}}$

7. *Probability density function and improper integration*
   Consider function

$$f(x) = 3e^{-3x}$$

defined on $[0, \infty)$. Find $F(x)$ and $P(X \geq 2)$. Show $\int_0^\infty f(x)\,dx = 1$.

(a) By substituting $u = -3x$, then $du = -3\,du$ and so

$$F(x) = \int 3e^{-3x}\,dx = -\int e^{-3x}(-3)\,dx = -\int e^u\,du = -e^u =$$

(i) $\boldsymbol{-e^{-3x}}$ (ii) $\boldsymbol{e^{-3x}}$ (iii) $\boldsymbol{-3e^{-3x}}$

then

$$F(x) = \int_0^x 3e^{-3t}\,dt = \int_0^x 3e^{-3t}\,dt = \left[-e^{-3t}\right]_{t=0}^{t=x} = \left(-e^{-3x} - (-e^{-k(0)})\right) =$$

(i) $\boldsymbol{-e^{-3x}+1}$ (ii) $\boldsymbol{e^{-3x}+1}$ (iii) $\boldsymbol{1-e^{-3x}}$

(b) And so

$$P(X \geq 2) = 1 - P(X < 2) = 1 - F(2) = 1 - \left(1 - e^{-3(2)}\right) \approx$$

(i) **0.002** (ii) **0.003** (iii) **0.004**

(c) (i) **True** (ii) **False**

$$\int_0^\infty 3e^{-3x}\,dx = \lim_{b\to\infty}\int_0^b 3e^{-3x}\,dx = \lim_{b\to\infty}\left[-e^{-3x}\right]_{x=0}^{x=b} = \lim_{b\to\infty}\left[-e^{-3b} - (-e^{x(0)})\right] = 1$$

or, equivalently,

$$\lim_{b\to\infty} F(b) = \lim_{b\to\infty} 1 - e^{-3b} = 1$$

11.2 Expected Value and Variance of Continuous Random Variables

The *expected value*, $E(X)$, of a *discrete* random variable, X is given by

$$E(X) = \sum_x xP(X = x).$$

The expected value, also called the *mean*, μ, $E(X) = \mu$, is, roughly, a weighted average of X.

The expected value, $E(X)$, of a *continuous* random variable, X is given by

$$E(X) = \int_{-\infty}^{\infty} xf(x)\,dx.$$

The *variance*, Var(X), is

$$\text{Var}(X) = \sigma^2 = E[(X-\mu)^2] = E(X^2) - [E(X)]^2 = E(X^2) - \mu^2$$

with associated *standard deviation*, $\sigma = \sqrt{\sigma^2}$, in both discrete and continuous cases. The *median* of X is the m that satisfies $P(X \leq m) \geq \frac{1}{2}$ *and* $P(X \geq m) \geq \frac{1}{2}$.

Exercise 11.2 (Expected Values for Continuous Random Variables)

1. *Discrete function: number of heads when flipping a coin twice.*
   Let the number of heads flipped in two flips of a coin be a random variable X with probability density function

x (number of heads)	0	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

   (a) *Expected number of flips of the coin*

$$\mu = E(X) = \sum_x xP(X=x) = 0 \times \frac{1}{4} + 1 \times \frac{2}{4} + 2 \times \frac{1}{4} =$$

   (i) $\mathbf{\frac{1}{2}}$ (ii) $\mathbf{\frac{3}{4}}$ (iii) **1** (iv) $\mathbf{\frac{3}{2}}$.

   (b) $E(X^2)$.

$$E\left(X^2\right) = \sum_x x^2 P(X=x) = 0^2 \times \frac{1}{4} + 1^2 \times \frac{2}{4} + 2^2 \times \frac{1}{4} =$$

   (i) $\mathbf{\frac{1}{2}}$ (ii) $\mathbf{\frac{3}{4}}$ (iii) **1** (iv) $\mathbf{\frac{3}{2}}$.

   (c) *Variance in number of flips of coin.*

$$\sigma^2 = \text{Var}(X) = E\left(X^2\right) - \mu^2 = \frac{3}{2} - (1)^2 =$$

   (i) $\mathbf{\frac{1}{2}}$ (ii) $\mathbf{\frac{3}{4}}$ (iii) **1** (iv) $\mathbf{\frac{3}{2}}$.

   (d) *Standard deviation in number of flips of coin.*

$$\sigma = \sqrt{\sigma^2} = \sqrt{\frac{1}{2}} \approx$$

   (i) **0.57** (ii) **0.61** (iii) **0.67** (iv) **0.71**.

(e) *Determine* $P(\mu - \sigma < X < \mu + \sigma)$

$P(\mu{-}\sigma < X < \mu{+}\sigma) \approx P\left(1 - 0.71 < X < 1 + 0.71\right) \approx P\left(0.29 < X < 1.71\right) =$

(i) $\mathbf{\frac{1}{2}}$ (ii) $\mathbf{\frac{3}{4}}$ (iii) $\mathbf{1}$ (iv) $\mathbf{\frac{3}{2}}$.
Look at the table above: how much probability is between 0.29 and 1.71?

(f) *Median.* Since

$$P(X \leq 1) = \frac{3}{4} > \frac{1}{2} \quad \text{and} \quad P(X \geq 1) = \frac{3}{4} > \frac{1}{2}$$

the median is $m =$ (i) $\mathbf{\frac{1}{2}}$ (ii) $\mathbf{\frac{3}{4}}$ (iii) $\mathbf{1}$ (iv) $\mathbf{\frac{3}{2}}$.

2. *Discrete function: number of heads when flipping an unfair coin twice.*
Let the number of heads flipped in two flips of a coin be a random variable X with probability density function

x (number of heads)	0	1	2
$P(X = x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

(a) *Expected number of flips of the coin*

$$\mu = E(X) = \sum_x xP(X = x) = 0 \times \frac{1}{2} + 1 \times \frac{1}{4} + 2 \times \frac{1}{4} =$$

(i) $\mathbf{\frac{11}{16}}$ (ii) $\mathbf{\frac{3}{4}}$ (iii) $\mathbf{1}$ (iv) $\mathbf{\frac{5}{4}}$.

(b) $E\left(X^2\right)$.

$$E\left(X^2\right) = \sum_x x^2 P(X = x) = 0^2 \times \frac{1}{2} + 1^2 \times \frac{1}{4} + 2^2 \times \frac{1}{4} =$$

(i) $\mathbf{\frac{11}{16}}$ (ii) $\mathbf{\frac{3}{4}}$ (iii) $\mathbf{1}$ (iv) $\mathbf{\frac{5}{4}}$.

(c) *Variance in number of flips of coin.*

$$\sigma^2 = \text{Var}(X) = E\left(X^2\right) - \mu^2 = \frac{5}{4} - \left(\frac{3}{4}\right)^2 =$$

(i) $\mathbf{\frac{11}{16}}$ (ii) $\mathbf{\frac{3}{4}}$ (iii) $\mathbf{1}$ (iv) $\mathbf{\frac{5}{4}}$.

(d) *Standard deviation in number of flips of coin.*

$$\sigma = \sqrt{\sigma^2} = \sqrt{\frac{11}{16}} \approx$$

(i) **0.77** (ii) **0.81** (iii) **0.83** (iv) **0.91**.

(e) *Determine* $P(\mu - \sigma < X < \mu + \sigma)$

$$P(\mu - \sigma < X < \mu + \sigma) \approx P\left(\frac{3}{4} - 0.83 < X < \frac{3}{4} + 0.83\right) \approx P(-0.08 < X < 1.58) =$$

(i) $\mathbf{\frac{1}{2}}$ (ii) $\mathbf{\frac{3}{4}}$ (iii) **1** (iv) $\mathbf{\frac{3}{2}}$.

Look at the table above: how much probability is between -0.08 and 1.58?

(f) *Median.* Since

$$P(X \leq 1) = \frac{3}{4} > \frac{1}{2} \quad \text{and} \quad P(X \geq 1) = \frac{1}{2} \geq \frac{1}{2}$$

the median is $m =$ (i) $\mathbf{\frac{1}{2}}$ (ii) $\mathbf{\frac{3}{4}}$ (iii) **1** (iv) $\mathbf{\frac{3}{2}}$.

3. *Continuous distribution function: Time and cost of cell phone.*
Let the amount of time spent (in minutes) on a cell phone call be represented by random variable X with the following probability density,

$$f(x) = \begin{cases} \frac{1}{6}x, & 2 \leq x \leq 4, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) *Expected value.* The expected amount of time on the call is,

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx = \int_2^4 x\left(\frac{1}{6}x\right)\, dx = \int_2^4 \frac{x^2}{6}\, dx = \left.\frac{x^3}{18}\right]_2^4 = \frac{4^3}{18} - \frac{2^3}{18} =$$

(i) $\mathbf{\frac{23}{9}}$ (ii) $\mathbf{\frac{28}{9}}$ (iii) $\mathbf{\frac{31}{9}}$ (iv) $\mathbf{\frac{35}{9}}$.

(b) $E(X^2)$.

$$E\left(X^2\right) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_2^4 x^2\left(\frac{1}{6}x\right)\, dx = \int_2^4 \frac{x^3}{6}\, dx = \left.\frac{x^4}{24}\right]_2^4 = \frac{4^4}{24} - \frac{2^4}{24} =$$

(i) **9** (ii) **10** (iii) **11** (iv) **12**.

(c) *Variance.* Variance in the amount of time on the call is,

$$\sigma^2 = \text{Var}(X) = E\left(X^2\right) - \mu^2 = 10 - \left(\frac{28}{9}\right)^2 =$$

(i) $\mathbf{\frac{23}{81}}$ (ii) $\mathbf{\frac{26}{81}}$ (iii) $\mathbf{\frac{31}{81}}$ (iv) $\mathbf{\frac{35}{81}}$.

(d) *Standard deviation.* Standard deviation in time spent on the call is,

$$\sigma = \sqrt{\sigma^2} = \sqrt{\frac{26}{81}} \approx$$

(i) **0.57** (ii) **0.61** (iii) **0.67** (iv) **0.73**.

(e) *Determine probability time falls within one standard deviation of mean.*

$$\begin{aligned}
P(\mu - \sigma < X < \mu + \sigma) &\approx P\left(\frac{28}{9} - 0.57 < X < \frac{28}{9} + 0.57\right) \\
&\approx P\left(2.54 < X < 3.68\right) \\
&= \int_{2.54}^{3.68} \frac{x}{6}\, dx \\
&= \left.\frac{x^2}{12}\right]_{2.54}^{3.68} \\
&= \frac{(3.68)^2}{12} - \frac{(2.54)^2}{12} \approx
\end{aligned}$$

(i) **0.49** (ii) **0.59** (iii) **0.69** (iv) **0.79**.

(f) *Median.* Since the distribution function is

$$F(x) = \int_2^x \frac{t}{6}\, dt = \left.\frac{t^2}{12}\right]_{t=2}^{t=x} = \frac{x^2}{12} - \frac{2^2}{12} = \frac{x^2-4}{12}$$

then median m occurs when

$$F(m) = P(X \leq m) = \frac{m^2-4}{12} = \frac{1}{2}$$

so

$$\begin{aligned}
\frac{m^2-4}{12} &= \frac{1}{2} \\
m^2 - 4 &= 6 \\
m^2 &= 10 \\
m &= \sqrt{10} \approx
\end{aligned}$$

(i) **1.16** (ii) **2.16** (iii) **3.16**

4. *Another continuous distribution example.*
   Let random variable X have the following probability density,

$$f(x) = \begin{cases} x, & 0 < x < 1, \\ 2 - x, & 1 \leq x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

   (a) *Expected value.*

$$\begin{aligned} \mu = E(X) &= \int_{-\infty}^{\infty} x f(x)\, dx \\ &= \int_0^1 x\,(x)\ dx + \int_1^2 x\,(2-x)\ dx \\ &= \int_0^1 x^2\ dx + \int_1^2 \left(2x - x^2\right)\ dx \\ &= \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_1^2 \\ &= \left(\frac{1^3}{3} - \frac{0^3}{3}\right) + \left(2^2 - \frac{2^3}{3}\right) - \left(1^2 - \frac{1^3}{3}\right) = \end{aligned}$$

   (i) **1** (ii) **2** (iii) **3** (iv) **4**.

   (b) $E\left(X^2\right)$.

$$\begin{aligned} E\left(X^2\right) &= \int_0^1 x^2\,(x)\ dx + \int_1^2 x^2\,(2-x)\ dx \\ &= \int_0^1 x^3\ dx + \int_1^2 \left(2x^2 - x^3\right)\ dx \\ &= \left[\frac{x^4}{4}\right]_0^1 + \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_1^2 \\ &= \left(\frac{1^4}{4} - \frac{0^4}{4}\right) + \left(\frac{2(2)^3}{3} - \frac{2^4}{4}\right) - \left(\frac{2(1)^3}{3} - \frac{1^4}{4}\right) = \end{aligned}$$

   (i) $\mathbf{\frac{4}{6}}$ (ii) $\mathbf{\frac{5}{6}}$ (iii) $\mathbf{\frac{6}{6}}$ (iv) $\mathbf{\frac{7}{6}}$.

   (c) *Variance.*

$$\sigma^2 = \text{Var}(X) = E\left(X^2\right) - \mu^2 = \frac{7}{6} - 1^2 =$$

   (i) $\mathbf{\frac{1}{3}}$ (ii) $\mathbf{\frac{1}{4}}$ (iii) $\mathbf{\frac{1}{5}}$ (iv) $\mathbf{\frac{1}{6}}$.

   (d) *Standard deviation.*

$$\sigma = \sqrt{\sigma^2} = \sqrt{\frac{1}{6}} \approx$$

(i) **0.27** (ii) **0.31** (iii) **0.41** (iv) **0.53**.

(e) *Determine* $P(\mu - \sigma < X < \mu + \sigma)$.

$$\begin{aligned} P(\mu - \sigma < X < \mu + \sigma) &\approx P(1 - 0.41 < X < 1 + 0.41) \\ &\approx P(0.59 < X < 1.41) \\ &= \int_{0.59}^{1} x \; dx + \int_{1}^{1.41} (2 - x) \; dx \\ &= \left[\frac{x^2}{2} \right]_{0.59}^{1} + \left[2x - \frac{x^2}{2} \right]_{1}^{1.41} \\ &= \left(\frac{1^2}{2} - \frac{0.59^2}{2} \right) + \left(2(1.41) - \frac{1.41^2}{2} \right) - \left(2(1) - \frac{1^2}{2} \right) \approx \end{aligned}$$

(i) **0.35** (ii) **0.45** (iii) **0.55** (iv) **0.65**.

(f) *Median.* Since the distribution function is

$$F(x) = \int_0^x t \; dt = \left. \frac{t^2}{2} \right]_{t=0}^{t=x} = \frac{x^2}{2} - \frac{0^2}{2} = \frac{x^2}{2}$$

then median m occurs when

$$F(m) = P(X \leq m) = \frac{m^2}{2} = \frac{1}{2}$$

so $m =$ (i) **1** (ii) **1.5** (iii) **2**

11.3 Special Probability Density Functions

Three special probability density functions are discussed: uniform, exponential and normal. The continuous *uniform* distribution of random variable X, defined on the interval $[a, b]$, has density

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b, \\ 0, & \text{elsewhere,} \end{cases}$$

distribution function,

$$F(x) = \begin{cases} 0 & x \leq a, \\ \frac{x-a}{b-a} & a < x < b, \\ 1 & x \geq b, \end{cases}$$

and its expected value (mean), variance and standard deviation are,

$$\mu = E(X) = \frac{a+b}{2}, \quad \sigma^2 = \text{Var}(X) = \frac{(b-a)^2}{12}, \quad \sigma = \sqrt{\text{Var}(X)}.$$

The continuous *exponential* random variable X has density

$$f(x) = \begin{cases} ae^{-ax}, & 0 \leq x < \infty, \\ 0, & \text{elsewhere}, \end{cases}$$

distribution function,

$$F(x) = \begin{cases} 0 & x < 0, \\ 1 - e^{-ax} & 0 \leq x < \infty, \end{cases}$$

and its expected value (mean), variance and standard deviation are,

$$\mu = E(X) = \frac{1}{a}, \quad \sigma^2 = V(Y) = \frac{1}{a^2}, \quad \sigma = \frac{1}{a}.$$

The continuous *normal* distribution of random variable X, defined on the interval $(-\infty, \infty)$, has density with parameters μ and σ,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(1/2)[(x-\mu)/\sigma]^2}$$

and its expected value (mean), variance and standard deviation are,

$$E(X) = \mu, \quad \text{Var}(X) = \sigma^2, \quad \sigma = \sqrt{\text{Var}(X)}.$$

A normal random variable, X, may be transformed to a *standard* normal, Z,

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2},$$

where $\mu = 0$ and $\sigma = 1$ using the following equation,

$$Z = \frac{X - \mu}{\sigma}.$$

The *distribution* of this density does not have a closed–form expression and so must be solved using numerical integration methods. We will be using the TI-84+, rather than tables, to obtain approximate numerical answers.

Exercise 11.3 (Special Probability Density Functions)

1. *Potatoes.* An automated process fills one bag after another with Idaho potatoes. Although each filled bag should weigh 50 pounds, in fact, because of the differing shapes and weights of each potato, each bag weighs anywhere from 49 pounds to 51 pounds, with the following uniform density:

$$f(x) = \begin{cases} 0.5, & 49 < x \leq 51, \\ 0, & \text{elsewhere}. \end{cases}$$

(a) Since $a = 49$ and $b = 51$, the distribution is

$$F(x) = \begin{cases} 0 & x < 49, \\ \frac{x-49}{51-49} & 49 \leq x < 51, \\ 1 & x \geq 52, \end{cases}$$

and so graphs of density and distribution are given in the figure.
(i) **True** (ii) **False**

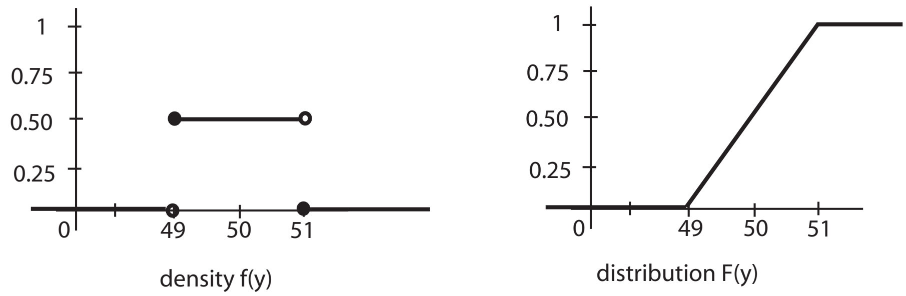

Figure 11.4: Distribution function: continuous uniform

(b) The chance the bags weight between 49.5 and 51 pounds is

$$P(49.5 < X < 51) = F(51) - F(49.5) = \frac{51-49}{51-49} - \frac{49.5-49}{51-49} =$$

(i) **0.25** (ii) **0.50** (iii) **0.75** (iv) **1**.
Notice $P(49.5 < X < 51) = \frac{51-49.5}{2} = 0.75$

(c) *Probability question.*

$$P(X > 49.5) = 1 - P(X \leq 49.5) = 1 - F(49.5) = 1 - \frac{49.5-49}{51-49} =$$

(i) **0.25** (ii) **0.50** (iii) **0.75** (iv) **1**.

(d) *Probability question.*

$$P(X \geq 49.5) = 1 - P(X < 49.5) = 1 - F(49.5) = 1 - \frac{49.5-49}{51-49} =$$

(i) **0.25** (ii) **0.50** (iii) **0.75** (iv) **1**.

(e) *Mean.* What is the mean weight of a bag of potatoes?

$$\mu = E(X) = \frac{a+b}{2} = \frac{49+51}{2} =$$

(i) **49** (ii) **50** (iii) **51** (iv) **52**.

(f) *What is the standard deviation in the weight of a bag of potatoes?*

$$\sigma = \sqrt{\frac{(b-a)^2}{12}} = \sqrt{\frac{(51-49)^2}{12}} =$$

(i) **0.44** (ii) **0.51** (iii) **0.55** (iv) **0.58**.

2. *Another example of a uniform distribution.*

$$f(x) = \begin{cases} k, & -4 < x \leq 8, \\ 0, & \text{elsewhere,} \end{cases}$$

where k is an unknown constant.

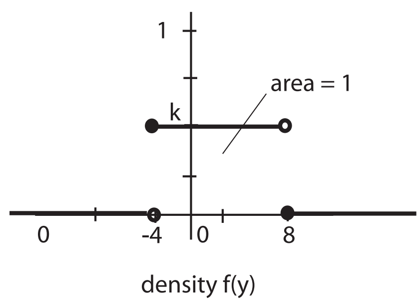

Figure 11.5: Distribution function: continuous uniform

(a) *What is k?*
Since the uniform has nonzero probability defined in the range $-4 \leq x < 8$ with length $8 - (-4) = 12$, and rectangular area under any uniform must be equal to 1,

$k =$ (i) $\mathbf{\frac{1}{11}}$ (ii) $\mathbf{\frac{1}{12}}$ (iii) $\mathbf{\frac{1}{13}}$ (iv) $\mathbf{\frac{1}{14}}$.

In other words,

$$f(x) = \begin{cases} \frac{1}{12}, & -4 < x \leq 8, \\ 0, & \text{elsewhere.} \end{cases}$$

(b) Since $a = -4$ and $b = 8$, the distribution is

$$F(x) = \frac{x - (-4)}{8 - (-4)} =$$

(i) $\mathbf{\frac{x-4}{8-4}}$ (ii) $\mathbf{\frac{x+4}{8-4}}$ (iii) $\mathbf{\frac{x-4}{8+4}}$ (iv) $\mathbf{\frac{x+4}{8+4}}$.

(c) *Probability.*

$$P(-7 < X < 0) = F(0) - F(-7) = \frac{0+4}{8+4} - 0 =$$

(i) $\mathbf{\frac{1}{12}}$ (ii) $\mathbf{\frac{2}{12}}$ (iii) $\mathbf{\frac{3}{12}}$ (iv) $\mathbf{\frac{4}{12}}$.
Notice $P(-7 < X < 0) = \frac{0-(-4)}{12}$

(d) *Probability.*

$$P(1 < X < 10) = F(10) - F(1) = 1 - \frac{1+4}{8+4} =$$

(i) $\mathbf{\frac{7}{12}}$ (ii) $\mathbf{\frac{8}{12}}$ (iii) $\mathbf{\frac{9}{12}}$ (iv) $\mathbf{\frac{10}{12}}$.

(e) *Mean.*

$$\mu = E(X) = \frac{a+b}{2} = \frac{-4+8}{2} =$$

(i) **1** (ii) **2** (iii) **3** (iv) **4**.

(f) *Standard deviation.*

$$\sigma = \sqrt{\frac{(b-a)^2}{12}} = \sqrt{\frac{(8-(-4))^2}{12}} \approx$$

(i) **1.93** (ii) **2.69** (iii) **3.46** (iv) **4.33**.

(g) *Probability between mean and 1 standard deviation above mean*

$$\begin{aligned} P(\mu < X < \mu + \sigma) &\approx P(2 < X < 2 + 3.46) \\ & \quad P(2 < X < 5.46) \\ &= F(5.46) - F(2) \\ &= \frac{5.46 - (-4)}{8+4} - \frac{2-(-4)}{8+4} = \end{aligned}$$

(i) **0.29** (ii) **0.35** (iii) **0.45** (iv) **0.51**.
Notice $P(2 < X < 5.46) = \frac{5.46-1}{12} = 0.28$

3. *Exponential: waiting time for emails.* Density is

$$f(x) = \begin{cases} ae^{-ax}, & 0 \le x < \infty, \\ 0, & \text{elsewhere,} \end{cases}$$

and corresponding distribution function is

$$F(x) = \begin{cases} 0 & x < 0, \\ 1 - e^{-ax} & 0 \leq x < \infty. \end{cases}$$

(a) If $a = \frac{1}{2}$, chance of waiting at most 1.1 minutes is

$$P(X \leq 1.1) = F(1.1) = 1 - e^{-\frac{1}{2}(1.1)} \approx$$

(i) **0.32** (ii) **0.42** (iii) **0.45** (iv) **0.48**.

(b) If $a = 3$,

$$P(X \leq 1.1) = F(1.1) = 1 - e^{-3(1.1)} \approx$$

(i) **0.32** (ii) **0.42** (iii) **0.75** (iv) **0.96**.

(c) If $a = 5$,

$$P(X < 1.1) = F(1.1) = 1 - e^{-5(1.1)} \approx$$

(i) **0.312** (ii) **0.432** (iii) **0.785** (iv) **0.996**.

(d) If $a = 3$, the chance of waiting at least 0.54 minutes

$$P(X > 0.54) = 1 - F(0.54) = 1 - \left(1 - e^{-(3)(0.54)}\right) = e^{-(3)(0.54)} \approx$$

(i) **0.20** (ii) **0.22** (iii) **0.29** (iv) **0.34**.

(e) If $a = 3$, chance of waiting between 1.13 minutes and 1.62 minutes,

$$\begin{aligned} P(1.13 < X < 1.62) &= F(1.62) - F(1.13) \\ &= \left(1 - e^{-(3)(1.62)}\right) - \left(1 - e^{-(3)(1.13)}\right) \\ &= e^{-(3)(1.13)} - e^{-(3)(1.62)} \approx \end{aligned}$$

(i) **0.014** (ii) **0.026** (iii) **0.034** (iv) **0.054**.

(f) *Expectation and Variance.*

For $a = 3$, $\mu = \frac{1}{a} =$ (i) $\mathbf{\frac{1}{2}}$ (ii) $\mathbf{\frac{1}{3}}$ (iii) $\mathbf{\frac{1}{4}}$ (iv) $\mathbf{\frac{1}{5}}$.

For $a = \frac{1}{3}$, $\mu = \frac{1}{a} =$ (i) **2** (ii) **3** (iii) **4** (iv) **5**.

For $a = 3$, $\sigma = \frac{1}{a} =$ (i) $\mathbf{\frac{1}{3}}$ (ii) $\mathbf{\frac{1}{5}}$ (iii) $\mathbf{\frac{1}{7}}$ (iv) $\mathbf{\frac{1}{9}}$.

For $a = \frac{1}{3}$, $\sigma = \frac{1}{a} =$ (i) **2** (ii) **3** (iii) **4** (iv) **5**.

(g) *Probability between mean and 1 standard deviation below mean*
If $a = 3$,

$$\begin{aligned}P(\mu - \sigma < X < \mu) &= P\left(\frac{1}{3} - \frac{1}{3} < X < \frac{1}{3}\right) \\ &= P\left(0 < X < \frac{1}{3}\right) \\ &= F(0) - F\left(\frac{1}{3}\right) \\ &= e^{-(3)(0)} - e^{-(3)\frac{1}{3}} \approx\end{aligned}$$

(i) **0.33** (ii) **0.43** (iii) **0.53** (iv) **0.63**.

4. *Nonstandard normal: IQ scores.*
It has been found that IQ scores, Y, can be distributed by a normal distribution. Densities of IQ scores for 16 year olds, Y_1, and 20 year olds, Y_2, are given by

$$\begin{aligned}f(y_1) &= \frac{1}{16\sqrt{2\pi}} e^{-(1/2)[(y-100)/16]^2}, \\ f(y_2) &= \frac{1}{20\sqrt{2\pi}} e^{-(1/2)[(y-120)/20]^2}.\end{aligned}$$

A graph of these two densities is given in the figure.

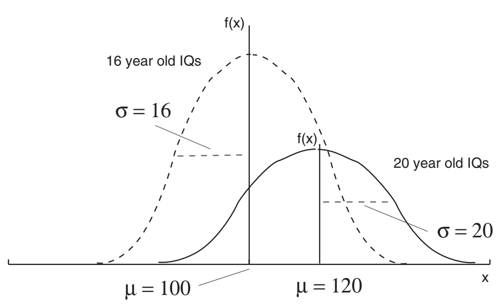

Figure 11.6: Normal distributions: IQ scores

(a) Mean IQ score for 20 year olds is
$\mu =$ (choose one) (i) **100** (ii) **120** (iii) **124** (iv) **136**.

(b) Average (or mean) IQ scores for 16 year olds is
$\mu =$ (choose one) (i) **100** (ii) **120** (iii) **124** (iv) **136**.

(c) Standard deviation in IQ scores for 20 year olds
$\sigma =$ (choose one) (i) **16** (ii) **20** (iii) **24** (iv) **36**.

(d) Standard deviation in IQ scores for 16 year olds is
$\sigma =$ (choose one) (i) **16** (ii) **20** (iii) **24** (iv) **36**.

(e) Normal density for 20 year old IQ scores is (choose one)
(i) **broader than normal density for 16 year old IQ scores.**
(ii) **as wide as normal density for 16 year old IQ scores.**
(iii) **narrower than normal density for 16 year old IQ scores.**

(f) Normal density for the 20 year old IQ scores is (choose one)
(i) **shorter than normal density for 16 year old IQ scores.**
(ii) **as tall as normal density for 16 year old IQ scores.**
(iii) **taller than normal density for 16 year old IQ scores.**

(g) Total area (probability) under normal density for 20's IQ scores is
(i) **smaller than area under normal density for 16's IQ scores.**
(ii) **the same as area under normal density for 16's IQ scores.**
(iii) **larger than area under normal density for 16's IQ scores.**

(h) Number of different normal densities: (choose one)
(i) **one** (ii) **two** (iii) **three** (iv) **infinity**.

5. *Percentages: IQ scores.*
Densities of IQ scores for 16 year olds, Y_1, and 20 year olds, Y_2, are given by

$$\begin{aligned} f(y_1) &= \frac{1}{16\sqrt{2\pi}} e^{-(1/2)[(y-100)/16]^2}, \\ f(y_2) &= \frac{1}{20\sqrt{2\pi}} e^{-(1/2)[(y-120)/20]^2}. \end{aligned}$$

(a) For the sixteen year old normal distribution, where $\mu = 100$ and $\sigma = 16$,

$$P(Y_1 < 84) = \int_{-\infty}^{84} \frac{1}{16\sqrt{2\pi}} e^{-(1/2)[(y-100)/16]^2} \, dy_1 \approx$$

(choose one) (i) **−0.1587** (ii) **0.1587** (iii) **0.3587** (iv) **0.8413**.

(2nd DISTR 2:normalcdf(− 2nd EE 99, 84, 100, 16) Notice the normalcdf function has four arguments: normalcdf(low, high, μ, σ). In this case, the "low" number is "− 2nd EE 99" and approximates negative infinity. The "high" number is 84. Finally, this is a nonstandard normal, where the μ and σ are 100 and 16, respectively..

(b) $P(Y_1 < 100) =$ (choose one)
(i) **0.4413** (ii) **0.5000** (iii) **0.6587** (iv) **0.8413**.
(2nd DISTR 2:normalcdf(− 2nd EE 99, 100, 100, 16).)

(c) $P(84 < Y_1 < 100) =$ (choose one)
(i) **0.3413** (ii) **0.4901** (iii) **0.5587** (iv) **0.7413**.
(2nd DISTR 2:normalcdf(84, 100, 100, 16).)

(d) For the *twenty* year old normal distribution, where $\mu = 120$ and $\sigma = 20$, $P(84 < Y_2 < 100) =$ (choose one)
(i) **0.0413** (ii) **0.1227** (iii) **0.3597** (iv) **0.5413**.
(2nd DISTR 2:normalcdf(84, 100, 120, 20).)

(e) Consider the following table of probabilities and possible values of probabilities. Use the figure.

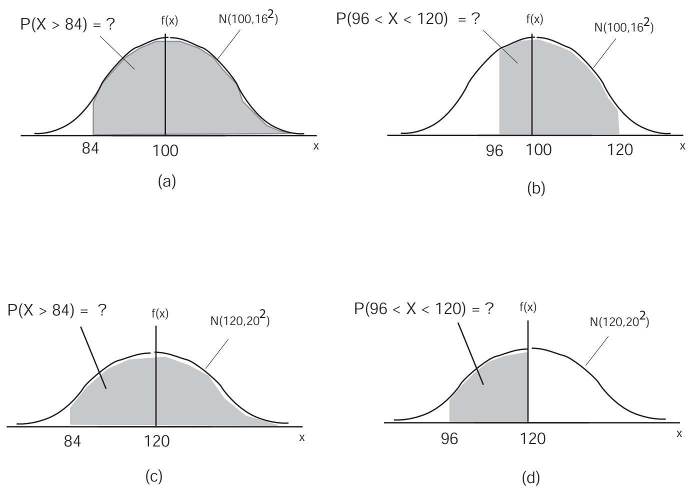

Figure 11.7: Normal probabilities: IQ scores

Column I	Column II
(a) $P(Y_1 > 84) \approx$ (2nd DISTR 2:normalcdf(84, 2nd EE 99, 100, 16))	**(a)** 0.4931
(b) $P(96 < Y_1 < 120) \approx$ (2nd DISTR 2:normalcdf(96, 120, 100, 16))	**(b)** 0.9641
(c) $P(Y_2 > 84) \approx$ (2nd DISTR 2:normalcdf(84, 2nd EE 99, 120, 20))	**(c)** 0.8413
(d) $P(96 < Y_2 < 120) \approx$ (2nd DISTR 2:normalcdf(96, 120, 120, 20))	**(d)** 0.3849

Using your calculator and the figure above, match the four items in column I with the items in column II.

Column I	(a)	(b)	(c)	(d)
Column II				

6. *Standard normal.*
Normal densities of IQ scores for 16 year olds, Y_1, and 20 year olds, Y_2, are

given by

$$f(y_1) = \frac{1}{16\sqrt{2\pi}} e^{-(1/2)[(y-100)/16]^2},$$
$$f(y_2) = \frac{1}{20\sqrt{2\pi}} e^{-(1/2)[(y-120)/20]^2}.$$

Both densities may be transformed to a *standard* normal with $\mu = 0$ and $\sigma = 1$ using the following equation,

$$Z = \frac{Y - \mu}{\sigma}.$$

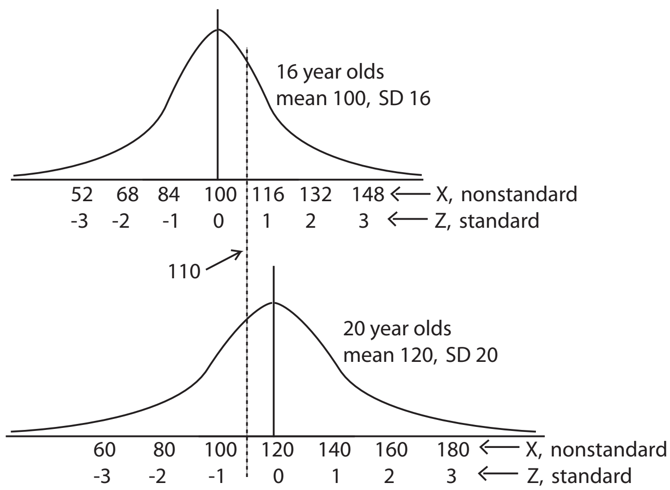

Figure 11.8: Standard normal and (nonstandard) normal

(a) Since $\mu = 100$ and $\sigma = 16$, a 16 year old who has an IQ of 132 is $z = \frac{132-100}{16} =$ (choose one) (i) **0** (ii) **1** (iii) **2** (iv) **3** standard deviations above the mean IQ, $\mu = 100$.

(b) A 16 year old who has an IQ of 84 is $z = \frac{84-100}{16} =$ (choose one) (i) $\mathbf{-2}$ (ii) $\mathbf{-1}$ (iii) **0** (iv) **1** standard deviations below the mean IQ, $\mu = 100$.

(c) Since $\mu = 120$ and $\sigma = 20$, a 20 year old who has an IQ of 180 is $z = \frac{180-120}{20} =$ (choose one) (i) **0** (ii) **1** (iii) **2** (iv) **3** standard deviations above the mean IQ, $\mu = 120$.

(d) A 20 year old who has an IQ of 100 is $z = \frac{100-120}{20} =$ (choose one) (i) $\mathbf{-3}$ (ii) $\mathbf{-2}$ (iii) $\mathbf{-1}$ (iv) **0** standard deviations below the mean IQ, $\mu = 120$.

(e) Although both the 20 year old and 16 year old scored the same, 110, on an IQ test, the 16 year old is clearly brighter relative to his/her age group than is the 20 year old relative his/her age group because

$$z_1 = \frac{110 - 100}{16} = 0.625 > z_2 = \frac{110 - 120}{20} = -0.5.$$

(i) **True** (ii) **False**

(f) The probability a 20 year old has an IQ greater than 90 is

$$P(Y_2 > 90) = P\left(Z_2 > \frac{90 - 120}{20}\right) = P\left(Z_2 > -1.5\right) =$$

(i) **0.93** (ii) **0.95** (iii) **0.97** (iv) **0.99**.

(nonstandard (x): 2nd DISTR 2:normalcdf(90, 2nd EE 99, 120, 20)

or standard (z): 2nd DISTR 2:normalcdf($\frac{90-120}{20}$, 2nd EE 99, 0, 1).)

(g) The probability a 20 year old has an IQ between 125 and 135 is

$$P(125 < Y_2 < 135) = P\left(\frac{125 - 120}{20} < Z_2 < \frac{135 - 120}{20}\right) = P\left(0.25 < Z_2 < 0.75\right) =$$

(i) **0.13** (ii) **0.17** (iii) **0.27** (iv) **0.31**.

(nonstandard (x): 2nd DISTR 2:normalcdf(125, 135, 120, 20)

or standard (z): 2nd DISTR 2:normalcdf $\left(\frac{125-120}{20}, \frac{135-120}{20}, 0, 1\right)$.)

(h) If a normal random variable Y with mean μ and standard deviation σ can be transformed to a standard one Z with mean $\mu = 0$ and standard deviation $\sigma = 1$ using

$$Z = \frac{Y - \mu}{\sigma},$$

then Z can be transformed to Y using

$$Y = \mu + \sigma Z.$$

(i) **True** (ii) **False**

(i) A 16 year old who has an IQ which is three (3) standards above the mean IQ has an IQ of $y_1 = 100 + 3(16) =$ (choose one)
(i) **116** (ii) **125** (iii) **132** (iv) **148**.

(j) A 20 year old who has an IQ which is two (2) standards below the mean IQ has an IQ of $y_2 = 120 - 2(20) =$ (choose one)
(i) **60** (ii) **80** (iii) **100** (iv) **110**.

(k) A 20 year old who has an IQ which is 1.5 standards below the mean IQ has an IQ of $y_2 = 120 - 1.5(20) =$ (choose one)
(i) **60** (ii) **80** (iii) **90** (iv) **95**.

(l) The probability a 20 year old has an IQ greater than one (1) standard deviation above the mean is

$$P(Z_2 > 1) = P\left(Y_2 > 120 + 1(20)\right) = P\left(Y_2 > 140\right) =$$

(choose one) (i) **0.11** (ii) **0.13** (iii) **0.16** (iv) **0.18**.

(standard (z): 2nd DISTR 2:normalcdf(1, 2nd EE 99, 0, 1)

or nonstandard (x): 2nd DISTR 2:normalcdf($120 + 1(20)$, 2nd EE 99, 120, 20).)