Chapter 12

Sequences and Series

A *sequence* is a function whose domain is the set of natural numbers; for example,

$$a(n) = 3n, \quad n = 1, 2, 3, \ldots$$

Sequence $a(n)$ is often written as a_n instead and a_n is called the *general term* or nth term. The *sum* of the elements of a sequence is called a *series*; for example,

$$S_n = \sum_{i=1}^{n} 3i.$$

Series are used in financial formulas.

12.1 Geometric Sequences

One type of sequence is a *geometric* sequence where each term in the sequence is found by multiplying the previous number by a constant r called the *common ratio*; that is, the ratio of any two consecutive terms is r:

$$r = \frac{a_{n+1}}{a_n}, \quad n \geq 1.$$

If the first term is a, then the general term of a geometric sequence is

$$a_n = ar^{n-1}, \quad a_n = a_{n-1}r.$$

The *sum* (series) of the first n terms of a geometric sequence is

$$S_n = \frac{a(r^n - 1)}{r - 1}, \quad r \neq 1.$$

We will find in the next section that geometric sequences are important because they are used for financial formulas such as, in particular, annuities and amortization.

Exercise 12.1 (Geometric Sequences)

1. *Identify if a geometric sequence and, if so, r and a_n of sequence.*

   (a) sequence
   $$2, 6, 18, 54, \ldots$$
   (i) **is** (ii) **is not** a *geometric* sequence where
   notice $r = \frac{6}{2} = \frac{18}{6} = \frac{54}{18} = 3$

   (i) $\boldsymbol{a_1 = 2, a_n = 2(3^{n-1})}$
   (ii) $\boldsymbol{a_1 = 2, a_n = 3(2^{n-1})}$
   (iii) r **is** *not* **constant**
   notice $2(3^{1-1}) = 2(1) = 2,\ 2(3)^{2-1} = 2(3) = 6,\ 2(3)^{3-1} = 2(3^2) = 18, \ldots$

   (b) sequence
   $$16, 8, 4, 2, \ldots$$
   (i) **is** (ii) **is not** a geometric sequence where
   notice $r = \frac{8}{16} = \frac{4}{8} = \frac{2}{4} = 0.5$

   (i) $\boldsymbol{a_1 = 16, a_n = 16(0.5^{n-1})}$
   (ii) $\boldsymbol{a_1 = 16, a_n = 16(2^{n-1})}$
   (iii) r **is** *not* **constant**
   notice $16(0.5^{1-1}) = 16(1) = 16,\ 160.5^{2-1} = 16(0.5) = 8,\ 160.5^{3-1} = 16(0.5^2) = 4, \ldots$

   (c) sequence
   $$2, -6, 18, -54, \ldots$$
   (i) **is** (ii) **is not** a geometric sequence where
   notice $r = \frac{-6}{2} = \frac{18}{-6} = \frac{-54}{18} = -3$

   (i) $\boldsymbol{a_1 = -2, r = 3, a_n = 2(3)^{n-1}}$
   (ii) $\boldsymbol{a_1 = 2, r = 3, a_n = 2(-3)^{n-1}}$
   (iii) r **is** *not* **constant**
   notice $2(-3)^{1-1} = 2(1) = 2,\ 2(-3)^{2-1} = 2(3) = 6,\ 2(-3)^{3-1} = 2(-3)^2 = 18, \ldots$

   (d) sequence
   $$2, 6, 24, 72, \ldots$$
   (i) **is** (ii) **is not** a geometric sequence where
   notice $r = \frac{6}{2} = 3,\ \frac{24}{6} = 4,\ \frac{72}{24} = 3$

   (i) $\boldsymbol{a_1 = -2, r = 3, a_n = 2(3)^{n-1}}$
   (ii) $\boldsymbol{a_1 = -2, r = 3, a_n = 2(4)^{n-1}}$
   (iii) r **is** *not* **constant**

2. *List geometric sequence, $a_n = ar^{n-1}$.*

   (a) $a_1 = 2, r = 3, n = 4$

   (i) $\mathbf{2, 6, 18, 48}$
   (ii) $\mathbf{2, 6, 18, 54, 162}$
   (iii) $\mathbf{2, 6, 18, 54}$
   notice $2(3)^{1-1} = 2(1) = 2,\ 2(3)^{2-1} = 2(3) = 6,\ 2(3)^{3-1} = 2(3)^2 = 18, \ldots$

   (b) $a_1 = \frac{1}{2}, r = 3, n = 5$

   (i) $\mathbf{\frac{1}{2}, \frac{3}{2}, \frac{9}{2}, \frac{27}{2}}$
   (ii) $\mathbf{\frac{1}{2}, \frac{3}{2}, \frac{9}{2}, \frac{27}{2}, \frac{81}{2}}$
   (iii) $\mathbf{2, 6, 18, 54}$
   notice $\frac{1}{2}(3)^{1-1} = \frac{1}{2}(1) = \frac{1}{2},\ \frac{1}{2}(3)^{2-1} = \frac{1}{2}(3) = \frac{3}{2},\ \frac{1}{2}(3)^{3-1} = \frac{1}{2}(3)^2 = \frac{9}{2}, \ldots$

   (c) $a_1 = 7, r = \frac{3}{2}, n = 6$

   (i) $\mathbf{7, \frac{21}{2}, \frac{63}{2}, \frac{189}{2}, \frac{567}{2}}$
   (ii) $\mathbf{7, \frac{21}{2}, \frac{63}{2}, \frac{189}{2}, \frac{567}{2}, \frac{1699}{2}}$
   (iii) $\mathbf{7, \frac{21}{2}, \frac{63}{2}, \frac{189}{2}, \frac{567}{2}, \frac{1701}{2}}$
   2nd LIST OPS seq$\left(7 * \left(\frac{3}{2}\right)^{X-1}, X, 1, 6\right)$, then MATH ENTER for fractions

3. *Find a_5 and $a_n = ar^{n-1} = a_{n-1}r$.*

   (a) $a_1 = 7, r = 4$

   (i) $\boldsymbol{a_5 = 1792, \quad a_n = 7(4)^{n-1}}$
   (ii) $\boldsymbol{a_5 = 448, \quad a_n = 7(4)^{n-1}}$
   (iii) $\boldsymbol{a_5 = 448, \quad a_n = 4(7)^{n-1}}$
   $a_5 = 7(4)^{5-1}$

   (b) $a_1 = -3, r = -4$

   (i) $\boldsymbol{a_5 = -768, \quad a_n = -3(-4)^{n-1}}$
   (ii) $\boldsymbol{a_5 = 768, \quad a_n = 3(-4)^{n-1}}$
   (iii) $\boldsymbol{a_5 = -768, \quad a_n = -4(-3)^{n-1}}$
   $a_5 = -3(-4)^{5-1}$

   (c) $a_1 = -3, r = \frac{1}{3}$

(i) $\boldsymbol{a_5 = -\frac{1}{27}, \quad a_n = -3\left(\frac{1}{3}\right)^{n-1}}$
(ii) $\boldsymbol{a_5 = \frac{1}{27}, \quad a_n = 3\left(-\frac{1}{3}\right)^{n-1}}$
(iii) $\boldsymbol{a_5 = -\frac{1}{27}, \quad a_n = \left(\frac{1}{3}\right)(-3)^{n-1}}$
$-3 * \left(\frac{1}{3}\right)^{5-1}$

4. *Find a_5 and $a_n = ar^{n-1} = a_{n-1}r$.*

(a) $a_4 = 16$ (notice a_4, not a_1), $r = 2$

(i) $\boldsymbol{a_5 = 32, \quad a_n = 2(2)^{n-1}}$
(ii) $\boldsymbol{a_5 = 64, \quad a_n = 4(2)^{n-1}}$
(iii) $\boldsymbol{a_5 = 32, \quad a_n = 2(-2)^{n-1}}$
notice $a_5 = a_4 r = 16(2) = 32$, and $a = \frac{a_n}{r^{n-1}} = \frac{a_5}{2^{5-1}} = \frac{32}{2^{5-1}} = 2$ and so $a_n = 2(2)^{n-1}$

(b) $a_3 = 16$, $r = 2$

(i) $\boldsymbol{a_5 = 32, \quad a_n = 2(2)^{n-1}}$
(ii) $\boldsymbol{a_5 = 64, \quad a_n = 4(2)^{n-1}}$
(iii) $\boldsymbol{a_5 = 32, \quad a_n = 2(-2)^{n-1}}$
notice $a_4 = a_3 r = 16(2) = 32$ and $a_5 = a_4 r = 32(2) = 64$,
and $a = \frac{a_n}{r^{n-1}} = \frac{a_5}{2^{5-1}} = \frac{64}{2^{5-1}} = 4$ and so $a_n = 4(2)^{n-1}$

5. *Sum the first five ($n = 5$) terms, $S_n = \sum_{i=1}^{n} ar^{i-1} = \frac{a(r^n-1)}{r-1}$.*

(a) $a_1 = 2, r = 3, n = 5$

$S_5 =$ (i) **241** (ii) **242** (iii) **243**
$S_5 = \sum_{i=1}^{5} 2(3)^{i-1} = 2(3)^{1-1} + 2(3)^{2-1} + \cdots + 2(3)^{5-1} = \frac{2(3^5-1)}{3-1} = 242$

(b) $a_1 = \frac{1}{2}, r = 3, n = 5$

$S_5 =$ (i) $\boldsymbol{\frac{120}{2}}$ (ii) $\boldsymbol{\frac{121}{2}}$ (iii) $\boldsymbol{\frac{122}{2}}$
$s_5 = \sum_{i=1}^{5} \frac{1}{2}(3)^{i-1} = \frac{1}{2}(3)^{1-1} + \frac{1}{2}(3)^{2-1} + \cdots + \frac{1}{2}(3)^{5-1} = \frac{\frac{1}{2}(3^5-1)}{3-1} = \frac{121}{2}$,
use MATH ENTER for fraction

(c) $a_4 = 16$ (notice a_4, not a_1), $r = 2$

$S_5 =$ (i) **61** (ii) **62** (iii) **63**
notice $a_5 = a_4 r = 16(2) = 32$, and $a = \frac{a_n}{r^{n-1}} = \frac{a_5}{2^{5-1}} = \frac{32}{2^{5-1}} = 2$ and so $a_n = 2(2)^{n-1}$

so $S_5 = \sum_{i=1}^{5} 2(2)^{i-1} = 2(2)^{1-1} + 2(2)^{2-1} + \cdots + 2(2)^{5-1} = \frac{2(2^5-1)}{2-1} = 62$

(d) sequence

$$2, 6, 18, 54, \ldots$$

$S_5 =$ (i) **241** (ii) **242** (iii) **243**

notice $2(3^{1-1}) = 2(1) = 2,\ 2(3)^{2-1} = 2(3) = 6,\ 2(3)^{3-1} = 2(3^2) = 18, \ldots$ so $a_n = 2(3)^{n-1}$

so $S_5 = \sum_{i=1}^{5} 2(3)^{i-1} = 2(3)^{1-1} + 2(3)^{2-1} + \cdots + 2(3)^{5-1} = \frac{2(3^5-1)}{3-1} = 242$

6. *Application: beetles.* A colony of beetles increases by 4.3% per day. What is the percent increase over the initial population after 6 days?

   Since $a_1 = 0.043, r = 1.043, n = 6$

   $S_6 \approx$ (i) **27%** (ii) **28%** (iii) **29%**

   $S_6 = \sum_{i=1}^{6} 0.043(1.043)^{i-1} = \frac{0.043(1.043^6-1)}{1.043-1}$

12.2 Annuities: An Application

Annuities are a sequence of equal payments made at regular time intervals; more specifically, an *ordinary annuity* is an annuity where frequency of payments same as frequency of compounding, where time between payments is the *payment period* and time from beginning of first period to end of last period is called *term of the annuity.* The (accumulated future) amount S of an annuity R made at the *end* of each period for $n = mt$ consecutive payment periods at a rate of interest of $i = \frac{r}{m}$ per period is

$$S = R\left[\frac{\left(1+\frac{r}{m}\right)^{mt} - 1}{\frac{r}{m}}\right] = R \cdot s_{\overline{mt|}\frac{r}{m}} = R\left[\frac{(1+i)^n - 1}{i}\right] = R \cdot s_{\overline{n|}i},$$

and so the *annuity payments required for a (accumulated future) sinking fund* is

$$R = S\left[\frac{\left(\frac{r}{m}\right)}{\left(1+\frac{r}{m}\right)^{mt} - 1}\right] = S\left[\frac{i}{(1+i)^n - 1}\right].$$

On the other hand, the *present value of a sequence of annuity payments* is

$$P = R\left[\frac{1-\left(1+\frac{r}{m}\right)^{-mt}}{\frac{r}{m}}\right] = R \cdot a_{\overline{mt|}\frac{r}{m}} = R\left[\frac{1-(1+i)^{-n}}{i}\right] = R \cdot a_{\overline{n|}i},$$

and so the *amortization, annuity payments required to retire a present loan* is

$$R = P\left[\frac{\left(\frac{r}{m}\right)}{1-\left(1+\frac{r}{m}\right)^{-mt}}\right] = P\left[\frac{i}{1-(1+i)^{-n}}\right].$$

Amortization determines sequence of payments (annuity) equivalent to *present* lump sum, whereas sinking fund determines annuity equivalent to *future* lump sum. *Amortization table* or *amortization schedule* is also discussed.

Exercise 12.2 (Annuities: An Application)

1. *Future value of annuity payments:* $S = R\left[\frac{\left(1+\frac{r}{m}\right)^{mt}-1}{\frac{r}{m}}\right] = R \cdot s_{\overline{n|}i}$

   (a) Future value of 5 year term annuity, \$100 paid each quarter, earning interest at 8.5% annually, compounded quarterly, is
   $S = R\left[\frac{\left(1+\frac{r}{m}\right)^{mt}-1}{\frac{r}{m}}\right] = 100\left[\frac{\left(1+\frac{0.085}{4}\right)^{4(5)}-1}{\frac{0.085}{4}}\right] \approx$
   (i) **2260.21** (ii) **2460.21**
   Calculator: $100 * ((1 + 0.085/4) \wedge (20) - 1)/(0.085/4)$

   (b) Future value of 3 year term annuity, \$120 paid each month, earning interest at 9.5% annually, compounded monthly, is
   $S = R\left[\frac{\left(1+\frac{r}{m}\right)^{mt}-1}{\frac{r}{m}}\right] = 120\left[\frac{\left(1+\frac{0.095}{12}\right)^{12(3)}-1}{\frac{0.095}{12}}\right] \approx$
   (i) **4975.89** (ii) **5075.89**
   Calculator: $120 * ((1 + 0.095/12) \wedge (36) - 1)/(0.095/12)$

   (c) Future value of 3.2 year term annuity, \$105 paid each day, earning interest at 6.5% annually, compounded daily (365 days), is
   $S = R\left[\frac{\left(1+\frac{r}{m}\right)^{mt}-1}{\frac{r}{m}}\right] = 105\left[\frac{\left(1+\frac{0.065}{365}\right)^{365(3.2)}-1}{\frac{0.065}{365}}\right] \approx$
   (i) **135, 313.40** (ii) **136, 313.40**
   Calculator: $105 * ((1 + 0.065/365) \wedge (365 * 3.2) - 1)/(0.065/365)$

   (d) Calculate ordinary annuity S if $R = \$115, i = 0.04, n = 9$
   $S = R\left[\frac{(1+i)^n-1}{i}\right] = 115\left[\frac{(1+0.04)^9-1}{0.04}\right] \approx$
   (i) **1217.02** (ii) **1218.02**
   Calculator: $115 * ((1 + 0.04) \wedge (9) - 1)/0.04$

2. *Annuity payments required for a sinking fund:* $R = S\left[\frac{\left(\frac{r}{m}\right)}{\left(1+\frac{r}{m}\right)^{mt}-1}\right]$

(a) Lab of computers replaced in 3 years time for anticipated (future) cost of \$25,000 where \$25,000 accumulated over 3 year period through equal installments made at end of each month. If yearly interest rate is 8.5%, size of each installment is
$R = S\left[\frac{\left(\frac{r}{m}\right)}{\left(1+\frac{r}{m}\right)^{mt}-1}\right] = 25000\left[\frac{\left(\frac{0.085}{12}\right)}{\left(1+\frac{0.085}{12}\right)^{12(3)}-1}\right] \approx$
(i) **612.11** (ii) **613.11** (iii) **614.11**
Calculator: $25000 * (0.085/12)/((1 + 0.085/12) \wedge (36) - 1)$

(b) Quarterly annuity required (future) sinking fund of \$30,000, needed after 5 years, if yearly interest rate is 7.5%, is
$p = A\left[\frac{\left(\frac{r}{m}\right)}{\left(1+\frac{r}{m}\right)^{mt}-1}\right] = 30000\left[\frac{\left(\frac{0.075}{4}\right)}{\left(1+\frac{0.075}{4}\right)^{4(5)}-1}\right] \approx$
(i) **1250.14** (ii) **1350.14**
Calculator: $30000 * (0.075/4)/((1 + 0.075/4) \wedge (20) - 1)$

(c) Calculate annuity payments R if $S = \$5,000, i = 0.04, n = 9$
$S = S\left[\frac{i}{(1+i)^n-1}\right] = 5000\left[\frac{0.04}{(1+0.04)^9-1}\right] \approx$
(i) **482.46** (ii) **472.46**
Calculator: $5000 * (0.04)/((1 + 0.04) \wedge (9) - 1)$

3. *Present value of annuity payments:* $P = R\left[\frac{1-\left(1+\frac{r}{m}\right)^{-mt}}{\frac{r}{m}}\right] = R \cdot a_{\overline{n}|i}$

(a) Present value of 5 year term annuity, \$100 paid each quarter, earning 8.5% yearly interest, compounded quarterly, is
$P = p\left[\frac{1-\left(1+\frac{r}{m}\right)^{-mt}}{\frac{r}{m}}\right] = 100\left[\frac{1-\left(1+\frac{0.085}{4}\right)^{-4(5)}}{\frac{0.085}{4}}\right] \approx$
(i) **1415.59** (ii) **1615.59**
Calculator: $100 * (1 - (1 + 0.085/4) \wedge (-20))/(0.085/4)$

(b) Present value of 3 year term annuity, \$120 paid monthly, earning 9.5% yearly interest, compounded monthly, is
$P = p\left[\frac{1-\left(1+\frac{r}{m}\right)^{-mt}}{\frac{r}{m}}\right] = 120\left[\frac{1-\left(1+\frac{0.095}{12}\right)^{-12(3)}}{\frac{0.095}{12}}\right] \approx$
(i) **3546.14** (ii) **3746.14**
Calculator: $120 * (1 - (1 + 0.095/12) \wedge (-36))/(0.095/12)$

(c) Present value of 7 year term annuity, \$97 paid monthly, earning 9.5% yearly interest, compounded daily (365 days), is

$P = p\left[\frac{1-\left(1+\frac{r}{m}\right)^{-mt}}{\frac{r}{m}}\right] = 97\left[\frac{1-\left(1+\frac{0.095}{365}\right)^{-365(7)}}{\frac{0.095}{365}}\right] \approx$

(i) **180,006** (ii) **181,006**

Calculator: $97 * (1 - (1 + 0.095/365) \wedge (-365 * 7))/(0.095/365)$

(d) Calculate present value P if $R = \$115, i = 0.04, n = 9$

$S = R\left[\frac{1-(1+i)^{-n}}{i}\right] = 115\left[\frac{1-(1+0.04)^{-9}}{0.04}\right] \approx$

(i) **755.06** (ii) **855.06**

Calculator: $115 * (1 - (1 + 0.04) \wedge (-9))/(0.04)$

4. *Amortization, annuity payments required to retire a loan:* $R = P\left[\frac{\left(\frac{r}{m}\right)}{1-\left(1+\frac{r}{m}\right)^{-mt}}\right]$

(a) Car loan of $25,000 repaid monthly over 3 year period, yearly interest 8.5%. Amount of each installment

$R = P\left[\frac{\left(\frac{r}{m}\right)}{1-\left(1+\frac{r}{m}\right)^{-mt}}\right] = 25000\left[\frac{\left(\frac{0.085}{12}\right)}{1-\left(1+\frac{0.085}{12}\right)^{-(12)3}}\right] \approx$

(i) **769.19** (ii) **789.19**

Calculator: $25000 * (0.085/12)/(1 - (1 + 0.085/12) \wedge (-36))$

(b) House loan of $125,000 repaid quarterly over 20 year period, yearly interest 7.5%. Amount of each installment

$R = P\left[\frac{\left(\frac{r}{m}\right)}{1-\left(1+\frac{r}{m}\right)^{-mt}}\right] = 125000\left[\frac{\left(\frac{0.075}{4}\right)}{1-\left(1+\frac{0.075}{4}\right)^{-(4)20}}\right] \approx$

(i) **1906.99** (ii) **3029.08**

Calculator: $125000 * (0.075/4)/(1 - (1 + 0.075/4) \wedge (-80))$

(c) Calculate annuity payments R if $P = \$5,000, i = 0.04, n = 9$

$S = P\left[\frac{i}{1-(1+i)^{-n}}\right] = 5000\left[\frac{0.04}{1-(1+0.04)^{-9}}\right] \approx$

(i) **672.46** (ii) **672.56**

Calculator: $5000 * (0.04)/(1 - (1 + 0.04) \wedge (-9))$

(d) *Amortization table.*

Loan of $5,000 repaid quarterly over 1.5 year period, yearly interest 8.5%. Amount of each installment

$R = P\left[\frac{\left(\frac{r}{m}\right)}{1-\left(1+\frac{r}{m}\right)^{-mt}}\right] = 5000\left[\frac{\left(\frac{0.085}{4}\right)}{1-\left(1+\frac{0.085}{4}\right)^{-(4)1.5}}\right] \approx$

(i) **896.40** (ii) **989.19**

Calculator: $5000 * (0.085/4)/(1 - (1 + 0.085/4) \wedge (-6))$

payment number	annuity payment	interest paid	principal paid	balance end of period
0				5000.00
1	896.40	106.25	790.15	4209.85
2	896.40	89.45	806.95	3402.90
3	896.40	72.31	824.09	2578.81
4	896.40	54.80	841.60	1737.21
5	896.40	36.92	859.48	877.73
6	896.40	18.65	877.75	0

To begin, interest $= 5000 \times \frac{0.085}{4} = 106.25$, then principal $= 896.40 - 106.25 = 790.15$, and balance $= 5000 - 790.15 = 4209.85$, then interest $= 4209.85 \times \frac{0.085}{4} \approx 89.45$ and so on.

Unpaid balance after $x = 2$ payments is equal to the present value of an annuity after $n - x = mt - x = mt - 2$ payments given by

$$R\left[\frac{1-\left(1+\frac{r}{m}\right)^{-(mt-2)}}{\frac{r}{m}}\right] = 896.40\left[\frac{1-\left(1+\frac{0.085}{4}\right)^{-(4\times 1.5-2)}}{\frac{0.085}{4}}\right] \approx$$

(i) **3402.92** (ii) **3502.92**

Calculator: $896.40 * (1 - (1 + 0.085/4) \wedge (-4))/(0.085/4)$

which, notice, is same, with some round-off error, as balance at end of 2nd period in table

(e) **True** / **False**. Amortization determines sequence of payments (annuity) equivalent to *present* lump sum, whereas sinking fund determines annuity equivalent to *future* lump sum.