Chapter 6

Applications of the Derivative

We continue to look at applications of the derivative, in particular *related rates*, and the techniques associated with these applications, such as identifying *absolute extrema*, and *implicit differentiation*.

6.1 Absolute Extrema

Function f at c, $f(c)$, in an interval has an

absolute maximum	if $f(x) \leq f(c)$,
absolute minimum	if $f(x) \geq f(c)$,

for all x in the interval. Technique for identifying absolute extrema depends on whether the interval is open or closed. For *closed* intervals, the *extreme value theorem* is used to identify absolute extrema. This theorem says a continuous function f on a closed interval $[a, b]$ will/must have both an absolute maximum and absolute minimum. Consequently, in the closed interval case,

1. evaluate f all critical numbers in (a, b),
2. evaluate f at endpoints a and b of $[a, b]$,
3. largest f is absolute maximum; smallest is absolute minimum.

If f is defined on an *open* interval (a, b), evaluate the *limit* of f as it approaches the endpoints; there is no absolute extrema if the limit is $\pm\infty$. In the special case when there is only *one* critical number c, the *critical point theorem* says for function f defined on (either open or closed) interval I,

if f has relative minimum at $x = c$	this relative minimum is an absolute minimum,
if f has relative maximum at $x = c$	this relative maximum is an absolute maximum.

Although treated in a similar manner, allowances are made for identifying absolute extrema for functions with discontinuities. A related question of finding the maximum of $g(x) = \frac{f(x)}{x}$ is given by finding x such that $f'(x) = \frac{f(x)}{x}$, in other words, finding the x where slope of tangent, $f'(x)$ *equals* slope of line from origin to point x.

Exercise 6.1 (Absolute Extrema)

1. *Extrema candidate points, endpoints and absolute extrema.*

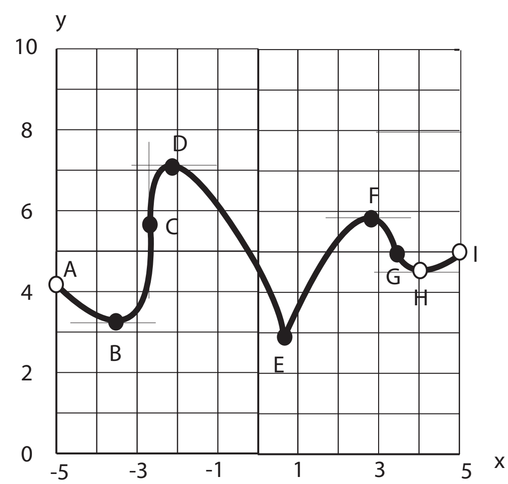

Figure 6.1 (Extrema candidate points, endpoints and absolute extrema)

(a) Endpoint(s) at (choose one or more)
(i) **A** (ii) **B** (iii) **C** (iv) **D** (v) **E**
(vi) **F** (vii) **G** (viii) **H** (ix) **I** (x) **none**
because function is defined on an *open* interval $(-5, 5)$ and so there are *no* endpoints

(b) Extrema candidate points(s) at (choose one or more)
(i) **A** (ii) **B** (iii) **C** (iv) **D** (v) **E**
(vi) **F** (vii) **G** (viii) **H** (ix) **I** (x) **none**
but not removable discontinuity point H because the function does not exist at this point

(c) Relative minimum at (choose one or more)
(i) **A** (ii) **B** (iii) **C** (iv) **D** (v) **E**
(vi) **F** (vii) **G** (viii) **H** (ix) **I** (x) **none**

(d) Absolute minimum at (choose one)
(i) **A** (ii) **B** (iii) **C** (iv) **D** (v) **E**
(vi) **F** (vii) **G** (viii) **H** (ix) **I** (x) **none**
because function value at point E is smallest of all the relative minima

(e) Relative maximum at (choose one or more)
(i) **A** (ii) **B** (iii) **C** (iv) **D** (v) **E**
(vi) **F** (vii) **G** (viii) **H** (ix) **I** (x) **none**

(f) Absolute maximum at (choose one)
(i) **A** (ii) **B** (iii) **C** (iv) **D** (v) **E**
(vi) **F** (vii) **G** (viii) **H** (ix) **I** (x) **none**
because function value at point D is largest of all the relative maxima

2. *More extrema candidate points, endpoints and absolute extrema.*

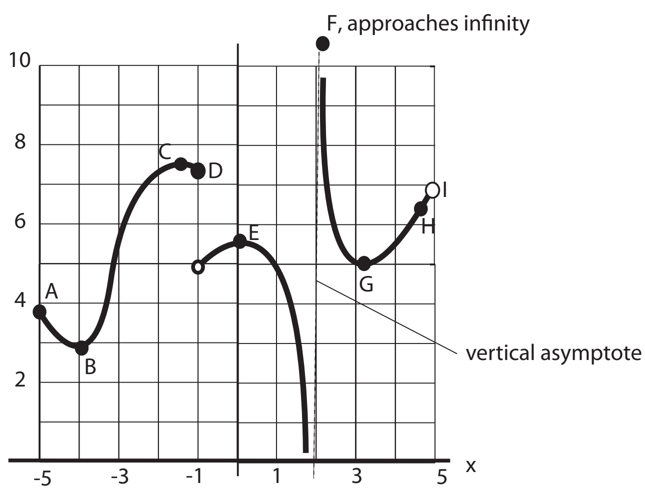

Figure 6.2 (More extrema candidate points, endpoints and absolute extrema)

(a) Endpoint(s) at (choose one or more)
(i) **A** (ii) **B** (iii) **C** (iv) **D** (v) **E**
(vi) **F** (vii) **G** (viii) **H** (ix) **I** (x) **none**
because function is defined on $[-5, 5), x \neq 2$

(b) Extrema candidate points(s) at (choose one or more)
(i) **A** (ii) **B** (iii) **C** (iv) **D** (v) **E**
(vi) **F** (vii) **G** (viii) **H** (ix) **I** (x) **none**
but not points D or F because
function is decreasing both before and after jump discontinuity point D
and, for point F, function approaches $\pm$ infinity, not a specific large number, when $x \to 2$

(c) Relative minimum at (choose one or more)
(i) **A** (ii) **B** (iii) **C** (iv) **D** (v) **E**
(vi) **F** (vii) **G** (viii) **H** (ix) **I** (x) **none**

(d) Absolute minimum at (choose one)
(i) **A** (ii) **B** (iii) **C** (iv) **D** (v) **E**
(vi) **F** (vii) **G** (viii) **H** (ix) **I** (x) **none**
because function value at point B is smallest of all the relative minima

(e) Relative maximum at (choose one or more)
(i) **A** (ii) **B** (iii) **C** (iv) **D** (v) **E**

(vi) **F** (vii) **G** (viii) **H** (ix) **I** (x) **none**

including endpoint A but not jump discontinuity point D

(f) Absolute maximum at (choose one)
(i) **A** (ii) **B** (iii) **C** (iv) **D** (v) **E**
(vi) **F** (vii) **G** (viii) **H** (ix) **I** (x) **none**

because function value at point C is larger than value of function at point E or endpoint A

3. *Absolute extrema:* $f(x) = 5x - 4$ *revisited*

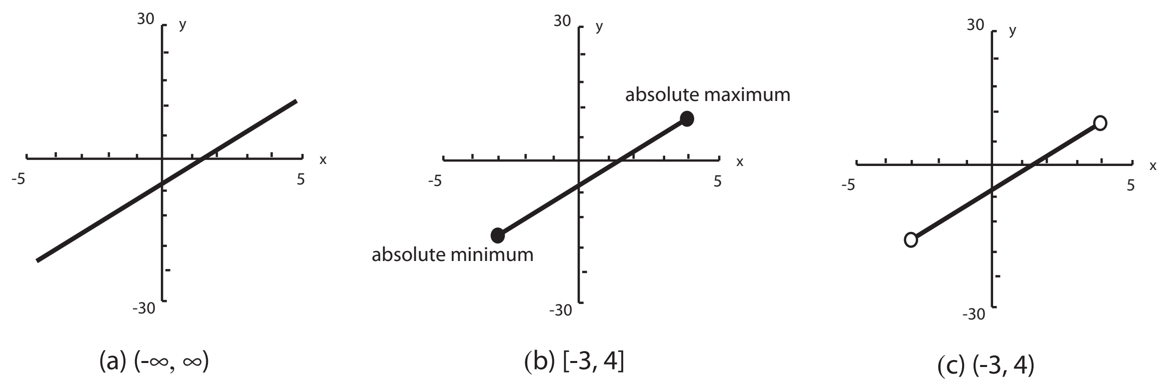

Figure 6.3 (Absolute extrema: $f(x) = 5x - 4$)

GRAPH using $Y_1 = 5x - 4$, with WINDOW -5 5 1 -30 30 1 1

(a) *Domain* $(-\infty, \infty)$.

i. *Critical numbers.*
Recall, since

$$f'(x) = 5(1)x^{1-1} = 5 > 0,$$

there (i) **is** (ii) **is no** critical number

because $f'(x) \neq 0$

ii. *Endpoints.*
lower endpoint. As $x \to -\infty$,

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} (5x - 4) =$$

(i) $-\boldsymbol{\infty}$ (ii) **0** (iii) $\boldsymbol{\infty}$

upper endpoint. As $x \to \infty$,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} (5x - 4) =$$

(i) $-\boldsymbol{\infty}$ (ii) **0** (iii) $\boldsymbol{\infty}$

iii. *Absolute extrema*
Since

candidate x	$f(x) = 5x - 4$
$x \to -\infty$	$f(x) \to -\infty$
$x \to \infty$	$f(x) \to \infty$

absolute minimum is
$f(x) =$ (i) **0** (ii) $\boldsymbol{-\infty}$ (iii) $\boldsymbol{\infty}$ (iv) **none**
because $f(x)$ approaches $-\infty$ rather than a particular number
and absolute maximum is
$f(x) =$ (i) **0** (ii) $\boldsymbol{-\infty}$ (iii) $\boldsymbol{\infty}$ (iv) **none**
because $f(x)$ grows arbitrarily large as $x \to \infty$

(b) *Domain: closed interval* $[-3, 4]$.

i. *Critical numbers.*
Recall, since

$$f'(x) = 5(1)x^{1-1} = 5 > 0,$$

there (i) **is** (ii) **is no** critical number
because $f'(x) \neq 0$

ii. *Endpoints.*
lower endpoint. At $x = -3$,
$f(-3) = 5(-3) - 4 =$ (i) $\boldsymbol{-\infty}$ (ii) $\boldsymbol{\infty}$ (iii) $\boldsymbol{-19}$
upper endpoint. At $x = 4$,
$f(4) = 5(4) - 4 =$ (i) **16** (ii) $\boldsymbol{-\infty}$ (iii) $\boldsymbol{\infty}$

iii. *Absolute extrema*
Since

candidate x	$f(x) = 5x - 4$
-3	-19
4	16

absolute minimum occurs at
$x =$ (i) $\boldsymbol{-\infty}$ (ii) $\boldsymbol{-3}$ (iii) $\boldsymbol{\infty}$ (iv) **none**
absolute maximum occurs at
$x =$ (i) **4** (ii) $\boldsymbol{-\infty}$ (iii) $\boldsymbol{\infty}$ (iv) **none**

iv. *Calculator check*
absolute minimum occurs at
$x =$ (i) $\boldsymbol{-3}$ (ii) $\boldsymbol{-\infty}$ (iii) $\boldsymbol{\infty}$ (iv) **none**
MATH FMin(ENTER $5X - 4$, X, -3, 4)
absolute maximum occurs at
$x =$ (i) **4** (ii) $\boldsymbol{-\infty}$ (iii) $\boldsymbol{\infty}$ (iv) **none**
MATH FMax(ENTER $5X - 4$, X, -3, 4)

(c) *Domain: open interval* $(-3, 4)$.

i. *Critical numbers.*
Recall, since
$$f'(x) = 5(1)x^{1-1} = 5 > 0,$$
there (i) **is** (ii) **is no** critical number
because $f'(x) \neq 0$

ii. *Endpoints.*
lower endpoint.
There (i) **is** (ii) **is no** lower endpoint
because $f(x)$ is defined on an *open interval* $(-3, 4)$

upper endpoint.
There (i) **is** (ii) **is no** upper endpoint
because $f(x)$ is defined on an *open interval* $(-3, 4)$

iii. *Absolute extrema*
Absolute minimum occurs at
$x =$ (i) $\mathbf{-3}$ (ii) $\mathbf{-\infty}$ (iii) $\mathbf{\infty}$ (iv) **none**
because there are no candidate extrema to choose from
Absolute maximum occurs at
$x =$ (i) **4** (ii) $\mathbf{-\infty}$ (iii) $\mathbf{\infty}$ (iv) **none**
because there are no candidate extrema to choose from

4. *Absolute extrema:* $f(x) = x^2 + 4x - 21$ *revisited*

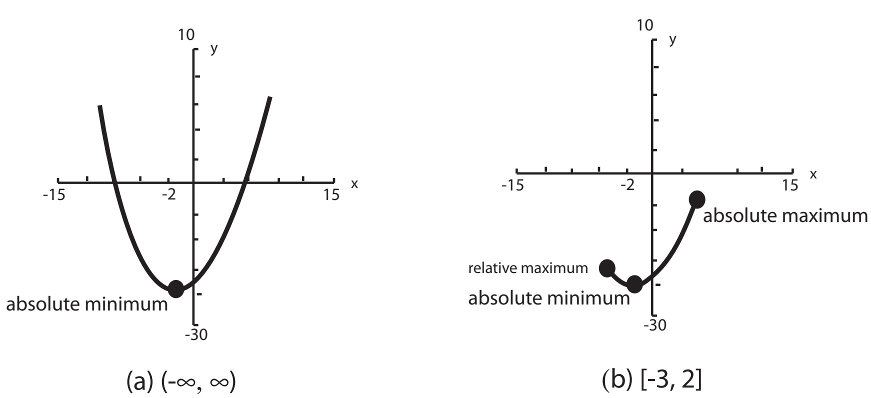

Figure 6.4 (Absolute extrema: $f(x) = x^2 + 4x - 21$)

GRAPH using $Y_2 = x^2 + 4x - 21$, with WINDOW -15 15 1 -30 10 1 1

(a) *Domain* $(-\infty, \infty)$.

i. *Critical numbers.*
Since
$$f'(x) = 2x^{2-1} + 4(1)x^{1-1} = 2x + 4 = 0,$$
there is *one* critical number at
$c = -\frac{4}{2} =$ (i) $\mathbf{-2}$ (ii) **2** (iii) **0**
and since $f(-2) = (-2)^2 + 4(-2) - 21 = -25$,
there is a critical *point* $(c, f(c)) = (-2, -25)$.

ii. *Endpoints.*
lower endpoint. As $x \to -\infty$,
$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \left(x^2 + 4x - 21\right) =$$
(i) $\mathbf{-\infty}$ (ii) **0** (iii) $\boldsymbol{\infty}$

lower endpoint. As $x \to \infty$,
$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \left(x^2 + 4x - 21\right) =$$
(i) $\mathbf{-\infty}$ (ii) **0** (iii) $\boldsymbol{\infty}$

iii. *Absolute extrema*
Since

candidate x	$f(x)$
-2	-25
$x \to -\infty$	$f(x) \to -\infty$
$x \to \infty$	$f(x) \to \infty$

absolute minimum is
$f(x) =$ (i) $\mathbf{-25}$ (ii) $\mathbf{-\infty}$ (iii) $\boldsymbol{\infty}$ (iv) **none**
and absolute maximum is
$f(x) =$ (i) $\mathbf{-25}$ (ii) $\mathbf{-\infty}$ (iii) $\boldsymbol{\infty}$ (iv) **none**
because $f(x)$ grows arbitrarily large as $x \to \infty$

iv. (i) **True** (ii) **False** This case is an example of the *critical value theorem* (which says relative extreme is absolute extreme), because there is only *one* critical value.

(b) *Domain* $[-3, 2]$.

i. *Critical numbers.*
Same as before.

ii. *Endpoints.*
lower endpoint. At $x = -3$,
$f(-3) = (-3)^2 + 4(-3) - 21 =$ (i) $\boldsymbol{\infty}$ (ii) $\mathbf{-24}$ (iii) $\mathbf{-19}$
upper endpoint. At $x = 2$,
$f(2) = (2)^2 + 4(2) - 21 =$ (i) $\mathbf{-\infty}$ (ii) $\mathbf{-9}$ (iii) $\boldsymbol{\infty}$

iii. *Absolute extrema*
Since

candidate x	$f(x)$
-3	-24
-2	-25
2	-9

absolute minimum is
$f(x) =$ (i) $\mathbf{-25}$ (ii) $\mathbf{-24}$ (iii) $\boldsymbol{\infty}$ (iv) **none**
and absolute maximum is
$f(x) =$ (i) $\mathbf{-25}$ (ii) $\mathbf{-9}$ (iii) $\boldsymbol{\infty}$ (iv) **none**

iv. *calculator check*
absolute minimum occurs at
$x =$ (i) $\mathbf{-25}$ (ii) $\mathbf{-2}$ (iii) $\boldsymbol{\infty}$ (iv) **none**
MATH FMin(ENTER $X^2 + 4X - 21$, X, -3, 2)
absolute maximum occurs at
$x =$ (i) $\mathbf{-2}$ (ii) $\mathbf{2}$ (iii) $\boldsymbol{\infty}$ (iv) **none**
MATH FMax(ENTER $X^2 + 4X - 21$, X, -3, 2)

5. *Absolute extrema:* $f(x) = 2x^3 + 3x^2 - 36x$ *revisited*

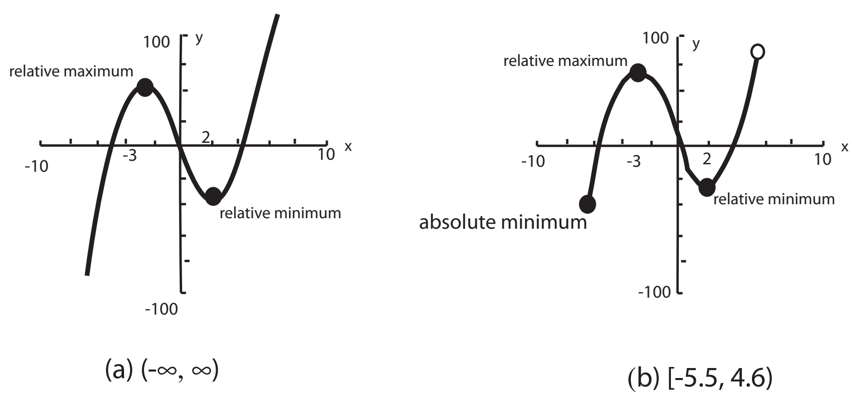

Figure 6.5 (Absolute extrema: $f(x) = 2x^3 + 3x^2 - 36x$)

GRAPH using $Y_3 = 2x^3 + 3x^2 - 36x$, with WINDOW -10 10 1 -100 100 1 1

(a) *Domain* $(-\infty, \infty)$.

i. *Critical numbers.*
Recall, since

$$f'(x) = 6x^2 + 6x - 36 = 6(x^2 + x - 6) = 6(x + 3)(x - 2) = 0,$$

there are *two* critical numbers at
$c =$ (i) $\mathbf{-3}$ (ii) $\mathbf{2}$ (iii) $\mathbf{6}$

and since $f(-3) = 2(-3)^3 + 3(-3)^2 - 36(-3) = 81$,
there is a critical *point* $(c, f(c)) = (-3, 81)$.
and since $f(2) = 2(2)^3 + 3(2)^2 - 36(2) = -44$,
there is also critical *point* $(c, f(c)) = (2, -44)$.

ii. *Endpoints.*
lower endpoint. As $x \to -\infty$,

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \left(2x^3 + 3x^2 - 36x\right) =$$

(i) $\mathbf{-\infty}$ (ii) **0** (iii) $\mathbf{\infty}$

upper endpoint. As $x \to \infty$,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \left(2x^3 + 3x^2 - 36x\right) =$$

(i) $\mathbf{-\infty}$ (ii) **0** (iii) $\mathbf{\infty}$

iii. *Absolute extrema*
Since

candidate x	$f(x)$
-3	81
2	-44
$x \to -\infty$	$f(x) \to -\infty$
$x \to \infty$	$f(x) \to \infty$

absolute minimum is
$f(x) =$ (i) **−44** (ii) $\mathbf{-\infty}$ (iii) $\mathbf{\infty}$ (iv) **none**
because $f(x) \to -\infty$ as $x \to -\infty$, even though there is a *relative* minimum
and absolute maximum is
$f(x) =$ (i) **81** (ii) $\mathbf{-\infty}$ (iii) $\mathbf{\infty}$ (iv) **none**
because $f(x) \to \infty$ as $x \to \infty$, even though there is a *relative* maximum

(b) *Domain* $[-5.5, 4.6]$.

i. *Critical numbers.*
Same as before

ii. *Endpoints.*
lower endpoint. At $x = -5.5$,
$f(-5.5) = 2(-5.5)^3 + 3(-5.5)^2 - 36(-5.5) =$
(i) **−44** (ii) **−41** (iii) **−19**
upper endpoint.
There (i) **is** (ii) **is no** upper endpoint
because $f(x)$ does not exist at the *open* point $x = 4.6$
However, at $x = 4.6$,
$f(4.6) = 2(4.6)^3 + 3(4.6)^2 - 36(4.6) \approx$
(i) **81** (ii) **92.6** (iii) **−19**

iii. *Absolute extrema*
Since

candidate x	$f(x)$
-5.5	-44
-3	81
2	-41
4.6^* (open point)	92.6^*

absolute minimum occurs at
$x =$ (i) $\mathbf{-3}$ (ii) $\mathbf{-5.5}$ (iii) $\mathbf{-44}$ (iv) **none**
and absolute maximum occurs at
$x =$ (i) $\mathbf{-5.5}$ (ii) $\mathbf{-3}$ (iii) $\mathbf{4.6}$ (iv) **none**
because even through $f(x)$ is an absolute maximum at $x = 4.6$, $f(x)$ does *not* exist at this *open* point, and, also, there is a *relative*, not absolute, maximum at $x = -3$ because $f(x) = 81$ is smaller than $f(4.6) \approx 92.6$

iv. *calculator check*
absolute minimum occurs at
$x =$ (i) $\mathbf{2}$ (ii) $\mathbf{-5.5}$ (iii) $\mathbf{-3}$ (iv) **none**
MATH FMin(ENTER $2X^3 + 3x^2 - 36X$, X, -5.5, 4.6), which is incorrect!
so better to GRAPH, 2nd CALC, minimum, -5.5 ENTER 0 ENTER -4 ENTER
and, as explained, an absolute maximum does not exist

6. $f(x) = \frac{2x^2+2}{x^2+5}$ *on* $(-\infty, \infty)$

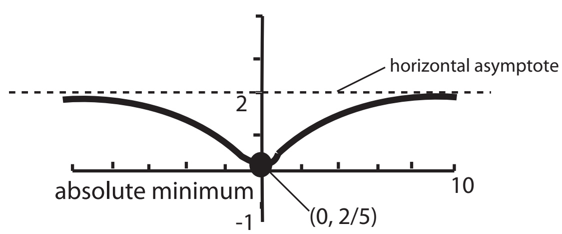

Figure 6.6 ($f(x) = \frac{2x^2+2}{x^2+5}$)

GRAPH using $Y_2 = \frac{2x^2+2}{x^2+5}$, with WINDOW -10 10 1 -1 3 1 1

(a) *Critical numbers.*
Let $u(x) = 2x^2 + 2$ and $v(x) = x^2 + 5$.
then, $u'(x) = 2(2)x^{2-1} = 4x$ and $v'(x) = 2x^{2-1} = 2x$
and so $v(x)u'(x) = (x^2 + 5)(4x)$ and $u(x)v'(x) = (2x^2 + 2)(2x)$
and so since

$$f'(x) = \frac{v(x) \cdot u'(x) - u(x) \cdot v'(x)}{[v(x)]^2} = \frac{(x^2 + 5)(4x) - (2x^2 + 2)(2x)}{[x^2 + 5]^2} = \frac{16x}{x^4 + 10x^2 + 25}$$

there is a *critical number* at
$c =$ (i) $\mathbf{-2}$ (ii) **2** (iii) **0**
because $16x = 0$ when $x = 0$
and since $f(0) = \frac{2(0)^2+2}{(0)^2+5} = \frac{2}{5}$,
there is a critical *point* $(c, f(c)) = \left(0, \frac{2}{5}\right)$.

(b) *Endpoints.*
lower endpoint. As $x \to -\infty$,

$$\lim_{x\to\infty} \frac{2x^2+2}{x^2+5} = \lim_{x\to\infty} \frac{\frac{2x^2}{x^2} + \frac{2}{x^2}}{\frac{x^2}{x^2} + \frac{5}{x^2}} = \lim_{x\to\infty} \frac{2 + \frac{2}{x^2}}{1 + \frac{5}{x^2}} =$$

(i) **0** (ii) **1** (iii)**2**.
upper endpoint. As $x \to \infty$,

$$\lim_{x\to-\infty} \frac{2x^2+2}{x^2+5} = \lim_{x\to-\infty} \frac{\frac{2x^2}{x^2} + \frac{2}{x^2}}{\frac{x^2}{x^2} + \frac{5}{x^2}} = \lim_{x\to-\infty} \frac{2 + \frac{2}{x^2}}{1 + \frac{5}{x^2}} =$$

(i) **0** (ii) **1** (iii) **2**.

(c) *Absolute extrema*
Since

candidate x	$f(x)$
0	$\frac{2}{5}$
$x \to -\infty$	$f(x) \to 2$
$x \to \infty$	$f(x) \to 2$

absolute minimum is
(i) **2** (ii) $\mathbf{\frac{2}{5}}$ (iii) **0** (iv) **none**
and absolute maximum is
(i) **2** (ii) $\mathbf{\frac{2}{5}}$ (iii) **0** (iv) **none**
because $f(x)$ does *not* equal any *particular* value, as $x \to \infty$

7. *Absolute extrema:* $f(x) = x^3(5.5)^x$ *on* $[-2, 1]$
Type Y=, then $\text{Y}_1 = x^3(5.5)^x$, WINDOW -2.5 1 1 -0.5 0.5 1 1, then GRAPH

(a) *Critical numbers.*
Let $u(x) = x^3$ and $v(x) = 5.5^x$.
then, $u'(x) = 2(2)x^{2-1} = 3x^2$ and $v'(x) = 2x^{2-1} = (\ln 5.5)5.5^x$
and so $v(x)u'(x) = (5.5^x)(3x^2)$ and $u(x)v'(x) = (x^3)(\ln 5.5)5.5^x$
and so since

$$f'(x) = v(x) \cdot u'(x) + u(x) \cdot v'(x) = \left(x^2\right)(5.5^x)(3 + x \ln 5.5)$$

there are *two* critical numbers at
$c =$ (i) $\mathbf{-\frac{3}{\ln 5.5} \approx -1.76}$ (ii) $\mathbf{0}$ (iii) $\mathbf{\frac{3}{\ln 5.5}}$
(Since $3 + x \ln 5.5 = 0, x = -\frac{3}{\ln 5.5} \approx -1.76$ and also $x^2 = 0$ when $x = 0$)
and since $f(0) = (0)^3(5.5)^0 = 0$,
there is a critical *point* $(c, f(c)) = (0, 0)$.
and since $f\left(-\frac{3}{\ln 5.5}\right) = \left(-\frac{3}{\ln 5.5}\right)^3 (5.5)^{-\frac{3}{\ln 5.5}} \approx -0.27$,
there is a critical *point* $(c, f(c)) \approx (-1.76, -0.27)$.

(b) *Endpoints.*
lower endpoint. At $x = -2$,
$f(-2) = (-2)^3(5.5)^{-2} \approx$ (i) $\mathbf{-0.27}$ (ii) $\mathbf{0.26}$ (iii) $\mathbf{0}$
upper endpoint. At $x = 1$,
$f(2) = (1)^3(5.5)^1 \approx$ (i) $\mathbf{5.5}$ (ii) $\mathbf{0}$ (iii) $\mathbf{-0.26}$

(c) *Absolute extrema*
Since

candidate x	$f(x)$
-2	-0.26
$-\frac{3}{\ln 5.5} \approx -1.76$	-0.27
0	0
1	5.5

absolute minimum is at
$x =$ (i) $\mathbf{-2}$ (ii) $\mathbf{-\frac{3}{\ln 5.5}}$ (iii) **1** (iv) **none**
and absolute maximum is at
$x =$ (i) $\mathbf{-2}$ (ii) $\mathbf{-\frac{3}{\ln 5.5}}$ (iii) **1** (iv) **none**

(d) *calculator check*
absolute minimum occurs at
$x =$ (i) $\mathbf{-2}$ (ii) **1** (iii) $\mathbf{-\frac{3}{\ln 5.5}}$ (iv) **none**
MATH FMin(ENTER $x^3(5.5)^x$, X, -2, 1)
absolute maximum occurs at
$x =$ (i) $\mathbf{-2}$ (ii) **1** (iii) **0** (iv) **none**
MATH FMax(ENTER $x^3(5.5)^x$, X, -2, 1)

8. *Application, graphical optimization: maximizing profit*

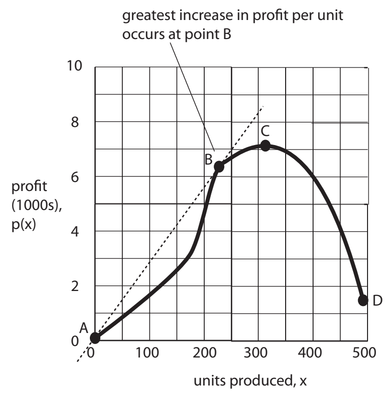

Figure 6.7 (Maximizing profit and profit per unit)

Figure gives profit (in 1000s of dollars) versus number of units produced. Determine maximum profit and also maximum profit *per unit*.

Maximum profit
Maximum profit occurs at point C, where
$p(x) \approx$ (i) **7.1** (ii) **6.5** (iii) **1.5**

Maximum profit *per unit* occurs at point B, where
profit $p(x) \approx$ (i) **7.1** (ii) **6.5** (iii) **0.026**
and number units $x \approx$ (i) **350** (ii) **250** (iii) **500**
and so maximum profit *per unit* is
$\frac{p(x)}{x} \approx \frac{6.5}{250} \approx$ (i) **7.1** (ii) **0.034** (iii) **0.026**
because to maximize profit/units, let $f(x) = \frac{p(x)}{x}$,
and to maximize this, set $f'(x) = \frac{xp'(x)-p(x)(1)}{x^2} = 0$, or $p'(x) = \frac{p(x)}{x}$

6.2 Applications of Extrema

We apply the techniques used in absolute minima and maxima problems to a number of applied topics. Steps to solving applied problems include:

- Determine *function*.
  draw a picture to make things clear if possible
  identify variable to maximize or minimize, express this variable as a function of *one* other variable
- Determine *domain* of function.
  identify if open or closed interval
- Determine *critical values and endpoints*.
- Identify *absolute extrema*.
  check results with calculator

Exercise 6.2 (Applications of Extrema)

1. *Maximizing area: garden fence.*

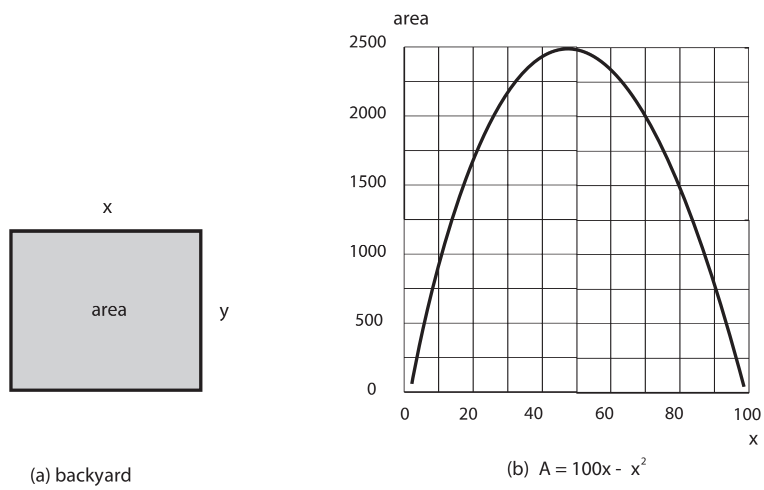

Figure 6.8 (Maximizing area: garden fence)

Darlene has 200 feet of rabbit fence to enclose a rectangular garden she wishes to put in her backyard. What should be the length and width of this garden to maximize area?

(a) *Function.*
Let x, y and A represent length, width and area, respectively, so

$$A = xy$$

Also, there is 200 feet of fence to enclose four sides to the garden (two lengths and two widths), so

(i) $\boldsymbol{x + y = 200}$ (ii) $\boldsymbol{2x + y = 200}$ (iii) $\boldsymbol{2x + 2y = 200}$

in other words, $2y = 200 - 2x$ or $y = 100 - x$, so

$$A = xy = x(100 - x) = 100x - x^2$$

(b) *Domain.*
Since length and area cannot be negative;
in other words, $x \geq 0$ and $A = x(100 - x) \geq 0$, or

(i) $\boldsymbol{x \geq 0, x \leq 100}$ or $\boldsymbol{[0, 100]}$

(ii) $\boldsymbol{x \leq 0, x \leq 100}$ or $\boldsymbol{(-\infty, 100]}$

(iii) $\boldsymbol{x \geq 0, x \geq 100}$ or $\boldsymbol{[0, \infty)}$

(c) *Critical numbers.*
Since

$$A'(x) = 100(1)x^{1-1} - 2x^{2-1} = 100 - 2x,$$

there is *one* critical number at
$c = \frac{100}{2} =$ (i) **50** (ii) **2** (iii) **100**
and since $A(50) = 100(50) - (50)^2 = 2500$,
there is a critical *point* $(c, A(c)) = (50, 2500)$.

(d) *Endpoints.*
lower endpoint. At $x = 0$,
$A(0) = 100(0) - (0)^2 =$ (i) **0** (ii) **100** (iii) **50**
upper endpoint. At $x = 100$,
$A(2) = 100(100) - (100)^2 =$ (i) **0** (ii) **50** (iii) $\boldsymbol{\infty}$

(e) *Absolute extrema*
Since

length candidates, x	area, A
0	0
50	2500
100	0

absolute maximum area is
(i) **0** (ii) **2500** (iii) **50** (iv) **none**
and occurs when length x is
(i) **0** (ii) **2500** (iii) **50** (iv) **none**

also, width $y = 100 - x = 100 - 50 = 50$

(f) *calculator check*
absolute maximum occurs at
$x =$ (i) **0** (ii) **2500** (iii) **50** (iv) **none**
MATH FMax(ENTER $100X - X^2$, X, 0, 100)

2. *Maximizing Volume: Pizza Box.*

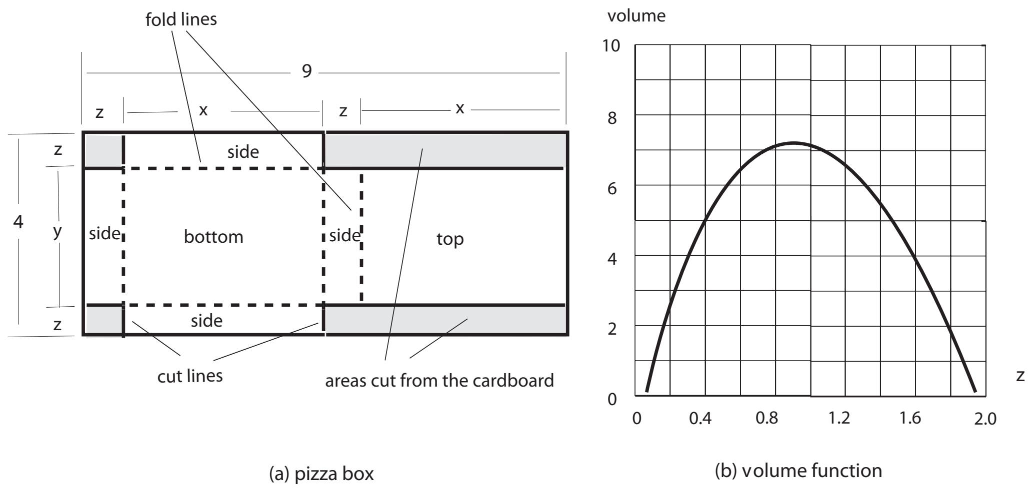

Figure 6.9 (Maximizing volume: pizza box)

Find the (x, y, z) dimensions to fold a 4 by 9 (width x length) piece of cardboard to make a pizza box with maximum volume.

(a) *Function.*
Let x, y, z and V represent length, width, depth and volume, so

$$V = xyz$$

also, since *width* is 4 feet, so (look at figure)

(i) $\boldsymbol{y + 2z = 4}$ (ii) $\boldsymbol{2x + 2z = 4}$ (iii) $\boldsymbol{2x + 2z = 9}$
in other words, $y = 4 - 2z$

and since *length* is 9 feet, so (look at figure)

(i) $\boldsymbol{y + 2z = 4}$ (ii) $\boldsymbol{2x + 2z = 4}$ (iii) $\boldsymbol{2x + 2z = 9}$
in other words, $2x = 9 - 2z$ or $x = \frac{9-2z}{2}$, so

$$V = xyz = \left(\frac{9 - 2z}{2}\right)(4 - 2z)z = \frac{1}{2}(9 - 2z)(4 - 2z)z = 2z^3 - 13z^2 + 18z$$

(b) *Domain.*
Since width, length and depth cannot be negative;

so, for width,

$$\begin{aligned} y = 4 - 2z &\geq 0 \\ 4 &\geq 2z \end{aligned}$$

so $z \leq 2$

and, for length,

$$\begin{aligned} x = \frac{9-2z}{2} &\geq 0 \\ 9-2z &\geq 0 \\ 9 &\geq 2z \end{aligned}$$

so $z \leq \frac{9}{2} = 4.5$

and for depth, $z \geq 0$, so combining,

(i) $\boldsymbol{z \geq 0, z \leq 4.5, z \leq 2}$ or $[\mathbf{0}, \mathbf{2}]$

(ii) $\boldsymbol{z \leq 0, z \leq 4.5, z \leq 2}$ or $(\boldsymbol{-\infty}, \mathbf{2}]$

(iii) $\boldsymbol{z \geq 0, z \geq 4.5, z \geq 2}$ or $[\mathbf{4.5}, \boldsymbol{\infty})$

(c) *Critical numbers.*
Since

$$V'(z) = 2(3)z^{3-1} - 13(2)z^{2-1} + 18(1)z^{1-1} = 6z^2 - 26z + 18 = 0,$$

there are *two* (possible) critical numbers at

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(26) \pm \sqrt{(-26)^2 - 4(6)(18)}}{2(6)} = \frac{26 \pm \sqrt{244}}{12} \approx$$

(i) **0.865** (ii) **3.468** (iii) **2**

but since critical value $c = 3.468 > 2$ is outside domain of function, it is not used
and since $f(0.865) = 2(0.865)^3 - 13(0.865)^2 + 18(0.865) \approx 7.138$,
there is a critical *point* $(c, f(c)) \approx (0.865, 7.138)$.

(d) *Endpoints.*
lower endpoint. At $z = 0$,
$V(0) = 2(0)^3 - 13(0)^2 + 18(0) =$ (i) **0** (ii) **100** (iii) **50**
upper endpoint. At $z = 2$,
$V(2) = 2(2)^3 - 13(2)^2 + 18(2) =$ (i) **0** (ii) **50** (iii) $\boldsymbol{\infty}$

(e) *Absolute extrema*
Since

depth candidates, z	volume, V
0	0
0.865	7.138
2	0

absolute maximum volume is
(i) **0** (ii) **7.138** (iii) **2** (iv) **none**
and occurs when depth z is
(i) **0.865** (ii) **2500** (iii) **50** (iv) **none**

also, width $x = \frac{9-2z}{2} \approx \frac{9-2(0.865)}{2} = 3.635$

and length $x = 4 - 2z \approx 4 - 2(0.865) = 2.27$

(f) *calculator check*
absolute maximum occurs at
$x =$ (i) **0** (ii) **0.865** (iii) **2** (iv) **none**
MATH FMax(ENTER $2X^3 - 13^X 2 + 18X$, X, 0, 2)

3. *Maximizing Profit.*

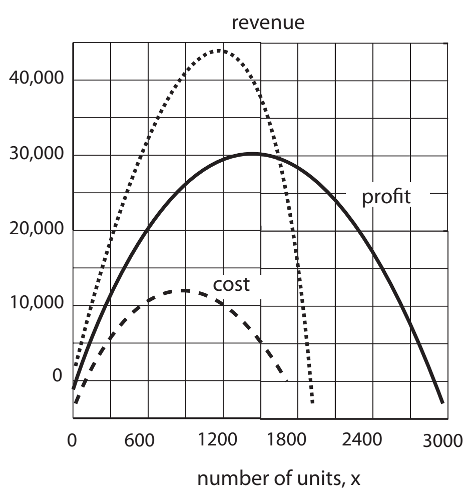

Figure 6.10 (Maximizing Profit)

Find the number of desks, x, that should be sold to maximize profit, P. The desk manufacturer sells x units if it sets unit *price* at $p(x) = -0.030x + 70$ dollars and has weekly costs of $C(x) = -0.015x^2 + 25x + 3100$.

(a) *Function.*
Profit equals *revenue*, $R = xp(x)$, minus costs (C)

$$P = R - C = x(-0.030x + 70) - (-0.015x^2 + 25x + 3100) =$$

(i) $\boldsymbol{-0.030x^2 + 70x}$
(ii) $\boldsymbol{-0.015x^2 + 25x + 3100}$
(iii) $\boldsymbol{-0.015x^2 + 45x - 3100}$

(b) *Domain.*
Assume number of desks sold is positive, so domain is
(i) $\boldsymbol{x \geq 0}$ or $\mathbf{[0, \infty)}$
(ii) $\boldsymbol{x \leq 0}$ or $\mathbf{(-\infty, 0]}$
(iii) $\mathbf{(-\infty, \infty)}$

(c) *Critical numbers.*
Since

$$P'(x) = -0.015(2)x^{2-1} + 45(1)x^{1-1} = -0.030x + 45 = 0,$$

there is *one* critical number at
$c =$ (i) **1000** (ii) **1250** (iii) **1500**
and since $P(1500) = -0.015(1500)^2 + 45(1500) - 3100 = 30650$,
there is a critical *point* $(c, A(c)) = (1500, 30650)$.

(d) *Endpoints.*
lower endpoint. At $x = 0$,
$P(0) = -0.015(0)^2 + 45(0) - 3100 =$ (i) $\mathbf{-3100}$ (ii) **100** (iii) **50**

upper endpoint. As $x \to \infty$,

$$\lim_{x\to\infty} P(x) = \lim_{x\to\infty}\left(-0.015x^2 + 25x + 3100\right) =$$

(i) $\boldsymbol{-\infty}$ (ii) **0** (iii) $\boldsymbol{\infty}$

(e) *Absolute extrema*
Since

units candidates, x	profit, P
0	-3100
1500	30650
$x \to \infty$	$P \to -\infty$

absolute maximum profit is
(i) **30650** (ii) **2500** (iii) **50** (iv) **none**
and occurs when number of units x is
(i) **0** (ii) **1500** (iii) $\boldsymbol{x \to \infty}$ (iv) **none**

(f) *calculator check*
It (i) **is** (ii) **is not** possible (or, at least, difficult) to use calculator
because one end of domain is open-ended, $[0, \infty)$

4. *Flu Epidemic.*

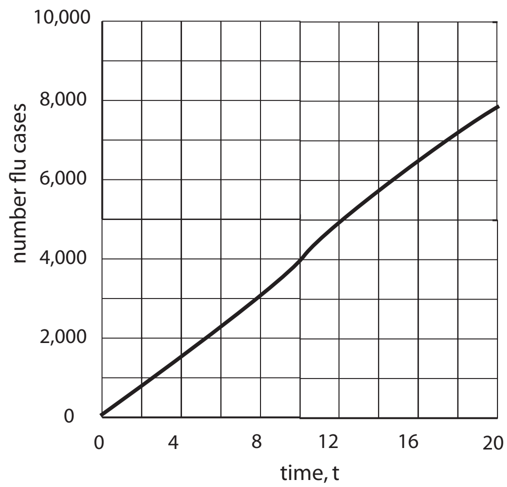

Figure 6.11 (Flu Epidemic)

The number of people in a city of size 50,000 who will contract the flu within the first t weeks after the first reported case can be predicted using the formula, $N(t) = -1.5t^3 + 50t^2$, $0 \leq t \leq 20$. Find the maximum number of people that get the flu and.

(a) *Function.*
Function is given as

$$N(t) = -1.5t^3 + 50t^2$$

(b) *Domain.*
Domain is given as
(i) $\boldsymbol{t \geq 0}$ or $\mathbf{[0, \infty)}$
(ii) $\boldsymbol{(-\infty, \infty)}$
(iii) $\mathbf{0 \leq \boldsymbol{t} \leq 20}$ or $\mathbf{[0, 20]}$

(c) *Critical numbers.*
Since

$$N'(t) = -1.5(3)t^{3-1} + 50(2)t^{2-1} = -4.5t^2 + 100t = t(-1.5t + 100) = 0,$$

there is *one* critical number at
$c =$ (i) $\mathbf{0}$ (ii) $\mathbf{\frac{100}{4.5} \approx 22.2}$ (iii) $\mathbf{100}$
because critical value $c \approx 22.2 > 20$, outside of domain $[0, 20]$
and since $N(0) = -1.5(0)^3 + 50(0)^2 = 0$,
there is a critical *point* $(c, N(c)) = (0, 0)$.

(d) *Endpoints.*
lower endpoint. At $x = 0$,
$N(0) = -1.5(0)^3 + 50(0)^2 =$ (i) $\mathbf{0}$ (ii) $\mathbf{100}$ (iii) $\mathbf{8000}$

upper endpoint. At $x = 20$,
$N(20) = -1.5(20)^3 + 50(20)^2 =$ (i) **100** (ii) **0** (iii) **8000**

(e) *Absolute extrema*
Since

time candidates, t	flu cases, N
0	0
20	8000

absolute maximum number of flu cases is
(i) **8000** (ii) **20** (iii) **50** (iv) **none**
and occurs at time $t =$
(i) **0** (ii) **20** (iii) $\boldsymbol{t \to \infty}$ (iv) **none**

(f) *calculator check*
absolute maximum number of flu cases occur at time
$t =$ (i) **0** (ii) **20** (iii) $\boldsymbol{t \to \infty}$ (iv) **none**
MATH FMax(ENTER $-1.5X^3 + 50X^2$, X, 0, 20)