Chapter 29

Repeated Measures and Related Measures

A repeated measures design involves a study where several or all treatments are applied to the same subject.

29.1 Elements of Repeated Measures Designs

Exercise 29.1 (Elements of Repeated Measures Designs)

1. *Single–factor with repeated measures on all treatments.*
   Five subjects are each subjected to three drugs; they receive the drugs in a random order.

subject →	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	
drug 1 (treatment $j = 1$)	5.90	5.92	5.91	5.89	5.88	$\overline{Y}_{1\cdot} \approx 5.90$
drug 2 (treatment $j = 2$)	5.50	5.50	5.50	5.49	5.50	$\overline{Y}_{2\cdot} = 5.50$
drug 3 (treatment $j = 3$)	5.01	5.00	4.99	4.98	5.02	$\overline{Y}_{3\cdot} \approx 5.00$

   This is a (choose one) **repeated measures / repeated observations** design because the same subject is measured repeatedly using different drugs[1].

2. *Repeated observations.*
   Five subjects are each subjected to three drugs; they receive the drugs in a random order.

[1]Each subject could also be thought of as a *block*; typically, though, a block would consist of *different* individuals but with some common characteristic, such as *age* or *health*, say.

subject →	1	2	3	4	5
drug 1 ($j = 1$)	5.90, 5.91	5.92, 5.90	5.91, 5.89	5.89, 5.91	5.88, 5.89
drug 2 ($j = 2$)	5.50, 5.49	5.50, 5.50	5.50, 5.49	5.49, 5.51	5.50, 5.50
drug 3 ($j = 3$)	5.01, 5.00	5.00, 5.00	4.99, 5.00	4.98, 4.99	5.02, 5.02

This is a (*choose none, one or more!*)
repeated measures / repeated observations
design because not only is the same subject measured repeatedly using different drugs but also each subject is observed twice for each drug.

3. *Single–factor without repeated measures on all treatments.*
   Fifteen (not five!) subjects are subjected to three drugs; the drugs are assigned to the subjects at random.

drug 1 (treatment $j = 1$)	5.90	5.92	5.91	5.89	5.88	$\bar{Y}_{1\cdot} \approx 5.90$
drug 2 (treatment $j = 2$)	5.50	5.50	5.50	5.49	5.50	$\bar{Y}_{2\cdot} = 5.50$
drug 3 (treatment $j = 3$)	5.01	5.00	4.99	4.98	5.02	$\bar{Y}_{3\cdot} \approx 5.00$

   This (choose one) **is / is not** a repeated measures design because the different subjects are measured using different drugs.

4. *Random Order of Repeated Measures.*
   Five subjects are each subjected to three drugs. A random assignment of drugs to subjects is most probably given by (choose one)

   (a) *Assignment A*

subject →	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$
treatment order ↓	drug 1	drug 1	drug 1	drug 1	drug 1
	drug 2	drug 2	drug 2	drug 2	drug 2
	drug 3	drug 3	drug 3	drug 3	drug 3

   (b) *Assignment B*

subject →	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$
treatment order ↓	drug 2	drug 3	drug 1	drug 2	drug 1
	drug 1	drug 2	drug 2	drug 1	drug 2
	drug 3	drug 1	drug 3	drug 3	drug 3

   The possible bias due to assigning the drugs to each subject in the same order is eliminated/reduced by assigning the drugs in a random order.

5. *Subjects random.*
   True / False
   Repeated measures are often used in behavioral and life sciences and, consequently, the subjects are treated as though they have been chosen at random from a larger population.

6. *Different arrangements of same data: 3–3 test scores.*
The following arrangements of STAT 514 test scores[2],

section, (A) j:	1			2			3		
test, (B) k:	1	2	3	1	2	3	1	2	3
instructor (subject) $i = 1$:	69	75	46	88	90	91	51	65	57
$i = 2$:	71	80	77	92	85	95	49	71	63

and[3]

	A_1 $(j = 1)$			A_2 $(j = 2)$			A_3 $(j = 3)$		
	$k = 1$	$k = 2$	$k = 3$	$k = 1$	$k = 2$	$k = 3$	$k = 1$	$k = 2$	$k = 3$
$i = 1$	69	75	46	88	90	91	92	85	95
$i = 2$	71	80	77	51	65	57	49	71	63

and

section	instructor (subject)	test		
		B_1 $(k = 1)$	B_2 $(k = 2)$	B_3 $(k = 3)$
A_1 $(j = 1)$	$i = 1$	69	75	46
	$i = 2$	71	80	77
A_2 $(j = 2)$	$i = 1$	88	90	91
	$i = 2$	92	85	95
A_3 $(j = 3)$	$i = 1$	51	65	57
	$i = 2$	49	71	63

are (choose one) **the same** / **different** data sets.
Call this the *3–3 test score* data set.

7. *Two–factor experiments with repeated measures on two factors: 3–3 test scores.*
Reconsider the 3–3 test score data set given above: test scores are given for three different tests and three different sections of STAT 514 taught by two different instructors (each instructor teaches all courses and all tests). In this case, the instructor (subject) is repeatedly measured on (choose one)

   (a) factor A, section

   (b) factor B, test

   (c) both factors A and B, section and test

[2]This arrangement tends to be used for models with repeated measurement on both factors.
[3]This arrangement tends to be used for models with repeated measurement on both factors.

8. *Random order of repeated measures on two factors: 3–3 test scores.* Reconsider the 3–3 test score data set given above. A random assignment of the *order* of the treatments to instructors (subjects) is most probably given by (choose one)

   (a) *Assignment A*

subject →	1	2
treatment order ↓	A_1B_1	A_1B_1
	A_2B_1	A_2B_1
	A_3B_1	A_3B_1
	A_1B_2	A_1B_2
	A_2B_2	A_2B_2
	A_3B_2	A_3B_2
	A_1B_3	A_1B_3
	A_2B_3	A_2B_3
	A_3B_3	A_3B_3

   (b) *Assignment B*

subject →	1	2
treatment order ↓	A_1B_1	A_2B_2
	A_3B_1	A_2B_1
	A_2B_1	A_3B_1
	A_1B_2	A_1B_2
	A_2B_2	A_1B_1
	A_3B_2	A_3B_2
	A_1B_3	A_1B_3
	A_2B_3	A_2B_3
	A_3B_3	A_3B_3

   The possible bias due to assigning the treatments to each instructor (subject) in the same order is eliminated/reduced by assigning them in a random order.

9. *More two–factor with repeated measures on both factors: 3–2 test scores.* Consider the *3–2 test score* data set where, as before, each instructor teaches all courses and all tests.

section, (Factor A) j:	1		2		3	
test, (Factor B) k:	1	2	1	2	1	2
instructor (subject) $i = 1$:	69	71	88	92	51	49
$i = 2$:	75	80	90	85	65	71
$i = 3$:	46	77	91	95	57	63

   The instructor (subject) is repeatedly measured on (choose one)

(a) factor A, section

(b) factor B, test

(c) both factors A and B, section and test

10. *More random order of repeated measures on two factors: 3–2 test scores.* Reconsider the 3–2 test score data set given above. A random assignment of the *order* of the treatments to instructors (subjects) is most probably given by (choose one)

(a) *Assignment A*

subject →	1	2	3
treatment order ↓	A_1B_1	A_1B_1	A_1B_1
	A_2B_1	A_2B_1	A_2B_1
	A_3B_1	A_3B_1	A_3B_1
	A_1B_2	A_1B_2	A_1B_2
	A_2B_2	A_2B_2	A_2B_2
	A_3B_2	A_3B_2	A_3B_2

(b) *Assignment B*

subject →	1	2	3
treatment order ↓	A_1B_1	A_2B_2	A_3B_1
	A_3B_1	A_2B_1	A_2B_1
	A_2B_1	A_3B_1	A_1B_1
	A_1B_2	A_1B_2	A_3B_2
	A_2B_2	A_1B_1	A_2B_2
	A_3B_2	A_3B_2	A_1B_2

The possible bias due to assigning the treatments to each test (subject) in the same order is eliminated/reduced by assigning them in a random order.

11. *Two–factor experiments with repeated measures on one factor: 3–2 test scores.* The test scores for *nine*[4] different instructors is given below.

[4]There is not three different instructors, as seems to be suggested by the notation!

section	instructor (subject)	test	
		B_1 ($k = 1$)	B_2 ($k = 2$)
	$i = 1$	69	71
A_1 ($j = 1$)	$i = 2$	75	80
	$i = 3$	46	77
	$i = 1$	88	92
A_2 ($j = 2$)	$i = 2$	90	85
	$i = 3$	91	95
	$i = 1$	51	49
A_3 ($j = 3$)	$i = 2$	65	71
	$i = 3$	57	63

The instructor (subject) is repeatedly measured on (choose one)

(a) factor A, section
(b) factor B, test
(c) both factors A and B, section and test

and, notice, is *nested* inside factor A, section. In other words, instructor is repeatedly measured on one factor and nested inside the other.

12. *Split plot design same as two–factor with repeated measures on one factor.* The wheat yields for three irrigation methods, two fertilizers and *nine*[5] fields are given below. On the one hand, each irrigation method is applied to a *whole* field (whole plot). On the other hand, each fertilizer is applied to half of a field (split plot); that is, fertilizer 1 is applied to one half of each field and fertilizer 2 is applied to the other half of each field in this case.

irrigation	field	fertilizer	
		B_1 ($k = 1$)	B_2 ($k = 2$)
	$i = 1$	69	71
A_1 ($j = 1$)	$i = 2$	75	80
	$i = 3$	46	77
	$i = 1$	88	92
A_2 ($j = 2$)	$i = 2$	90	85
	$i = 3$	91	95
	$i = 1$	51	49
A_3 ($j = 3$)	$i = 2$	65	71
	$i = 3$	57	63

The split–plot design is just an example of a two–factor with repeated measures on one factor design where, in this case, field is repeatedly measured for (choose one)

[5] Although implied by the notation, there are *nine*, not three, different instructors!

(a) factor A, irrigation

(b) factor B, fertilizer

(c) both factors A and B, irrigation and fertilizer

and, notice, is nested inside factor A, irrigation.

13. *More two–factor with repeated measures on one factor: 2–3 test scores.* The test scores for *six*[6] different instructors is given below. Call this the *2–3 test score* data set.

section	instructor (subject)	test		
		B_1 $(k = 1)$	B_2 $(k = 2)$	B_3 $(k = 3)$
	$i = 1$	69	88	51
A_1 $(j = 1)$	$i = 2$	75	90	65
	$i = 3$	46	91	57
	$i = 1$	71	92	49
A_2 $(j = 2)$	$i = 2$	80	85	71
	$i = 3$	77	95	63

The instructor (subject) is repeatedly measured on (choose one)

(a) factor A, section

(b) factor B, test

(c) both factors A and B, section and test

and, notice, is *nested* inside factor A, section.

29.2 Single–Factor Experiments with Repeated Measures on All Treatments

SAS program: att11-29-2-drugs-repeated-one

Exercise 29.2 (Single–Factor Repeated Measures on All Treatments) Five subjects are each subjected to three drugs, given in a random order.

subject →	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	
drug 1 (treatment $j = 1$)	5.90	5.92	5.91	5.89	5.88	$\overline{Y}_{1\cdot} \approx 5.90$
drug 2 (treatment $j = 2$)	5.50	5.50	5.50	5.49	5.50	$\overline{Y}_{2\cdot} = 5.50$
drug 3 (treatment $j = 3$)	5.01	5.00	4.99	4.98	5.02	$\overline{Y}_{3\cdot} \approx 5.00$

[6]There is not three different instructors, as seems to be suggested by the notation!

1. *Single–factor with repeated measures model*
Using the following repeated measures model, match appropriately,

$$Y_{ij} = \mu_{\cdot\cdot} + \rho_i + \tau_j + \varepsilon_{ij}$$

model	example
(a) Y_{ijk}	**(a)** error
(b) $\mu_{\cdot\cdot}$	**(b)** subject (random) effect
(c) ρ_i	**(c)** drug (treatment, fixed) effect
(d) τ_j	**(d)** (grand) average of all responses
(e) ε_{ijk}	**(e)** individual response

model	(a)	(b)	(c)	(d)	(e)
example					

2. *Residuals*
From SAS,
$e_{ij} = e_{12} =$ (choose one) **−0.0002** / **−0.034** / **−0.43**.
The residuals versus fitted values plot appears to be randomly scattered and so this indicates (choose one) **constant** / **non–constant** variance.
The normal probability plot appears to be a straight line and so this indicates (choose one) **normality** / **non–normality**.

3. *Treatment Plot*
True / **False**
From SAS, the response increases from drug 1, to 2 to 3 and, although it is difficult to tell, there appears to be an interaction between subjects and drugs, but it does not appear to be significant.

4. *ANOVA Table.*
The ANOVA table is given by,

Source	*df*	*SS*	*MS*	$E\{MS\}$
Subjects	$n-1$	*SSS*	*MSS*	$\sigma^2 + r\sigma_\rho^2$
Treatments	$r-1$	*SSTR*	*MSTR*	$\sigma^2 + n\frac{\sum \tau_j^2}{r-1}$
Error	$(n-1)(r-1)$	*SSTR.S*	*MSTR.S*	σ^2
Total	$nr-1$	*SSTO*		

where, in this case,

Source	*df*	*SS*	*MS*
Subject	4	0.00069	0.000173
Drug (Treatment)	2	2.033	1.01634
Error	8	0.00139	0.000173
Total	14	?	

where $SSTO =$ (choose one) **0.0002 / 1.034 / 2.0347**.

5. *Test of drug (treatment).*
$H_0 : \tau_i = 0$ versus
$H_a :$ at least one $\tau_i \neq 0,\ i = 1, 2, 3.$
since p–value $P(F > \frac{MSTR}{MSTR.S} = \frac{1.01634}{0.000173} = 5875; 2, 8) = 0 < \alpha = 0.05$
reject null; that is, the drug effect (choose one) **is / is not** significant
(mean responses different for different drugs)

6. *Pairwise Comparison, Bonferroni.*
Estimate $L_1 = \mu_{\cdot 1} - \mu_{\cdot 2}$ using the Bonferroni procedure with a 95 percent family confidence coefficient where this is one of $g = 3$ contrasts (and, of course, there are $n = 5$ subjects).

   Using SAS and the ANOVA table above,
$\hat{L} = \overline{Y}_{\cdot 1} - \overline{Y}_{\cdot 2} = 5.9 - 5.498 = 0.402,$
$s\{\hat{L}\} = \sqrt{MSTR.S\left(\frac{1}{n} + \frac{1}{n}\right)} = \sqrt{0.000173\left(\frac{1}{5} + \frac{1}{5}\right)} \approx 0.0083187$
$B = t(1 - \frac{\alpha}{2g}; df_{SSTR.S}) = t(1 - \frac{0.05}{2(3)}; 8) = t(0.9917; 8) \approx 3.0184$
and so the CI is
$0.402 \pm 3.0184(0.0083187) \approx$ (choose one)
$\mathbf{0.379 \leq L_1 \leq 0.425}$ / $\mathbf{0.377 \leq L_1 \leq 0.427}$ / $\mathbf{-7.94 \leq L_1 \leq -5.66}$

7. *Efficiency*

$$\begin{aligned} \hat{E} &= \frac{(n-1)MSS + n(r-1)MSTR.S}{(nr-1)MSTR.S} \\ &= \frac{(5-1)(0.000173) + 5(3-1)(0.000173)}{(5(3)-1)(0.000173)} \\ &= \end{aligned}$$

   0.857 / 0.957 / 1.000
Exactly the same number of replications would be required for a completely randomized design as are used in a comparable repeated measures design to achieve the same error variance. In other words, the repeated measures design is not any better than a comparable completely randomized design in this case.

29.3 Two–Factor Experiments with Repeated Measures on Both Factors

SAS program: att11-29-3-teach-repeated-twotwo

Exercise 29.3 (Two–Factor . . . Repeated Measures on Both Factors)
Reconsider the *3–2 test score* data set where, as before, each instructor teaches all courses and all tests. That is, the instructor (subject) is repeatedly measured on both section and instructor factors.

section, (Factor A) j:	1		2		3	
test, (Factor B) k:	1	2	1	2	1	2
instructor (subject) $i = 1$:	69	71	88	92	51	49
$i = 2$:	75	80	90	85	65	71
$i = 3$:	46	77	91	95	57	63

1. *Two–factor with repeated measures on both factors model*
   Using the following two–factor with repeated measures on both factors model, match appropriately,

$$Y_{ijk} = \mu_{\dots} + \rho_i + \alpha_j + \beta_k + (\alpha\beta)_{jk} + \varepsilon_{ijk}$$

model	example
(a) Y_{ijk}	**(a)** error
(b) $\mu_{\dots}$	**(b)** subject (random) effect
(c) ρ_i	**(c)** interaction (treatment, random) effect
(d) α_j	**(d)** (grand) average of all responses
(e) β_k	**(e)** section (treatment, fixed) effect
(f) $(\alpha\beta)_{jk}$	**(f)** test (treatment, fixed) effect
(g) ε_{ijk}	**(g)** individual response

model	(a)	(b)	(c)	(d)	(e)	(f)	(g)
example							

2. *Residuals*
   From SAS,
   $e_{ijk} = e_{231} =$ (choose one) **−0.0002 / 1.034 / 2.7222**.
   The residuals versus fitted values plot appears to be randomly scattered and so this indicates (choose one) **constant / non–constant** variance.
   The normal probability plot appears to be a straight line and so this indicates (choose one) **normality / non–normality**.

3. *ANOVA Table.*
   The ANOVA table is given by,

Source	*df*	*SS*	*MS*	$E\{MS\}$
Subjects	$n-1$	*SSS*	*MSS*	$\sigma^2 + ab\sigma_\rho^2$
Factor A	$a-1$	*SSA*	*MSA*	$\sigma^2 + nb\frac{\sum \alpha_j^2}{a-1}$
Factor B	$b-1$	*SSB*	*MSB*	$\sigma^2 + na\frac{\sum \beta_k^2}{b-1}$
AB interactions	$(a-1)(b-1)$	*SSAB*	*MSAB*	$\sigma^2 + n\frac{\sum\sum (\alpha\beta)_{jk}}{(a-1)(b-1)}$
Error	$(n-1)(ab-1)$	*SSTR.S*	*MSTR.S*	σ^2
Total	$abn-1$	*SSTO*		

True / **False** where, in this case,

Source	*df*	*SS*	*MS*
Instructor (Subject)	2	198.11	99.06
Section (Factor A)	2	2955.44	1477.72
Test (Factor B)	1	144.5	144.5
Interaction (AB Interaction)	2	114.33	57.167
Error	10	716.55	71.66
Total	17	4128.94	

4. *Test if interaction is significant at* $\alpha = 0.05$.
   $H_0 : \alpha\beta_{jk} = 0$ versus
   $H_a :$ at least one $\alpha\beta_{jk} \neq 0,\ j = 1,2,3; k = 1,2$.
   since p–value $P(F > \frac{MSAB}{MSTR.S} = \frac{57.167}{71.66} = 0.798; 2, 10) = 0.48 > \alpha = 0.05$
   accept null; that is, there (choose one) **is an** / **is no** interaction effect[7]

5. *Test if section (factor A) is significant at* $\alpha = 0.05$.
   $H_0 : \alpha_j = 0$ versus
   $H_a :$ at least one $\alpha_j \neq 0,\ j = 1,2,3$.
   since p–value $P(F > \frac{MSA}{MSTR.S} = \frac{1477.72}{71.66} = 20.62; 2, 10) = 0.00028 < \alpha = 0.05$
   reject null; that is, there (choose one) **is an** / **is no** section effect
   (the mean test scores are different for different sections)

6. *Pairwise comparison of two tests in first section, bonferroni.*
   Estimate $L = \mu_{\cdot 11} - \mu_{\cdot 12}$ using the Bonferroni procedure with a 95 percent family confidence coefficient where this is one of $g = 3$ contrasts and there are $n = 3$ instructors per test in the first section.

   Using SAS and the ANOVA table above,
   $\hat{L} = \overline{Y}_{\cdot 11} - \overline{Y}_{\cdot 12} = 63.33 - 76.00 = -12.67$,
   $s\{\hat{L}\} = \sqrt{MSTR.S\left(\frac{1}{n} + \frac{1}{n}\right)} = \sqrt{71.66\left(\frac{1}{3} + \frac{1}{3}\right)} \approx 6.912$
   $B = t(1 - \frac{\alpha}{2g}; df_{MSTR.S}) = t(1 - \frac{0.05}{2(3)}; 10) = t(0.9917; 10) \approx 2.87$

[7] This allows us to test the main effects.

and so the CI is
$-12.67 \pm 2.87(6.912) \approx$ (choose one)
$\boldsymbol{-32.51 \leq L_1 \leq 7.17}$ / $\boldsymbol{-32.44 \leq L_1 \leq 7.24}$ / $\boldsymbol{-7.94 \leq L_1 \leq -5.66}$

7. *Test if instructor (subject) effect is significant.*
   $H_0 : \sigma_\rho^2 = 0$ versus $H_a : \sigma_\rho^2 > 0$.
   since $P(F > \frac{MSS}{MSTR.S} = \frac{99.06}{71.66} = 1.38; 2, 10) = 0.295 > \alpha = 0.01$
   accept null; that is, there the instructor effect (choose one) **is** / **is not** significant (there is no difference in the mean test scores between instructors)

29.4 Two–Factor Experiments with Repeated Measures on One Factor

SAS program: att11-29-4-teach-repeated-onetwo

Exercise 29.4 (Two–Factor . . . Repeated Measures on One Factor)
Reconsider the *3–2 test score* data set above, where, this time, there are *nine* instructors where each instructor is repeatedly measured for the test factor, factor B (and, consequently, nested inside the section factor, factor A).

section	instructor (subject)	test	
		B_1 $(k=1)$	B_2 $(k=2)$
	$i=1$	69	71
A_1 $(j=1)$	$i=2$	75	80
	$i=3$	46	77
	$i=1$	88	92
A_2 $(j=2)$	$i=2$	90	85
	$i=3$	91	95
	$i=1$	51	49
A_3 $(j=3)$	$i=2$	65	71
	$i=3$	57	63

1. *Two–factor with repeated measures on one factor model*
   Using the following two–factor with repeated measures on factor B (and nested in factor A) model, match appropriately,

$$Y_{ijk} = \mu_{\cdots} + \rho_{i(j)} + \alpha_j + \beta_k + (\alpha\beta)_{jk} + \varepsilon_{ijk}$$

model	example
(a) Y_{ijk}	**(a)** error
(b) $\mu_{\cdots}$	**(b)** subject (random, nested in section) effect
(c) $\rho_{i(j)}$	**(c)** interaction (treatment, random) effect
(d) α_j	**(d)** (grand) average of all responses
(e) β_k	**(e)** section (treatment, fixed) effect
(f) $(\alpha\beta)_{jk}$	**(f)** test (treatment, fixed) effect
(g) ε_{ijk}	**(g)** individual response

model	(a)	(b)	(c)	(d)	(e)	(f)	(g)
example							

2. *Residuals*
   From SAS, $e_{ij} = e_{332} =$ (choose one) **−0.0002** / **1.3333** / **2.7222**.
   The residuals versus fitted values plot appears to be randomly scattered and so this indicates (choose one) **constant** / **non–constant** variance.
   The normal probability plot appears to be a straight line and so this indicates (choose one) **normality** / **non–normality**.

3. *ANOVA Table.*
   The ANOVA table is given by,

Source	df	SS	MS	$E\{MS\}$
Factor A	$a-1$	SSA	MSA	$\sigma^2 + b\sigma_\rho^2 + nb\frac{\sum \alpha_j^2}{a-1}$
Subjects (within Factor A)	$a(n-1)$	$SSS(A)$	$MSS(A)$	$\sigma^2 + b\sigma_\rho^2$
Factor B	$b-1$	SSB	MSB	$\sigma^2 + na\frac{\sum \beta_k^2}{b-1}$
AB interactions	$(a-1)(b-1)$	$SSAB$	$MSAB$	$\sigma^2 + n\frac{\sum\sum (\alpha\beta)_{jk}}{(a-1)(b-1)}$
Error (B×S(A))	$(n-1)(ab-1)$	$SSTR.S(A)$	$MSTR.S(A)$	σ^2
Total	$abn-1$	$SSTO$		

True / **False** where

Source	df	SS	MS
Section (Factor A)	2	2955.44	1477.72
Test(Section)	6	612.0	102
Instructor (Factor B)	1	144.5	144.5
Interaction (AB Interaction)	2	114.33	57.167
Error (Instructor×Test(Section))	6	302.67	50.44
Total	17	4128.94	

4. *Test if interaction AB is significant at* $\alpha = 0.05$.
   $H_0 : \alpha\beta_{jk} = 0$ versus
   $H_a :$ at least one $\alpha\beta_{jk} \neq 0,\ j = 1, 2, 3; k = 1, 2$.
   since p–value $P(F > \frac{MSAB}{MSTR.S(A)} = \frac{57.167}{50.44} = 1.133; 2, 6) = 0.38 > \alpha = 0.05$
   accept null; that is, there (choose one) **is an** / **is no** interaction effect[8]

[8]This allows us to test the main effects.

5. *Test if section (factor A) is significant at* $\alpha = 0.05$.
   $H_0 : \alpha_j = 0$ versus
   $H_a :$ at least one $\alpha_j \neq 0,\ j = 1, 2, 3$.
   since p–value $P(F > \frac{MSA}{MSS(A)} = \frac{1477.72}{102} = 14.49; 2, 6) = 0.005 < \alpha = 0.05$
   reject null; that is, there (choose one) **is an / is no** section effect
   (the mean test scores are different for different sections)

6. *Test if test (factor B) is significant at* $\alpha = 0.05$.
   $H_0 : \beta_k = 0$ versus
   $H_a :$ at least one $\beta_k \neq 0,\ j = 1, 2$.
   since p–value $P(F > \frac{MSB}{SSTR.S(A)} = \frac{144.5}{50.44} = 2.86; 1, 6) = 0.14 > \alpha = 0.05$
   accept null; that is, there (choose one) **is an / is no** section effect
   (the mean test scores are same for different tests)

7. *Pairwise comparison of test scores in first two sections, bonferroni.*
   Estimate $L = \mu_{.1.} - \mu_{.2.}$ using the Bonferroni procedure with a 95 percent family confidence coefficient where this is one of $g = 2$ contrasts and there are $n = 6$ instructors in each section.

   Using the SAS output and the ANOVA table above,
   $\hat{L} = \overline{Y}_{.1.} - \overline{Y}_{.2.} = 69.67 - 90.17 = -20.5$,
   $s\{\hat{L}\} = \sqrt{MSS(A)\left(\frac{1}{n} + \frac{1}{n}\right)} = \sqrt{\frac{2(102)}{6}} \approx 5.83$
   $B = t(1 - \frac{\alpha}{2g}; df) = t(1 - \frac{0.05}{2(2)}; 6) = t(0.9875; 6) \approx 2.97$
   and so the CI is
   $-20.5 \pm 2.97(5.83) \approx$ (choose one)
   $\mathbf{-37.81 \leq L_1 \leq -3.18}$ / $\mathbf{-32.44 \leq L_1 \leq 7.24}$ / $\mathbf{-7.94 \leq L_1 \leq -5.66}$

29.5 Regression Approach to Repeated Measures Designs

The regression approach is, as discussed previously, used for *un*balanced designs, when there, for example, missing observations.

29.6 Split–Plot Designs

SAS program: att11-29-6-field-splitplot

A split–plot design is an example of two–factor experiments with repeated measures on one factor.

Exercise 29.5 (Split–Plot Designs)
The wheat yields for three irrigation methods, two fertilizers and *nine* fields are given below. On the one hand, each irrigation method is applied to a *whole* field (whole plot). On the other hand, each fertilizer is applied to half of a field (split plot); that is, fertilizer 1 is applied to one half of each field and fertilizer 2 is applied to the other half of each field in this case.

irrigation	field	fertilizer	
		B_1 $(k=1)$	B_2 $(k=2)$
	$i=1$	69	71
A_1 $(j=1)$	$i=2$	75	80
	$i=3$	46	77
	$i=1$	88	92
A_2 $(j=2)$	$i=2$	90	85
	$i=3$	91	95
	$i=1$	51	49
A_3 $(j=3)$	$i=2$	65	71
	$i=3$	57	63

1. *Split–plot design*
   True / False
   The split plot design is identical to the two–factor with repeated measures on factor B (and nested in factor A) model,

$$Y_{ijk} = \mu_{\dots} + \rho_{i(j)} + \alpha_j + \beta_k + (\alpha\beta)_{jk} + \varepsilon_{ijk}$$

2. *ANOVA Table.*
   The ANOVA table is given by,

Source	*df*	*SS*	*MS*	$E\{MS\}$
Factor A	$a-1$	SSA	MSA	$\sigma^2 + b\sigma_\rho^2 + nb\frac{\sum \alpha_j^2}{a-1}$
Whole–plot error	$a(n-1)$	$SSW(A)$	$MSW(A)$	$\sigma^2 + b\sigma_\rho^2$
Factor B	$b-1$	SSB	MSB	$\sigma^2 + na\frac{\sum \beta_k^2}{b-1}$
AB interactions	$(a-1)(b-1)$	$SSAB$	$MSAB$	$\sigma^2 + n\frac{\sum\sum (\alpha\beta)_{jk}}{(a-1)(b-1)}$
Split–plot error	$(n-1)(ab-1)$	$SSTR.W(A)$	$MSTR.W(A)$	σ^2
Total	$abn-1$	$SSTO$		

   True / False where

Source	*df*	*SS*	*MS*
Irrigation (Factor A)	2	2955.44	1477.72
Whole–plot error	6	612.0	102
Instructor (Factor B)	1	144.5	144.5
Interaction (AB Interaction)	2	114.33	57.167
Split–plot error	6	302.67	50.44
Total	17	4128.94	

3. *Why split–plot designs?*
   True / False
   The split–plot design is useful here because it may be difficult (impossible) to quickly adjust the irrigation method (involving a system of hoses, values and pipes and so on) to fit on different sized field, whereas it is easy to apply the different fertilizer methods over different plot sizes. In general, split–plot designs are used when one factor requires larger experimental units than another.