Chapter 7

Integration

The "reverse" of differentiation is called *integration*. If $F'(x) = f(x)$, then $F(x)$ is the *antiderivative* of $f(x)$; or

$$\int f(x)\,dx = F(x) + C,$$

where $\int$ in the *integral sign*, $f(x)$ is the *integrand* and $\int f(x)\,dx$ is the *indefinite integral*. Whereas differentiation determines the *slope* of a tangent line to a curve, integration determines the *area* under a curve.

7.1 Antiderivatives

- *power rule* $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$
- *constant multiple rule* $\int k \cdot f(x)\,dx = k \int f(x)\,dx + C$
- *sum or difference rule* $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
- *exponential functions*
  1. $\int e^{kx}\,dx = \frac{e^{kx}}{k} + C$, $k \neq 0$
  2. $\int a^{kx}\,dx = \frac{a^{kx}}{k(\ln a)} + C$, $a > 0, a \neq 1$
- $\int \frac{1}{x}\,dx = \int x^{-1}\,dx = \int \frac{dx}{x} = \ln|x| + C$

Use boundary conditions to determine the constant of integration, C.

Exercise 7.1 (Antiderivatives)

1. *Antiderivatives, derivatives and constants of integration.*

(a) (i) **True** (ii) **False**
If $F(x) = 5x$, then $F'(x) = 5(1)x^{1-1} = 5x^0 = 5$ is the derivative and so the original function, $F(x) = 5x$, is an *anti*derivative.

(b) An antiderivative of $F'(x) = 5$ is $F(x) =$ (i) $\boldsymbol{5x^2}$ (ii) $\boldsymbol{5x}$ (iii) $\mathbf{5}$
Check if $F(x) = 5x$ is the antiderivative of $F'(x) = 5$: $\frac{d}{dx}F(x) = \frac{d}{dx}(5x) = 5$, so, yes, it is

(c) An antiderivative of $F'(x) = -3$ is $F(x) =$ (i) $\boldsymbol{-3x^2}$ (ii) $\mathbf{-3}$ (iii) $\boldsymbol{-3x}$
$F(x) = -3x$ is the antiderivative of $F'(x) = -3$ because $\frac{d}{dx}F(x) = \frac{d}{dx}(-3x) = -3$

(d) An antiderivative of $F'(x) = x$ is $F(x) =$ (i) $\boldsymbol{\frac{3}{2}x^2}$ (ii) $\boldsymbol{\frac{1}{2}x^2}$ (iii) $\boldsymbol{\frac{1}{2}}$
$F(x) = \frac{1}{2}x^2$ is the antiderivative of $F'(x) = x$ because $\frac{d}{dx}F(x) = \frac{d}{dx}\left(\frac{1}{2}x^2\right) = \frac{1}{2}(2)x^{2-1} = x$

(e) An antiderivative of $F'(x) = x$ is $F(x) =$ (i) $\boldsymbol{\frac{1}{2}x^2 + 7}$ (ii) $\boldsymbol{\frac{3}{2}x^2}$ (iii) $\boldsymbol{\frac{1}{2}}$
$F(x) = \frac{1}{2}x^2 + 7$ is the antiderivative of $F'(x) = x$ because $\frac{d}{dx}\left(\frac{1}{2}x^2 + 7\right) = \frac{1}{2}(2)x^{2-1} + 0 = x$

(f) An antiderivative of $F'(x) = x$ is (i) $\boldsymbol{\frac{1}{2}x^2 - 39}$ (ii) $\boldsymbol{\frac{3}{2}x^2}$ (iii) $\mathbf{-39}$
$F(x) = \frac{1}{2}x^2 - 39$ is the antiderivative of $F'(x) = x$ because $\frac{d}{dx}F(x) = \frac{d}{dx}\left(\frac{1}{2}x^2 - 39\right) = x$

(g) The antiderivative of $F'(x) = x$ is (i) $\boldsymbol{\frac{3}{2}x + C}$ (ii) $\boldsymbol{\frac{1}{2}x^2 + C}$ (iii) $\boldsymbol{\frac{1}{2}}$
$F(x) = \frac{1}{2}x^2$ antiderivative of $F'(x) = x + C$ because $\frac{d}{dx}F(x) = \frac{d}{dx}\left(\frac{1}{2}x^2 + C\right) = x$, *if* C is a *constant*

(h) The antiderivative of $F'(x) = 5x$ is (i) $\boldsymbol{\frac{5}{2}x^2}$ (ii) $\boldsymbol{C}$ (iii) $\boldsymbol{\frac{5}{2}x^2 + C}$
$F(x) = 5$ antiderivative of $F'(x) = 5x$ because $\frac{d}{dx}F(x) = \frac{d}{dx}\left(\frac{5}{2}x^2 + C\right) = 5$, where C constant

2. *Power rule rule, constant multiple rule and notation.*

(a) The antiderivative of $F'(x) = 5 = 5x^0$, or, equivalently, the *integral* of $5x^0$

$$F(x) = \int f(x)\,dx = \int 5x^0\,dx = 5\int x^0\,dx = 5\left(\frac{1}{0+1}x^{0+1} + C\right) =$$

(i) $\boldsymbol{5C}$ (ii) $\boldsymbol{5 + 5C}$ (iii) $\boldsymbol{5x + C}$
since C is *any* constant, $5 \times C$ is also any constant, so just keep calling it C;
also, $F(x) = 5x + C$ integral of 5 because $\frac{d}{dx}F(x) = \frac{d}{dx}(5x + C) = 5$, where C constant

(b) The integral of $4 = 4x^0$

$$F(x) = \int 4x^0\,dx = 4\int x^0\,dx = 4\left(\frac{1}{0+1}x^{0+1} + C\right) =$$

(i) $\boldsymbol{C}$ (ii) $\boldsymbol{4 + C}$ (iii) $\boldsymbol{4x + C}$

This is an example of the *power rule* listed above

(c) The integral of k, k a constant,

$$\int k\,dx = \int kx^0\,dx = k\int x^0\,dx = k\left(\frac{1}{0+1}x^{0+1} + C\right) =$$

(i) $\boldsymbol{C}$ (ii) $\boldsymbol{k + C}$ (iii) $\boldsymbol{kx + C}$

This is an example of the power rule and also *multiple constant rule* listed above.

(d) The integral of $f(x) = x = x^1$,

$$\int x^1\,dx = \frac{1}{1+1}x^{1+1} + C =$$

(i) $\boldsymbol{C}$ (ii) $\boldsymbol{\frac{1}{2}x^2}$ (iii) $\boldsymbol{\frac{1}{2}x^2 + C}$

$F(x) = \frac{1}{2}x^2 + C$, where C constant, integral of x^2 because $\frac{d}{dx}\left(\frac{1}{2}x^2 + C\right) = \frac{1}{2}(2)x^{1-1} + 0 = x$

(e) The integral of $f(x) = x^2$,

$$\int x^2\,dx = \frac{1}{2+1}x^{2+1} + C =$$

(i) $\boldsymbol{C}$ (ii) $\boldsymbol{\frac{1}{3}x^3}$ (iii) $\boldsymbol{\frac{1}{3}x^3 + C}$

A word on notation: *Both* the integral sign "$\int$" and the differential "dx" are necessary components to say you want to integrate the function enclosed between them, in this case, x^2. Neither "$\int$" nor "dx" are part of the function. They do not have to be "solved" or "calculated" or "determined" in any sense. They are simply the *notation* used to say the function is to be integrated.

(f) Integral of $f = x^{10}$,

$$\int x^{10}\,dx = \frac{1}{10+1}x^{10+1} + C =$$

(i) $\boldsymbol{\frac{1}{11}x^5}$ (ii) $\boldsymbol{\frac{1}{5}x^{11} + 11}$ (iii) $\boldsymbol{\frac{1}{11}x^{11} + C}$

(g) Integral of $f = x^{-5}$,

$$\int x^{-5}\,dx = \frac{1}{-5+1}x^{-5+1} + C =$$

(i) $\boldsymbol{-\frac{1}{4}x^{-4} + C}$ (ii) $\boldsymbol{-\frac{1}{6}x^{-5} + C}$ (iii) $\boldsymbol{-\frac{1}{5}x^{-6} + C}$

(h) Integral of $f = x^{-7}$,

$$\int x^{-7}\,dx = \frac{1}{-7+1}x^{-7+1} + C =$$

(i) $\boldsymbol{-\frac{1}{6}x^{-8}+C}$ (ii) $\boldsymbol{-\frac{1}{8}x^{-8}+C}$ (iii) $\boldsymbol{-\frac{1}{6}x^{-6}+C}$

(i) Integral of $f = 3x^{10}$,

$$\int 3x^{10}\,dx = 3\int x^{10}\,dx = 3\left(\frac{1}{10+1}x^{10+1} + C\right) =$$

(i) $\boldsymbol{\frac{3}{11}x^{3}+C}$ (ii) $\boldsymbol{\frac{1}{3}x^{11}+C}$ (iii) $\boldsymbol{\frac{3}{11}x^{11}+C}$

(j) Integral of $f = -3x^{10}$,

$$\int \left(-3x^{10}\right)\,dx = -3\int x^{10}\,dx = -3\left(\frac{1}{10+1}x^{10+1} + C\right) =$$

(i) $\boldsymbol{-\frac{3}{11}x^{3}+C}$ (ii) $\boldsymbol{\frac{1}{3}x^{11}-3C}$ (iii) $\boldsymbol{-\frac{3}{11}x^{11}+C}$

(k) Integral of $f = kx$, k a constant,

$$\int kx\,dx = \int kx^1\,dx = k\left(\frac{1}{1+1}x^{1+1} + C\right) =$$

(i) $\boldsymbol{C}$ (ii) $\boldsymbol{k+C}$ (iii) $\boldsymbol{\frac{k}{2}x^{2}+C}$

(l) Integral of $f = \sqrt[4]{x}$,

$$\int \sqrt[4]{x}\,dx = \int x^{\frac{1}{4}}\,dx = \frac{1}{\frac{1}{4}+1}x^{\frac{1}{4}+1} + C = \frac{1}{\frac{1}{4}+\frac{4}{4}}x^{\frac{1}{4}+\frac{4}{4}} + C =$$

(i) $\boldsymbol{C}$ (ii) $\boldsymbol{\frac{5}{4}x^{\frac{5}{4}}+C}$ (iii) $\boldsymbol{\frac{4}{5}x^{\frac{5}{4}}+C}$

(m) Integral of $f = 6\sqrt[5]{x}$,

$$\int 6\sqrt[5]{x}\,dx = \int 6x^{\frac{1}{5}}\,dx = 6\left(\frac{1}{\frac{1}{5}+1}x^{\frac{1}{5}+1} + C\right) = 6\left(\frac{1}{\frac{6}{5}}x^{\frac{6}{5}} + C\right) =$$

(i) $\boldsymbol{x^{\frac{6}{5}}+C}$ (ii) $\boldsymbol{6x^{\frac{6}{5}}+C}$ (iii) $\boldsymbol{5x^{\frac{6}{5}}+C}$

(n) Integral of $f = 6\sqrt[5]{x}$,

$$\int 6\sqrt[5]{x}\,dx = \int 6x^{\frac{1}{5}}\,dx =$$

(i) $\boldsymbol{5x^{\frac{6}{5}} + m}$ (ii) $\boldsymbol{5x^{\frac{6}{5}} + k}$ (iii) $\boldsymbol{5x^{\frac{6}{5}} + C}$

All are correct as long as k, m and C are all constants.

3. *Integration for exponential, logarithm and other functions.*

(a) Integral of $f = \frac{1}{x} = x^{-1}$,

$$\int x^{-1}\,dx =$$

(i) $\boldsymbol{-2x^{-2} + C}$ (ii) $\boldsymbol{\ln|x| + C}$ (iii) $\boldsymbol{3\ln|x| + C}$

This is the $\frac{1}{x}$ integration rule above;
but *not* the power rule because $\int x^{-1}\,dx = \frac{x^{-1+1}}{-1+1} = \frac{x^0}{0}$ which does not exist.

(b) Integral of $f = \frac{3}{x} = 3x^{-1}$,

$$\int 3x^{-1}\,dx =$$

(i) $\boldsymbol{-6x^{-2} + C}$ (ii) $\boldsymbol{\ln|x| + C}$ (iii) $\boldsymbol{3\ln|x| + C}$

(c) Integral of $f = e^x$,

$$\int e^x\,dx =$$

(i) $\boldsymbol{C}$ (ii) $\boldsymbol{e^x + C}$ (iii) $\boldsymbol{e^x}$

This is one of the exponential function integration rules above

(d) Integral of $f = 5e^{3x}$,

$$\int 5e^{3x}\,dx = 5\int e^{3x}\,dx = 5\left(\frac{1}{3}e^{3x} + C\right) =$$

(i) $\boldsymbol{\frac{1}{3}e^{3x} + C}$ (ii) $\boldsymbol{\frac{1}{3}x^{11} + C}$ (iii) $\boldsymbol{\frac{5}{3}e^{3x} + C}$

This is another one of the exponential function integration rules above

(e) Integral of $f = 5e^{-3x}$,

$$\int e^{3x}\,dx = 5\left(\frac{1}{-3}e^{-3x} + C\right) =$$

(i) $\boldsymbol{\frac{1}{3}e^{3x} + C}$ (ii) $\boldsymbol{-\frac{5}{3}e^{x} + C}$ (iii) $\boldsymbol{-5e^{3x} + C}$

(f) Integral of $5x - 7x^3$,

$$\int \left(5x - 7x^3\right)\, dx = 5\int x\, dx - 7\int x^3\, dx = 5\left(\frac{1}{1+1}x^{1+1}\right) - 7\left(\frac{1}{3+1}x^{3+1}\right) + C =$$

(i) $\mathbf{5\left(\frac{x^3}{3}\right) - 7\left(\frac{x^4}{4}\right) + C}$ (ii) $\mathbf{5\left(\frac{x^2}{2}\right) + 7\left(\frac{x^4}{4}\right) + C}$ (iii) $\mathbf{5\left(\frac{x^2}{2}\right) - 7\left(\frac{x^4}{4}\right) + C}$

(g) Integral of $3x^{-1} + 1$,

$$\int \left(3x^{-1} + 1\right)\, dx = 3\int x^{-1}\, dx + \int x^0\, dx = 3\ln|x| + \left(\frac{1}{0+1}x^{0+1}\right) + C =$$

(i) $\mathbf{3\ln|x| + x^2 + C}$ (ii) $\mathbf{3\ln|x| + x + C}$ (iii) $\mathbf{3e^x + x^2 + C}$

(h) Integral of $6e^{3x} + \sqrt[3]{x}$,

$$\int \left(6e^{3x} + \sqrt[3]{x}\right)\, dx = 6\int e^{3x}\, dx + \int x^{\frac{1}{3}}\, dx = 6\left(\frac{1}{3}e^{3x}\right) + \frac{1}{\frac{1}{3}+1}x^{\frac{1}{3}+1} + C =$$

(i) $\mathbf{2e^{3x} + \frac{3}{4}e^{\frac{4}{3}} + C}$ (ii) $\mathbf{e^{3x} + \frac{3}{4}e^{\frac{4}{3}} + C}$ (iii) $\mathbf{2e^{3x} + \frac{4}{3}e^{\frac{4}{3}} + C}$

4. *Integration with boundary or initial conditions: determining C.*

(a) Integrate $f'(x) = 5x^2$, where $f(-1) = 5$.

Since $\int (5x^2)\, dx = 5\left(\frac{1}{2+1}x^{2+1}\right) + C =$
(i) $\mathbf{\frac{1}{3}e^{3x} + C}$ (ii) $\mathbf{\frac{5}{3}x^3 + C}$ (iii) $\mathbf{5e^{3x} + C}$

and since $f(-1) = 5$, then $f(-1) = \frac{5}{3}(-1)^3 + C = 5$, or
$C =$ (i) $\mathbf{5}$ (ii) $\mathbf{\frac{20}{3}}$ (iii) $\mathbf{5e^{3x}}$

and so $f(x) = \frac{5}{3}x^3 + C =$
(i) $\mathbf{\frac{5}{3}x^3 + C}$ (ii) $\mathbf{\frac{5}{3}x^3 + \frac{15}{3}}$ (iii) $\mathbf{\frac{5}{3}x^3 + \frac{20}{3}}$

(b) Integrate $f'(x) = 5x^2$, where $f(-1) = 6$.

Since $\int (5x^2)\, dx = 5\left(\frac{1}{2+1}x^{2+1}\right) + C =$
(i) $\mathbf{\frac{1}{3}e^{3x} + C}$ (ii) $\mathbf{\frac{5}{3}x^3 + C}$ (iii) $\mathbf{5e^{3x} + C}$

and since $f(-1) = 6$, then $f(-1) = \frac{5}{3}(-1)^3 + C = 6$, or
$C =$ (i) $\mathbf{6}$ (ii) $\mathbf{\frac{23}{3}}$ (iii) $\mathbf{\frac{21}{3}}$

and so $f(x) = \frac{5}{3}x^3 + C =$
(i) $\boldsymbol{\frac{5}{3}x^3 + C}$ (ii) $\boldsymbol{\frac{5}{3}x^3 + \frac{21}{3}}$ (iii) $\boldsymbol{\frac{5}{3}x^3 + \frac{23}{3}}$

(c) Integrate $f'(x) = 6x^{-1}$, where $f(2) = 4$.

Since $\int (6x^{-1})\, dx = 6\,(\ln|x|) + C =$
(i) $\boldsymbol{6x^{-1} + C}$ (ii) $\boldsymbol{-6\ln x + C}$ (iii) $\boldsymbol{6\ln|x| + C}$

and since $f(2) = 4$, then $f(2) = 6\ln|2| + C = 4$, or
$C =$ (i) $\mathbf{4 + 6(2)}$ (ii) $\mathbf{4 - 6\ln 2}$ (iii) $\mathbf{4 + 6\ln 2}$

and so $f(x) = 6\ln|2| + C =$
(i) $\boldsymbol{-\frac{6}{2}x^{-2} + \frac{19}{4}}$ (ii) $\boldsymbol{6\ln|x| + 4 - 6\ln(2)}$ (iii) $\boldsymbol{-\frac{6}{2}x^{-2} + \frac{21}{4}}$

5. *Application: economics.* Find total cost function, $C(x)$, such that marginal cost is $C'(x) = x^2 - 2x$ and where fixed costs are \$45 (in other words, $C(0) = 45$).

(a) Since $C(x) = \int C'(x)\, dx = \int (x^2 - 2x)\, dx = \frac{1}{2+1}x^{2+1} - \frac{2}{1+1}x^{1+1} + k =$
(i) $\boldsymbol{\frac{1}{3}e^{3x} + k}$ (ii) $\boldsymbol{\frac{1}{3}x^3 - x^2 + k}$ (iii) $\boldsymbol{5e^{3x} + k}$
Let's use constant k instead of C, to avoid confusion with cost C.

(b) and $C(0) = 45$, then $C(0) = \frac{1}{3}(0)^3 - (0)^2 + k = 45$, or,
$k =$ (i) $\boldsymbol{5 + \frac{5}{3}}$ (ii) $\boldsymbol{\frac{20}{3}}$ (iii) $\mathbf{45}$

(c) and so $C(x) = \int C'(x)\, dx = \frac{1}{3}x^3 - x^2 + k =$
(i) $\boldsymbol{\frac{5}{3}x^3 + C}$ (ii) $\boldsymbol{\frac{1}{3}x^3 - x^2}$ (iii) $\boldsymbol{\frac{1}{3}x^3 - x^2 + 45}$

6. *Application: physics.* Find position function $s(t)$ of a rolling ball such that velocity function is $v(t) = s'(t) = 6t^3$ and where the ball is 9 meters from the start position at time zero ($s(0) = 9$).

(a) Since $s(t) = \int s'(t)\, dt = \int (6t^3)\, dt = \frac{6}{3+1}t^{3+1} + C =$
(i) $\boldsymbol{\frac{1}{3}e^{3t} + C}$ (ii) $\boldsymbol{\frac{3}{2}t^4 + C}$ (iii) $\boldsymbol{5e^{3t} + C}$

(b) and $s(0) = 9$, then $s(0) = \frac{3}{2}(0)^4 + C = 9$, or,
$C =$ (i) $\mathbf{9}$ (ii) $\boldsymbol{\frac{9}{3}}$ (iii) $\boldsymbol{e^9}$

(c) and so $s(t) = \int s'(t)\, dx = \frac{3}{2}t^4 + C =$
(i) $\boldsymbol{\frac{3}{2}t^4 - 5}$ (ii) $\boldsymbol{\frac{3}{2}t^4 + 0}$ (iii) $\boldsymbol{\frac{3}{2}t^4 + 9}$

7. *Application: more physics.* Find velocity function $v(t)$ of a rolling ball such that acceleration function is $a(t) = v'(t) = -7t^4$ and where the ball has velocity -3 meters per second at time 2 ($v(2) = -3$).

   (a) Since $v(t) = \int v'(t)\,dt = \int (-7t^4)\,dt = \frac{-7}{4+1}t^{4+1} + C =$
   (i) $\boldsymbol{\frac{1}{3}e^{3t} + C}$ (ii) $\boldsymbol{-\frac{7}{5}t^5 + C}$ (iii) $\boldsymbol{5e^{3t} + C}$

   (b) and $v(2) = -3$, then $v(2) = -\frac{7}{5}(2)^5 + C = -3$, or,
   $C =$ (i) $\mathbf{9}$ (ii) $\boldsymbol{-\frac{120}{5}}$ (iii) $\boldsymbol{\frac{209}{5}}$

   (c) and so $v(t) = \int v'(t)\,dx = -\frac{7}{5}t^5 + C =$
   (i) (i) $\boldsymbol{-\frac{7}{5}t^5 - \frac{120}{5}}$ (ii) $\boldsymbol{-\frac{7}{5}t^5}$ (iii) $\boldsymbol{-\frac{7}{5}t^5 + \frac{209}{5}}$

7.2 Substitution

We look at an integration technique called *substitution*, which often simplifies a complicated integration. Roughly, the substitution integration technique is the reverse of the chain rule differentiation technique. We use the following formulas as a basis for the substitution technique, after substituting $u = f(x)$ (and so $du = f'(x)dx$).

- $\int [f(x)]^n f'(x)\,dx$ becomes $\int u^n\,du = \frac{u^{n+1}}{n+1} + C$, $n \neq 1$
- $\int e^{f(x)} f'(x)\,dx$ becomes $\int e^u\,du = e^u + C$
- $\int \frac{f'(x)}{f(x)}\,dx$ becomes $\int \frac{1}{u}\,du = \int u^{-1}\,du = \ln|u| + C$

Substitution method typically concerned with three cases; chose substitution u to be

- quantity under root or raised to a power
- quantity in denominator
- exponent of e

and allow for constants. We also look at how to deal with fractions in integration.

Exercise 7.2 (Substitution)

1. *Power Function and Integral Substitution Technique.*

(a) Find $\int f(x)\,dx = \int \frac{3}{2}\sqrt{3x+x^2}(3+2x)\,dx = \frac{3}{2}\int (3x+x^2)^{\frac{1}{2}}(3+2x)\,dx$.

guess $u = 3x + x^2$
then $\frac{du}{dx} = 3(1)x^{1-1} + 2x^{2-1} = 3 + 2x$ or $du = (3+2x)\,dx$
substituting u and du into $\int f(x)\,dx$,

$$\frac{3}{2}\int \left(3x+x^2\right)^{\frac{1}{2}}(3+2x)\,dx = \frac{3}{2}\int u^{\frac{1}{2}}\,du = \frac{3}{2}\left(\frac{1}{\frac{1}{2}+1}u^{\frac{1}{2}+1} + C\right) =$$

(i) $\boldsymbol{\frac{3}{2} + C}$ (ii) $\boldsymbol{\frac{2}{3}u^{\frac{1}{2}} + C}$ (iii) $\boldsymbol{u^{\frac{3}{2}} + C}$

but $u = 3x + x^2$, so

$$\int f(x)\,dx = u^{\frac{3}{2}} + C =$$

(i) $\boldsymbol{(3+2x)^{\frac{3}{2}} + C}$ (ii) $\boldsymbol{(3x+x^2)^{\frac{3}{2}} + C}$ (iii) $\boldsymbol{(3x^2+x^3)^{\frac{3}{2}} + C}$

(b) Find $\int f(x)\,dx = \int \frac{5}{2}(3x+x^2)^{\frac{3}{2}}(3+2x)\,dx$.

guess $u = 3x + x^2$
then $\frac{du}{dx} = 3(1)x^{1-1} + 2x^{2-1} = 3 + 2x$ or $du = (3+2x)\,dx$
substituting u and du into $\int f(x)\,dx$,

$$\frac{5}{2}\int \left(3x+x^2\right)^{\frac{3}{2}}(3+2x)\,dx = \frac{5}{2}\int u^{\frac{3}{2}}\,du = \frac{5}{2}\left(\frac{1}{\frac{3}{2}+1}u^{\frac{3}{2}+1} + C\right) =$$

(i) $\boldsymbol{\frac{5}{2} + C}$ (ii) $\boldsymbol{\frac{2}{5}u^{\frac{3}{2}} + C}$ (iii) $\boldsymbol{u^{\frac{5}{2}} + C}$

but $u = 3x + x^2$, so

$$\int f(x)\,dx = u^{\frac{5}{2}} + C =$$

(i) $\boldsymbol{(3+2x)^{\frac{3}{2}} + C}$ (ii) $\boldsymbol{(3x+x^2)^{\frac{3}{2}} + C}$ (iii) $\boldsymbol{(3x+x^3)^{\frac{5}{2}} + C}$

(c) Find $\int f(x)\,dx = \int \frac{1}{2}\frac{3+2x}{\sqrt{3x+x^2}}\,dx = \frac{1}{2}\int (3x+x^2)^{-\frac{1}{2}}(3+2x)\,dx$.

guess $u =$ (i) $\boldsymbol{3x+x^2}$ (ii) $\boldsymbol{3+2x}$ (iii) $\boldsymbol{\sqrt{3x+x^2}}$

then $\frac{du}{dx} = 3(1)x^{1-1} + 2x^{2-1} = 3 + 2x$ or $du = (3+2x)\,dx$
substituting u and du into $\int f(x)\,dx$,

$$\frac{1}{2}\int \left(3x+x^2\right)^{-\frac{1}{2}}(3+2x)\,dx = \frac{1}{2}\int u^{-\frac{1}{2}}\,du = \frac{1}{2}\left(\frac{1}{-\frac{1}{2}+1}u^{-\frac{1}{2}+1} + C\right) =$$

(i) $\mathbf{\frac{3}{2} + C}$ (ii) $\mathbf{\frac{1}{2}u^{\frac{3}{2}} + C}$ (iii) $\mathbf{u^{\frac{1}{2}} + C}$

but $u = 3x + x^2$, so

$$\int f(x)\, dx = u^{\frac{1}{2}} + C =$$

(i) $\mathbf{(3 + 2x)^{\frac{3}{2}} + C}$ (ii) $\mathbf{(3x + x^2)^{\frac{1}{2}} + C}$ (iii) $\mathbf{(3x + x^3)^{\frac{5}{2}} + C}$

(d) Find $\int \frac{9+6x}{\sqrt{3x+x^2}}\, dx = \int (3x + x^2)^{-\frac{1}{2}}(9 + 6x)\, dx$.

guess $u =$ (i) $\mathbf{3x + x^2}$ (ii) $\mathbf{3 + 2x}$ (iii) $\mathbf{\sqrt{3x + x^2}}$

then $\frac{du}{dx} = 3(1)x^{1-1} + 2x^{2-1} = 3 + 2x$ or $du = (3 + 2x)\, dx$
substituting u and du into $\int f(x)\, dx$,

$$\begin{aligned}
\int \left(3x + x^2\right)^{-\frac{1}{2}}(9 + 6x)\, dx &= \int \left(3x + x^2\right)^{-\frac{1}{2}}(3)(3 + 2x)\, dx \\
&= \int u^{-\frac{1}{2}}\,(3)du \;=\; 3\int u^{-\frac{1}{2}}\, du \\
&= 3\left(\frac{1}{-\frac{1}{2}+1}u^{-\frac{1}{2}+1} + C\right) =
\end{aligned}$$

(i) $\mathbf{\frac{3}{2} + C}$ (ii) $\mathbf{\frac{3}{2}u^{\frac{3}{2}} + C}$ (iii) $\mathbf{6u^{\frac{1}{2}} + C}$

but $u = 3x + x^2$, so

$$\int f(x)\, dx = 6u^{\frac{1}{2}} + C =$$

(i) $\mathbf{\frac{3}{2}(3 + 2x)^{\frac{3}{2}} + C}$ (ii) $\mathbf{6(3x + x^2)^{\frac{1}{2}} + C}$ (iii) $\mathbf{(3x + x^3)^{\frac{5}{2}} + C}$

(e) Find $\int 5\left(-2x^4 + 7x\right)^4\left(-8x^3 + 7\right) dx$.

guess $u =$ (i) $\mathbf{(-2x^4 + 7x)^4}$ (ii) $\mathbf{-8x^3 + 7}$ (iii) $\mathbf{-2x^4 + 7x}$

then $\frac{du}{dx} = -2(4)x^{4-1} + 7(1)x^{1-1} = -8x^3 + 7$ or $du = (-8x^3 + 7)\, dx$
substituting u and du into $\int f(x)\, dx$,

$$5\int \left(-2x^4 + 7x\right)^4(-8x^3 + 7)\, dx = 5\int u^4\, du = 5\left(\frac{1}{4+1}u^{4+1} + C\right) =$$

(i) $\mathbf{5 + C}$ (ii) $\mathbf{5u^5 + C}$ (iii) $\mathbf{u^5 + C}$

but $u = -2x^4 + 7x$, so

$$\int f(x)\, dx = u^5 + C =$$

(i) $\boldsymbol{(-2x^4+7x)^5+C}$ (ii) $\boldsymbol{(-2x^4+7x)^6+C}$ (iii) $\boldsymbol{5(-2x^4+7x)^5+C}$

(f) Find $\int f(x)\,dx = \int (3x^3+2x^2-4x)^6\,(9x^2+4x-4)\ dx$.

guess $u =$ (i) $\boldsymbol{3x^3+2x^2-4x}$ (ii) $\boldsymbol{9x^2+4x-4}$

then $du = (3(3)x^{3-1}+2(2)x^{2-1}-4(1)x^{1-1})\,dx = (9x^2+4x+4)\,dx$
substituting u and du into $\int f(x)\,dx$,

$$\int \left(3x^3+2x^2-4x\right)^6 \left(9x^2+4x-4\right)\ dx = \int u^6\,du = \left(\frac{1}{6+1}u^{6+1}+C\right) =$$

(i) $\boldsymbol{\frac{1}{7}u^7+C}$ (ii) $\boldsymbol{\frac{1}{7}u^6+C}$ (iii) $\boldsymbol{u^7+C}$

but $u = 3x^3+2x^2-4x$, so

$$\int f(x)\,dx = \frac{1}{7}u^7+C =$$

(i) $\boldsymbol{\frac{1}{7}\left(3x^3+2x^2-4x\right)^7+C}$
(ii) $\boldsymbol{\left(3x^3+2x^2-4x\right)^7+C}$
(iii) $\boldsymbol{7\left(3x^3+2x^2-4x\right)^7+C}$

(g) Find $\int (1+4x^2)^6\,(15x)\ dx$.

guess $u =$ (i) $\boldsymbol{1+4x^2}$ (ii) $\boldsymbol{15x}$

then $du = (0+4(2)x^{2-1})\,dx =$ (i) $\boldsymbol{(8x)\,dx}$ (ii) $\boldsymbol{(1+8x)\,dx}$

substituting u and du into $\int f(x)\,dx$,

$$\int \left(1+4x^2\right)^6 (15x)\ dx = \int \left(1+4x^2\right)^6 \left(\frac{15}{8}\right)(8x)\ dx = \frac{15}{8}\int u^6\,du = \frac{15}{8}\left(\frac{1}{6+1}u^{6+1}+C\right) =$$

(i) $\boldsymbol{\frac{15}{8}u^7+C}$ (ii) $\boldsymbol{\frac{15}{56}u^7+C}$ (iii) $\boldsymbol{u^7+C}$

but $u = 1+4x^2$, so

$$\int f(x)\,dx = \frac{15}{56}u^7+C =$$

(i) $\boldsymbol{\frac{1}{56}\left(1+4x^2\right)^7+C}$
(ii) $\boldsymbol{\frac{15}{56}\left(1+4x^2\right)^7+C}$
(iii) $\boldsymbol{15\left(1+4x^2\right)^7+C}$

(h) Find $\int f(x)\,dx = \int x\sqrt{x+5}\,dx = \int x(x+5)^{\frac{1}{2}}\,dx$.

guess $u =$ (i) $\boldsymbol{\sqrt{x+5}}$ (ii) $\boldsymbol{x+5}$

then $du = dx$ and also $x = u - 5$

substituting u, du and x into $\int f(x)\,dx$,

$$\begin{aligned}\int x(x+5)^{\frac{1}{2}}\,dx &= \int (u-5)(u)^{\frac{1}{2}}\,du \\ &= \int \left(u^{\frac{3}{2}} - 5u^{\frac{1}{2}}\right)\,du \\ &= \int u^{\frac{3}{2}}\,du - 5\int u^{\frac{1}{2}}\,du \\ &= \frac{1}{\frac{3}{2}+1}u^{\frac{3}{2}+1} - 5\left(\frac{1}{\frac{1}{2}+1}u^{\frac{1}{2}+1}\right) + C =\end{aligned}$$

(i) $\boldsymbol{\frac{2}{5}u^{\frac{5}{2}} + C}$ (ii) $\boldsymbol{\frac{2}{5}u^{\frac{5}{2}} - \frac{10}{3}u^{\frac{3}{2}} + C}$ (iii) $\boldsymbol{10u^{\frac{3}{2}} + C}$

but $u = x + 5$, so

$$\int f(x)\,dx = \frac{2}{5}u^{\frac{5}{2}} - \frac{10}{3}u^{\frac{3}{2}} + C =$$

(i) $\boldsymbol{\frac{2}{5}(x+5)^{\frac{5}{2}} - \frac{10}{3}(x+5)^{\frac{3}{2}} + C}$
(ii) $\boldsymbol{-\frac{10}{3}(x+5)^{\frac{3}{2}} + C}$
(iii) $\boldsymbol{\frac{2}{5}(x+5)^{\frac{5}{2}} + C}$

2. *Exponential Function and Substitution Technique.*

(a) Find $\int e^{(7+x^3)}\,(3x^2)\,dx$.

guess $u =$ (i) $\boldsymbol{7 + x^3}$ (ii) $\boldsymbol{3x^2}$

then $du = (0 + 3x^{3-1})\,dx =$ (i) $\boldsymbol{(1 + 3x^2)\,dx}$ (ii) $\boldsymbol{(3x^2)\,dx}$

substituting u and du into $\int f(x)\,dx$,

$$\int e^{(7+x^3)}\left(3x^2\right)\,dx = \int e^u\,du = e^u + C$$

but $u = 7 + x^3$, so

$$\int f(x)\,dx = e^u + C =$$

(i) $\boldsymbol{x^3 e^{7+x^3} + C}$ (ii) $\boldsymbol{7e^{7+x^3} + C}$ (iii) $\boldsymbol{e^{7+x^3} + C}$

(b) Find $\int f(x)\,dx = \int -\frac{7}{2}\left[\frac{e^{(-7\sqrt{x})}}{\sqrt{x}}\right]\,dx = \int e^{\left(-7x^{\frac{1}{2}}\right)}\left(-\frac{7}{2}x^{-\frac{1}{2}}\right)\,dx.$

guess $u =$ (i) $\boldsymbol{-7x^{\frac{1}{2}}}$ (ii) $\boldsymbol{-\frac{7}{2}x^{-\frac{1}{2}}}$

then $du = \left(-7\left(\frac{1}{2}\right)x^{\frac{1}{2}-1}\right)\,dx =$ (i) $\boldsymbol{\left(-\frac{7}{2}x^{-\frac{1}{2}}\right)\,dx}$ (ii) $\boldsymbol{\left(\frac{7}{2}x^{-\frac{1}{2}}\right)\,dx}$

substituting u and du into $\int f(x)\,dx$,

$$\int e^{\left(-7x^{\frac{1}{2}}\right)}\left(-\frac{7}{2}x^{-\frac{1}{2}}\right)\,dx = \int e^u\,du = e^u + C$$

but $u = -7x^{\frac{1}{2}} = -7\sqrt{x}$, so

$$\int f(x)\,dx = e^u + C =$$

(i) $\boldsymbol{\sqrt{x}e^{-7\sqrt{x}} + C}$ (ii) $\boldsymbol{-7e^{-7\sqrt{x}} + C}$ (iii) $\boldsymbol{e^{-7\sqrt{x}} + C}$

(c) Find $\int f(x)\,dx = \int \frac{e^{(-7\sqrt{x})}}{\sqrt{x}}\,dx = \int e^{\left(-7x^{\frac{1}{2}}\right)}\left(x^{-\frac{1}{2}}\right)\,dx.$

guess $u =$ (i) $\boldsymbol{-7x^{\frac{1}{2}}}$ (ii) $\boldsymbol{x^{-\frac{1}{2}}}$

then $du = \left(-7\left(\frac{1}{2}\right)x^{\frac{1}{2}-1}\right)\,dx =$ (i) $\boldsymbol{\left(-\frac{7}{2}x^{-\frac{1}{2}}\right)\,dx}$ (ii) $\boldsymbol{\left(-\frac{1}{2}x^{-\frac{3}{2}}\right)\,dx}$

substituting u and du into $\int f(x)\,dx$,

$$\int e^{\left(-7x^{\frac{1}{2}}\right)}\left(x^{-\frac{1}{2}}\right)\,dx = \int e^{\left(-7x^{\frac{1}{2}}\right)}\left(-\frac{2}{7}\right)\left(-\frac{7}{2}x^{-\frac{1}{2}}\right)\,dx = \int e^u\left(-\frac{2}{7}\right)\,du =$$

(i) $\boldsymbol{-\frac{2}{7}e^u + C}$ (ii) $\boldsymbol{-7e^u + C}$ (iii) $\boldsymbol{e^u + C}$

but $u = -7x^{\frac{1}{2}} = -7\sqrt{x}$, so

$$\int f(x)\,dx = -\frac{2}{7}e^u + C =$$

(i) $\boldsymbol{-\frac{2}{7}e^{-7\sqrt{x}} + C}$ (ii) $\boldsymbol{-7e^{-7\sqrt{x}} + C}$ (iii) $\boldsymbol{e^{-7\sqrt{x}} + C}$

(d) Find $\int f(x)\,dx = \int xe^{x^2}\,dx = \int e^{x^2}\,(x)\,dx.$

guess $u =$ (i) $\boldsymbol{e^x}$ (ii) $\boldsymbol{x^2}$

then $du = (2x^{2-1})\,dx =$ (i) $\boldsymbol{(2x)\,dx}$ (ii) $\boldsymbol{(e^x)\,dx}$

substituting u and du into $\int f(x)\,dx$,

$$\int e^{x^2}(x)\ dx = \int e^{x^2}\left(\frac{1}{2}\right)(2x)\ dx = \int e^{u}\left(\frac{1}{2}\right)\ du =$$

(i) $\boldsymbol{-\frac{1}{2}e^u + C}$ (ii) $\boldsymbol{\frac{1}{2}e^u + C}$ (iii) $\boldsymbol{e^u + C}$

but $u = x^2$, so

$$\int f(x)\,dx = \frac{1}{2}e^u + C =$$

(i) $\boldsymbol{-\frac{1}{2}e^{x^2} + C}$ (ii) $\boldsymbol{\frac{1}{2}e^{x^2} + C}$ (iii) $\boldsymbol{e^{x^2} + C}$

(e) Find $\int f(x)\,dx = \int 2x7^{x^2}\,dx = \int 7^{x^2}(2x)\ dx$.

notice if $y = 7^{x^2}$, then $\ln y = \ln 7^{x^2} = x^2 \ln 7$ or $y = e^{x^2 \ln 7}$, so $\int f(x)\,dx = \int 7^{x^2}(2x)\ dx = \int e^{x^2 \ln 7}(2x)\ dx$

guess $u =$ (i) $\boldsymbol{x^2 \ln 7}$ (ii) $\boldsymbol{x^2}$

then $du = (2x^{2-1}\ln 7)\ dx =$ (i) $\boldsymbol{(2x \ln 7)\ dx}$ (ii) $\boldsymbol{(e^{\ln 7})\ dx}$

substituting u and du into $\int f(x)\,dx$,

$$\int e^{x^2 \ln 7}(2x)\ dx = \int e^{x^2 \ln 7}\frac{1}{\ln 7}(2x \ln 7)\ dx = \int e^{u}\left(\frac{1}{\ln 7}\right)\ du =$$

(i) $\boldsymbol{-\frac{1}{\ln 7}e^u + C}$ (ii) $\boldsymbol{\frac{1}{\ln 7}e^u + C}$ (iii) $\boldsymbol{e^u + C}$

but $u = x^2 \ln 7$, so

$$\int f(x)\,dx = \frac{1}{\ln 7}e^u + C =$$

(i) $\boldsymbol{-\frac{1}{\ln 7}e^{x^2 \ln 7} + C}$ (ii) $\boldsymbol{\frac{1}{\ln 7}e^{x^2 \ln 7} + C}$ (iii) $\boldsymbol{e^{x^2 \ln 7} + C}$
which is $\frac{1}{\ln 7}7^{x^2} + C$

3. *Logarithmic Function and Substitution Technique.*

(a) Find $\int \frac{2+2x}{2x+x^2}\,dx = \int (2x + x^2)^{-1}(2 + 2x)\,dx$.

guess $u =$ (i) $\boldsymbol{2 + 2x}$ (ii) $\boldsymbol{2x + x^2}$

then $du = (2(1)x^{1-1} + 2x^{2-1})\ dx =$ (i) $\boldsymbol{(2 + 2x^2)\ dx}$ (ii) $\boldsymbol{(2 + 2x)\ dx}$

substituting u and du into $\int f(x)\,dx$,

$$\int \left(2x+x^2\right)^{-1}(2+2x)\,dx = \int u^{-1}\,du = \ln|u| + C$$

but $u = 2x + x^2$, so

$$\int f(x)\,dx = \ln|u| + C =$$

(i) $\boldsymbol{2\ln|2x+x^2|+C}$ (ii) $\boldsymbol{\ln|2x+x^2|+C}$ (iii) $\boldsymbol{x\ln|2x+x^2|+C}$

(b) Find $\int \frac{4+4x}{2x+x^2}\,dx = \int (2x+x^2)^{-1}(4+4x)\,dx$.

guess $u =$ (i) $\boldsymbol{4+4x}$ (ii) $\boldsymbol{2x+x^2}$

then $du = \left(2(1)x^{1-1} + 2x^{2-1}\right)\,dx =$ (i) $\boldsymbol{(2+2x^2)\;dx}$ (ii) $\boldsymbol{(2+2x)\;dx}$

substituting u and du into $\int f(x)\,dx$,

$$\int \left(2x+x^2\right)^{-1}(4+4x)\,dx = \int \left(2x+x^2\right)^{-1}(2)(2+2x)\,dx = \int u^{-1}(2)\,du =$$

(i) $\boldsymbol{-\ln|u|+C}$ (ii) $\boldsymbol{\ln|u|+C}$ (iii) $\boldsymbol{2\ln|u|+C}$

but $u = 2x + x^2$, so

$$\int f(x)\,dx = 2\ln|u| + C =$$

(i) $\boldsymbol{2\ln|2x+x^2|+C}$ (ii) $\boldsymbol{\ln|2x+x^2|+C}$ (iii) $\boldsymbol{x\ln|2x+x^2|+C}$

(c) Find $\int \frac{\ln x}{x}\,dx = \int \ln x\,(x^{-1})\;dx$.

guess $u =$ (i) $\boldsymbol{\ln x}$ (ii) $\boldsymbol{x^{-1}}$

then $du = (x^{-1})\;dx =$ (i) $\boldsymbol{(x^{-1})\;dx}$ (ii) $\boldsymbol{(-x^{-2})\;dx}$

substituting u and du into $\int f(x)\,dx$,

$$\int \ln x\left(x^{-1}\right)\;dx =$$

(i) $\boldsymbol{\int u\,du = \frac{1}{1+1}u^{1+1} + C = \frac{u^2}{2} + C}$
(ii) $\boldsymbol{\int u^{-1}\,du = \ln|u| + C}$
(iii) $\boldsymbol{2\int u\,du = 2\left(\frac{1}{1+1}u^{1+1} + C\right) = u^2 + C}$

but $u = \ln x$, so

$$\int f(x)\,dx = \frac{u^2}{2} + C =$$

(i) $\boldsymbol{\frac{1}{2}(\ln x)^2 + C}$ (ii) $\boldsymbol{\ln|\ln x| + C}$ (iii) $\boldsymbol{(\ln x)^2 + C}$

(d) Find $\int \frac{\ln 8x}{x}\,dx = \int \ln 8x\,(x^{-1})\,dx$.

guess $u =$ (i) $\boldsymbol{\ln 8x}$ (ii) $\boldsymbol{x^{-1}}$

recall if $u = f[g(x)] = \ln 8x$ and $g(x) = 8x$ and $f(x) = \ln x$

and $g'(x) =$ (i) $\boldsymbol{8x}$ (ii) $\mathbf{8}$ (iii) $\boldsymbol{\ln 8x}$
and $f'(x) =$ (i) $\boldsymbol{\frac{1}{x^2}}$ (ii) $\boldsymbol{\frac{1}{x}}$ (iii) $\boldsymbol{\frac{1}{8x}}$
and so by chain rule

$$\frac{du}{dx} = f'[g(x)] \cdot g'(x) = f'[8x] \cdot (2) = \frac{1}{8x}(8) =$$

(i) $\boldsymbol{\frac{1}{x} = x^{-1}}$ (ii) $\boldsymbol{\frac{2}{x} = 2x^{-1}}$ (iii) $\boldsymbol{\frac{1}{8x}}$

in other words $du =$ (i) $\boldsymbol{(x^{-1})\ dx}$ (ii) $\boldsymbol{(-x^{-2})\ dx}$

substituting u and du into $\int f(x)\,dx$,

$$\int \ln 8x \left(x^{-1}\right)\,dx =$$

(i) $\boldsymbol{\int u\,du = \frac{1}{1+1}u^{1+1} + C = \frac{u^2}{2} + C}$
(ii) $\boldsymbol{\int u^{-1}\,du = \ln|u| + C}$
(iii) $\boldsymbol{2\int u\,du = 2\left(\frac{1}{1+1}u^{1+1} + C\right) = u^2 + C}$

but $u = \ln 8x$, so

$$\int f(x)\,dx = \frac{u^2}{2} + C =$$

(i) $\boldsymbol{\frac{1}{2}(\ln 8x)^2 + C}$ (ii) $\boldsymbol{\ln|\ln 8x| + C}$ (iii) $\boldsymbol{(\ln 8x)^2 + C}$

(e) Find $\int \frac{x-5}{x-4}\,dx$.

notice

$$\frac{x-5}{x-4} = A + \frac{B}{x-4}$$

$$= \frac{A(x-4)+B}{x-4}$$
$$= \frac{Ax+(B-4A)}{x-4}$$

and so $A = 1$ and $B - 4A = -5$
and so $B = -5 + 4A = -5 + 4(1) = -1$
and so $\frac{x-5}{x-4} = A + \frac{B}{x-4} =$ (i) $\mathbf{1 + \frac{1}{x-4}}$ (ii) $\mathbf{1 - \frac{1}{x-4}}$ (iii) $\mathbf{1 + \frac{1}{x+4}}$

in other words,

$$\begin{aligned}
\int \frac{x-5}{x-4}\,dx &= \int \left(1 - \frac{1}{x-4}\right)\,dx \\
&= \int 1\,dx - \int \frac{1}{x-4}\,dx \\
&= \int x^0\,dx - \int \frac{1}{x-4}\,dx \\
&= \frac{1}{0+1}x^{\frac{1}{0+1}} - \int \frac{1}{x-4}\,dx \\
&= x - \int \frac{1}{x-4}\,dx + C
\end{aligned}$$

where, for the second integral, guess $u =$ (i) $\boldsymbol{x-4}$ (ii) $\boldsymbol{x}$

so $du =$ (i) $\boldsymbol{(x^{-1})\;dx}$ (ii) $\boldsymbol{dx}$

substituting u and du into second integral,

$$x - \int \frac{1}{x-4}\,dx + C =$$

(i) $\boldsymbol{x - \int u^{-1}\,du = x - \ln u + C}$
(ii) $\boldsymbol{\int u^{-1}\,du = \ln|u| + C}$
(iii) $\boldsymbol{2\int u\,du = 2\left(\frac{1}{1+1}u^{1+1} + C\right) = u^2 + C}$

but $u = x - 4$, so

$$\int f(x)\,dx = x - \ln u + C =$$

(i) $\boldsymbol{\frac{1}{2}(x-4)^2 + C}$ (ii) $\boldsymbol{x - \ln(x-4) + C}$ (iii) $\boldsymbol{(x-4)^2 + C}$

7.3 Area and the Definite Integral

So far, we have looked at *in*definite integrals; now, we turn to *definite* integrals. An *in*definite integral determines the area under a curve; a *definite* integral determines a *specific* area under a curve between a lower bound a and an upper bound b.

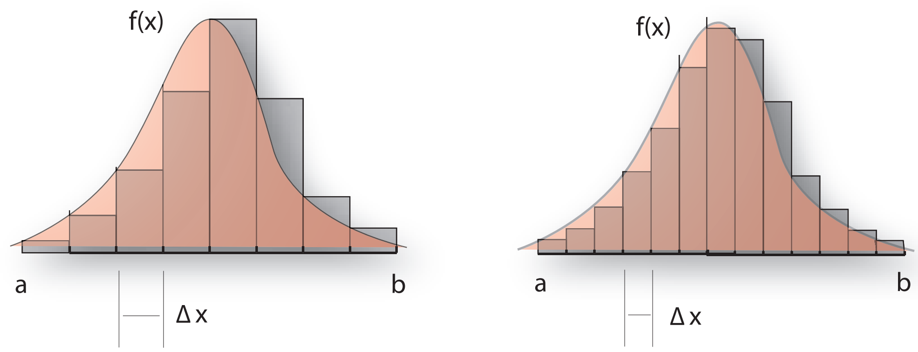

Figure 7.1 (Approximating area with sum of rectangles)

As shown in the figure, the area under the curve, between points a and b, can be approximated by adding the area of n rectangles and this approximation improves the greater the number of increasingly narrow rectangles. If f is defined on interval $[a, b]$ the definite integral is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where the limit exists, $\Delta x = \frac{b-a}{n}$ and x_i is (somewhere, possibly to the left or to the right) in the ith interval. We look at different ways the rectangles are summed, whether using the left endpoints or from the right endpoints or from the middle endpoints or left and right total areas are averaged. In economic applications, the definite integral is called *total change*.

Exercise 7.3 (Area and the Definite Integral)

1. *Review of Summation Notation.* Consider the following $n = 10$ temperatures in Celsius degrees,

$$x_1 = 0,\quad x_2 = 0,\quad x_3 = 0,\quad x_4 = 1,\quad x_5 = 1,$$
$$x_6 = 2,\quad x_7 = 2,\quad x_8 = 3,\quad x_9 = 3,\quad x_{10} = 4$$

and, remember, formula to convert Celsius degrees, x, to Fahrenheit degrees, $f(x)$, is $f(x) = \frac{9}{5}x + 32$. Also, let difference between two temperatures be $\Delta x_i = x_{i+1} - x_i$.

(a) *Sum* of ten temperatures, in Celsius degrees, is

$$\sum_{i=1}^{10} x_i = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} =$$

(i) **16** (ii) **18** (ii) **20**

STAT ENTER EDIT ENTER; type ten temperatures, 0,0,0,1,1,2,2,3,3,4 into L_1, then 1–Var Stats L_1 ENTER, read $\sum = 16$

(b) The sum of temperatures $i = 3$ to $i = 8$ is $\sum_{i=3}^{8} x_i = x_3 + x_4 + x_5 + x_6 + x_7 + x_8 =$ (i) **6** (ii) **9** (iii) **12**

(c) For $n = 4$, $\sum_{i=1}^{n} x_i =$ (i) **1** (ii) **2** (iii) **3**

(d) $\sum_{i=1}^{5} x_i =$ (i) **2** (ii) **5** (iii) **9**

(e) $\sum_{i=10}^{10} x_i =$ (i) **2** (ii) **4** (iii) **9**

(f) The *average* of the $n = 10$ temperatures is $\frac{1}{n}\sum_{i=1}^{n} x_i =$ (i) **1.6** (ii) **4** (iii) **9**

1–Var Stats L_1 ENTER, read $\bar{x} = 1.6$.

(g) $\sum_{i=1}^{4} i = 1 + 2 + 3 + 4 =$ (i) **9** (ii) **10** (iii) **11**

(h) $\sum_{i=1}^{10} i =$ (i) **55** (ii) **56** (iii) **57**

Type 1,2,3,4,5,6,7,8,9, 10 into L_2, then 1–Var Stats L_2 ENTER gives 55 *or* MATH summation X 10 X ENTER gives $\sum = 55$.

(i) $\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2 =$ (i) **20** (ii) **30** (iii) **40**

(j) $\sum_{i=1}^{4}(i \cdot x_i) = (1 \cdot 0) + (2 \cdot 0) + (3 \cdot 0) + (4 \cdot 1) =$ (i) **4** (ii) **6** (iii) **7**

(k) $\sum_{i=1}^{10}(i \cdot x_i) =$ (i) **125** (ii) **126** (iii) **127**

STAT ENTER, define $L_3 = L_2 \times L_1$, then 1–Var Stats L_3, read $\sum = 126$

(l) $\sum_{i=1}^{4}(i + x_i) = (1 + 0) + (2 + 0) + (3 + 0) + (4 + 1) =$ (i) **10** (ii) **11** (iii) **12**

(m) Since $\Delta x_i = x_{i+1} - x_i$, sum of *differences* for temperatures 1 to 6 is

$$\begin{aligned}\sum_{i=1}^{5} \Delta x_i &= \Delta x_1 + \Delta x_2 + \Delta x_3 + \Delta x_4 + \Delta x_5 \\ &= (x_2 - x_1) + (x_3 - x_2) + (x_4 - x_3) + (x_5 - x_4) + (x_6 - x_5) \\ &= (0 - 0) + (0 - 0) + (1 - 0) + (1 - 1) + (2 - 1) \\ &= 0 + 0 + 1 + 0 + 1 =\end{aligned}$$

(i) **2** (ii) **3** (iii) **4**

Notice sum of differences for temperatures 1 to 6 involves five (5), *not* six (6) differences; *always* one less difference than number of numbers (which are temperatures in this case)

(n) $\sum_{i=1}^{9} \Delta x_i =$ (i) **3** (ii) **4** (iii) **8**

$L_3 = \Delta\text{List}L_1$ (type 2nd LIST OPS 7:ΔList), then 1–Var Stats L_3, read $\sum = 4$.

(o) $\sum_{i=1}^{5} \Delta {x_i}^2 = 0^2 + 0^2 + 1^2 + 0^2 + 1^2 =$ (i) **0** (ii) **1** (iii) **2**

(p) $\sum_{i=1}^{5} i\Delta x_i = 1(0) + 2(0) + 3(1) + 4(0) + 5(1) =$ (i) **3** (ii) **5** (iii) **8**

(q) Since $f(x) = \frac{9}{5}x + 32$, sum of ten temperatures, in Fahrenheit degrees,

$$\begin{aligned}
\sum_{i=1}^{10} f(x_i) &= f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_{10}) \\
&= f(0) + f(0) + f(0) + \cdots + f(4) \\
&= \left(\frac{9}{5}(0) + 32\right) + \left(\frac{9}{5}(0) + 32\right) + \left(\frac{9}{5}(0) + 32\right) + \cdots + \left(\frac{9}{5}(4) + 32\right) \\
&= 32 + 32 + 32 + \cdots + 39.2 =
\end{aligned}$$

(i) **161.4** (ii) **218.2** (iii) **348.8**

STAT ENTER, define $L_4 = (9/5)L_1 + 32$, then 1–Var Stats L_4, read $\sum = 348.8$

(r) Sum of *squares* of ten temperatures, in Fahrenheit degrees, is

$$\begin{aligned}
\sum_{i=1}^{10} f(x_i)^2 &= f(x_1)^2 + f(x_2)^2 + f(x_3)^2 + \cdots + f(x_{10})^2 \\
&= f(0)^2 + f(0)^2 + f(0)^2 + \cdots + f(4)^2 \\
&= 32^2 + 32^2 + 32^2 + \cdots + 39.2^2 =
\end{aligned}$$

(i) **12225.76** (ii) **12245.54** (iii) **12343.32**

Read $\sum x^2$.

(s) $\sum_{i=1}^{10} f(x_i)x_i =$

$$\begin{aligned}
&= f(x_1)x_1 + f(x_2)x_2 + f(x_3)x_3 + \cdots + f(x_{10})x_{10} \\
&= (f(0) \cdot 0) + (f(0) \cdot 0) + (f(0) \cdot 0) + \cdots + (f(4) \cdot 4) \\
&= (32 \cdot 0) + (32 \cdot 0) + (32 \cdot 0) + \cdots + (39.2 \cdot 4) =
\end{aligned}$$

(i) **589.2** (ii) **591.2** (iii) **643.3**

Define $L_5 = L_4 \times L_1$, then 1–Var Stats L_5, read $\sum$.

(t) $\sum_{i=1}^{9} f(x_i)\Delta x_i =$

$$\begin{aligned}
&= f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + f(x_3)\Delta x_3 + \cdots + f(x_9)\Delta x_9 \\
&= (f(0) \cdot 0) + (f(0) \cdot 0) + (f(0) \cdot 1) + \cdots + (f(3) \cdot 1) \\
&= (32 \cdot 0) + (32 \cdot 0) + (32 \cdot 1) + \cdots + (37.4 \cdot 1)
\end{aligned}$$

(i) **138.8** (ii) **291.2** (iii) **343.3**

since 10 temperatures, $f(x_i)$, but only 9 differences, Δx_i, one $f(x_i)$ must be deleted to multiply them; deleting last temperature $f(x_{10}) = 39.2$ means differences matched with *left endpoint* temperatures; deleting first temperature $f(x_1) = 32$ means differences matched with *right endpoint* temperatures; delete $f(x_{10}) = 39.2$ in L_4, define $L_5 = L_4 \times L_3$, then 1–Var Stats L_5, read $\sum$

2. *Different methods of approximating area.*

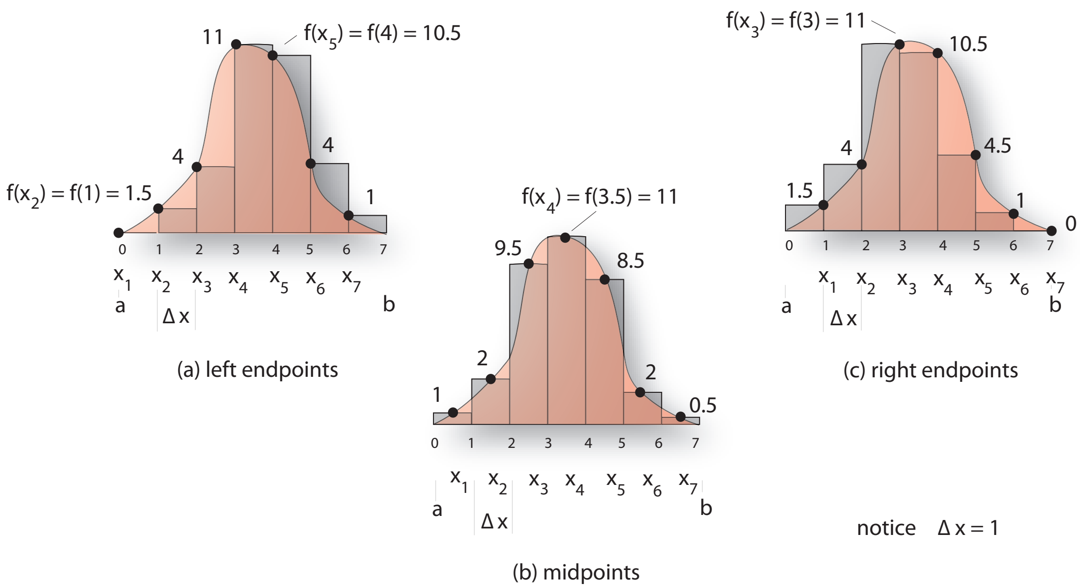

Figure 7.2 (Different methods of approximating area)

Area of smooth function $f(x)$ can be approximated with summed area of seven rectangles in different ways. Three methods are given in figure, each giving slightly different approximations, depending on where (to the left, in the middle, to the right) the x_i are defined in each equally spaced interval, $\Delta x = 1$, between a and b.

(a) *Left endpoint rule: x_i defined at left endpoints of each Δx*

Sum of areas of all seven rectangles is

$$\begin{aligned} &= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_7)\Delta x \\ &= \Delta x\,[f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(x_6) + f(x_7)] \\ &= 1\,[f(0) + f(1) + f(2) + f(3) + f(4) + f(5) + f(6)] \\ &= 1\,[0 + 1.5 + 4 + 11 + 10.5 + 5 + 1] = \end{aligned}$$

(i) **32** (ii) **32.25** (iii) **32.5** (iv) **34.5**

STAT ENTER then type $f(x_i) = 0, 1.5, 4, 11, 10.5, 4, 1$ into L_1,
then STAT CALC ENTER 2nd L_1 ENTER, read $\sum = 32$

(b) *Midpoint rule: x_i defined at midpoints of each Δx*

Sum of areas of all seven rectangles is

$$\begin{aligned} &= \Delta x\,[f(x_1)+f(x_2)+f(x_3)+f(x_4)+f(x_5)+f(x_6)+f(x_7)] \\ &= 1\,[f(0.5)+f(1.5)+f(2.5)+f(3.5)+f(4.5)+f(5.5)+f(6.5)] \\ &= 1\,[1+2+9.5+11+8.5+2+0.5] = \end{aligned}$$

(i) **32** (ii) **32.25** (iii) **32.5** (iv) **34.5**

STAT ENTER then type $f(x_i) = 1, 2, 9.5, 11, 8.5, 2, 0.5$ into L_1,
then STAT CALC ENTER 2nd L_2 ENTER, read $\sum$

(c) *Right endpoint rule: x_i defined at right endpoints of each Δx*

Sum of areas of all seven rectangles is

$$\begin{aligned} &= \Delta x\,[f(x_1)+f(x_2)+f(x_3)+f(x_4)+f(x_5)+f(x_6)+f(x_7)] \\ &= 1\,[f(1)+f(2)+f(3)+f(4)+f(5)+f(6)+f(7)] \\ &= 1\,[1.5+4+11+10.5+4.5+1+0] = \end{aligned}$$

(i) **32** (ii) **32.25** (iii) **32.5** (iv) **34.5**

STAT ENTER then type $f(x_i) = 1.5, 4, 11, 10.5, 4.5, 1, 0$ into L_3,
then STAT CALC ENTER 2nd L_3 ENTER, read $\sum$

(d) *Trapezoid rule: average of left and right endpoints rules.*

Average area from left and right endpoint rules:

$$\frac{\text{area left endpoint rule} + \text{area right endpoint rule}}{2} = \frac{32+32.5}{2} =$$

(i) **32** (ii) **32.25** (iii) **32.5** (iv) **34.5**

(e) (i) **True** (ii) **False**
As number of rectangles increases, the different areas resulting from the different rules to sum the areas of these rectangles, tend to equal not only one another, but tend to equal the actual area under the smooth function $f(x)$ between a and b.

3. *Approximating area of triangle, $f(x) = 3x$.*

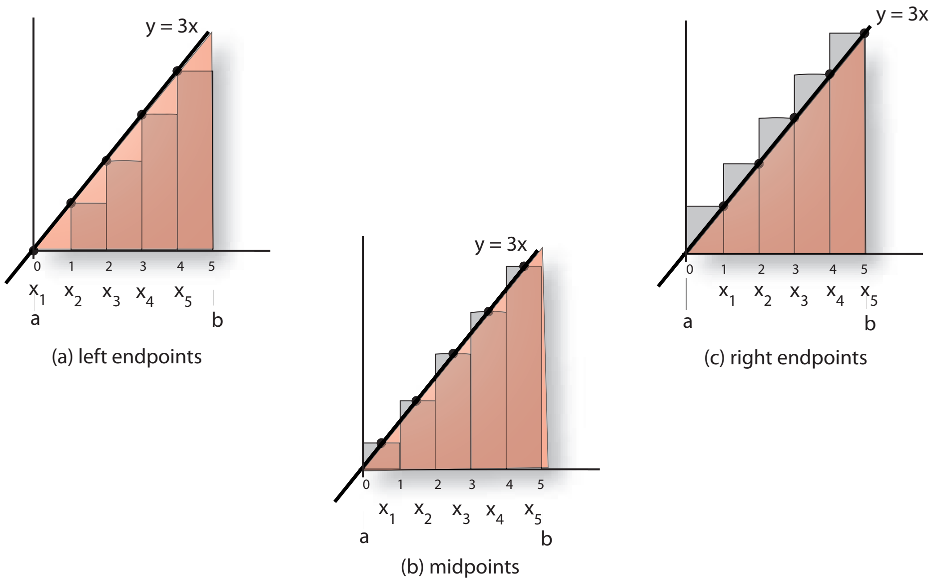

Figure 7.3 (Approximating area of triangle)

Calculate area of triangle under $f(x) = 3x$, $0 \leq x \leq 5$, both exactly and also using the four approximation methods where $n = 5$, $\Delta x = 1$, then where $n = 50$, so $\Delta x = 0.1$, and then where $n = 500$, so $\Delta x = 0.01$.

(a) *exact area*

$$\text{area } = \frac{1}{2}(5)f(5) = \frac{1}{2}(5)(3(5)) =$$

(i) **30** (ii) **37.5** (iii) **45**

area of triangle equal $\frac{1}{2}$ times base times height.

(b) $n = 5$, $\Delta x = 1$

i. *Left endpoint rule: x_i defined at left endpoints of each Δx*

Sum of areas of all five rectangles is

$$\begin{aligned}
&= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x \\
&= \Delta x\,[f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)] \\
&= 1\,[f(0) + f(1) + f(2) + f(3) + f(4)] \\
&= 1\,[3(0) + 3(1) + 3(2) + 3(3) + 3(4)] =
\end{aligned}$$

(i) **30** (ii) **37.5** (iii) **45**

2nd LIST OPS seq 3X, X, 0, 4, 1) STO 2nd L_1 ENTER,

then STAT CALC ENTER 2nd L_1 ENTER, read $\sum = 30$

ii. *Midpoint rule: x_i defined at midpoints of each Δx*

Sum of areas of all five rectangles is

$$\begin{aligned} &= \Delta x\,[f(x_1)+f(x_2)+f(x_3)+f(x_4)+f(x_5)] \\ &= 1\,[f(0.5)+f(1.5)+f(2.5)+f(3.5)+f(4.5)] \\ &= 1\,[3(0.5)+3(1.5)+3(2.5)+3(3.5)+3(4.5)] = \end{aligned}$$

(i) **30** (ii) **37.5** (iii) **45**

2nd LIST OPS seq 3X, X, 0.5, 4.5, 1) STO 2nd L_2 ENTER,
then STAT CALC ENTER 2nd L_2 ENTER, read $\sum$

iii. *Right endpoint rule: x_i defined at right endpoints of each Δx*

Sum of areas of all five rectangles is

$$\begin{aligned} &= \Delta x\,[f(x_1)+f(x_2)+f(x_3)+f(x_4)+f(x_5)] \\ &= 1\,[f(1)+f(2)+f(3)+f(4)+f(5)] \\ &= 1\,[3(1)+3(2)+3(3)+3(4)+3(5)] = \end{aligned}$$

(i) **30** (ii) **37.5** (iii) **45**

2nd LIST OPS seq 3X, X, 1, 5, 1) STO 2nd L_3 ENTER,
then STAT CALC ENTER 2nd L_3 ENTER, read $\sum$

iv. *Trapezoid rule: average of left and right endpoints rules.*

Average area from left and right endpoint rules:

$$\frac{\text{area left endpoint rule} + \text{area right endpoint rule}}{2} = \frac{30+45}{2} =$$

(i) **30** (ii) **37.5** (iii) **45**

(c) $n = 50$, $\Delta x = 0.1$

i. *Left endpoint rule*

Sum of areas of all 50 rectangles is

$$\begin{aligned} &= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_{50})\Delta x \\ &= \Delta x\,[f(x_1)+f(x_2)+f(x_3)+\cdots+f(x_{50})] \\ &= 0.1\,[f(0)+f(0.1)+f(0.2)+\cdots+f(4.9)] \\ &= 0.1\,[3(0)+3(0.1)+3(0.2)+\cdots+3(4.9)] = \end{aligned}$$

(i) **36.75** (ii) **37.5** (iii) **38.25**

2nd LIST OPS seq 3X, X, 0, 4.9, 0.1) STO 2nd L_1 ENTER,
then 2nd LIST MATH sum ENTER 2nd L_1) × 0.1 ENTER

ii. *Midpoint rule*

Sum of areas of all 50 rectangles is

$$\begin{aligned} &= \Delta x\,[f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_{50})] \\ &= 0.1\,[f(0.05) + f(0.15) + f(0.25) + \cdots + f(4.95)] \\ &= 0.1\,[3(0.05) + 3(0.15) + 3(0.25) + \cdots + 3(4.95)] = \end{aligned}$$

(i) **36.75** (ii) **37.5** (iii) **38.25**

2nd LIST OPS seq 3X, X, 0.05, 4.95, 0.1) STO 2nd L_2 ENTER,
then 2nd LIST MATH sum ENTER 2nd L_2) × 0.1 ENTER

iii. *Right endpoint rule*

Sum of areas of all 50 rectangles is

$$\begin{aligned} &= \Delta x\,[f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_{50})] \\ &= 0.1\,[f(0.1) + f(0.2) + f(0.3) + \cdots + f(5)] \\ &= 0.1\,[3(0.1) + 3(0.2) + 3(0.3) + \cdots + 3(5)] = \end{aligned}$$

(i) **36.75** (ii) **37.5** (iii) **38.25**

2nd LIST OPS seq 3X, X, 0.1, 5, 0.1) STO 2nd L_3 STAT ENTER,
then 2nd LIST MATH sum ENTER 2nd L_3) × 0.1 ENTER

iv. *Trapezoid rule*

Average area from left and right endpoint rules:

$$\frac{\text{area left endpoint rule} + \text{area right endpoint rule}}{2} = \frac{36.75 + 38.25}{2} =$$

(i) **36.75** (ii) **37.5** (iii) **38.25**

(d) $n = 500$, $\Delta x = 0.01$

i. *Left endpoint rule*

Sum of areas of all 500 rectangles is

$$= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_{500})\Delta x$$

$$\begin{aligned} &= \Delta x \left[f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_{500})\right] \\ &= 0.01\left[f(0) + f(0.01) + f(0.02) + \cdots + f(4.99)\right] \\ &= 0.01\left[3(0) + 3(0.01) + 3(0.02) + \cdots + 3(4.99)\right] = \end{aligned}$$

(i) **37.425** (ii) **37.5** (iii) **37.575**

2nd LIST OPS seq 3X, X, 0, 4.99, 0.01) STO 2nd L_1 STAT ENTER,
then 2nd LIST MATH sum ENTER 2nd L_1) × 0.01 ENTER

ii. *Midpoint rule*

Sum of areas of all 500 rectangles is

$$\begin{aligned} &= \Delta x \left[f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_{500})\right] \\ &= 0.01\left[f(0.005) + f(0.015) + f(0.025) + \cdots + f(4.995)\right] \\ &= 0.01\left[3(0.005) + 3(0.015) + 3(0.025) + \cdots + 3(4.995)\right] = \end{aligned}$$

(i) **37.425** (ii) **37.5** (iii) **37.575**

2nd LIST OPS seq 3X, X, 0.005, 4.995, 0.01) STO 2nd L_2 STAT ENTER,
then 2nd LIST MATH sum ENTER 2nd L_2) × 0.01 ENTER

iii. *Right endpoint rule*

Sum of areas of all 500 rectangles is

$$\begin{aligned} &= \Delta x \left[f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_{500})\right] \\ &= 0.01\left[f(0.01) + f(0.02) + f(0.03) + \cdots + f(5)\right] \\ &= 0.01\left[3(0.01) + 3(0.02) + 3(0.03) + \cdots + 3(5)\right] = \end{aligned}$$

(i) **37.425** (ii) **37.5** (iii) **37.575**

2nd LIST OPS seq 3X, X, 0.01, 5, 0.01) STO 2nd L_3 STAT ENTER,
then 2nd LIST MATH sum ENTER 2nd L_3) × 0.01 ENTER

iv. *Trapezoid rule*

Average area from left and right endpoint rules:

$$\frac{\text{area left endpoint rule} + \text{area right endpoint rule}}{2} = \frac{37.425 + 37.575}{2} =$$

(i) **37.425** (ii) **37.5** (iii) **37.575**

(e) *Summary.* Recall, exact area under $y = 3x$ and between $a = 0$ and $b = 5$ is 37.5. Summary of approximations:

$n, \Delta x$	left endpoint	midpoint	right endpoint	trapezoid
5, 1	30	37.5	45	37.5
50, 0.1	36.75	37.5	38.25	37.5
500, 0.01	37.425	37.5	37.575	37.5

As number, n, of rectangles increases, all approximations tend
(i) **closer to** (ii) **away from** exact value 37.5.
Best (closest) approximation (choose two):
(i) **left endpoint** (ii) **midpoint** (iii) **right endpoint** (iv) **trapezoid**