12.3 Taylor Polynomials at 0

The *Taylor polynomial of degree* n for differentiable function f at $x = 0$ is

$$P_n(x) = f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n = \sum_{i=0}^{n} \frac{f^{(n)}(0)}{i!}x^i.$$

When $x = 0$, the Taylor polynomial $P_n(x)$ equals the function f exactly, $P_n(0) = f(0)$. For values of x *close* to 0, $P_n(x) \approx f(x)$. If f is *analytic*, for larger n, $P_n(x) \approx f(x)$ for x farther away from 0.

Exercise 12.3 (Taylor Polynomials at 0)

1. *Review of factorial notation.*

   (a) Special mathematical notation, called *factorial notation*, denoted by an exclamation mark, "!", is used in Taylor polynomials. For example,

   $$5! = 5 \times 4 \times 3 \times 2 \times 1 =$$

   (i) **100** (ii) **110** (iii) **120**.

   (Use your calculator: type five (5), then MATH PRB 4:! ENTER.)

   (b) $7! =$ (choose one or more)

   i. $7 \times 6!$
   ii. 5040
   iii. $7 \times 6 \times 5!$
   iv. $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

   (Use your calculator: type seven (7), then MATH PRB 4:! ENTER.)

   (c) $\frac{7!}{5!} =$ (choose one or more)

   i. $\frac{7\times6\times5\times4\times3\times2\times1}{5\times4\times3\times2\times1}$
   ii. 7×6
   iii. 42

   (d) $\frac{7!}{5!3!} =$ (choose one or more)

   i. $\frac{7\times6\times5\times4\times3\times2\times1}{(5\times4\times3\times2\times1)(3\times2\times1)}$
   ii. $\frac{7\times6\times5\times4}{5\times4\times3\times2\times1}$
   iii. $\frac{7\times6}{3\times2\times1}$
   iv. $\frac{42}{6}$

(e) $(7-3)! =$ (i) $\mathbf{7! - 3!}$ (ii) $\mathbf{4!}$ (iii) $\mathbf{4 \times 3 \times 2 \times 1}$ (iv) **24**.

(f) $\frac{7!}{(7-3)!} =$ (choose one or more)

i. $\frac{7\times6\times5\times4\times3\times2\times1}{4\times3\times2\times1}$

ii. $7 \times 6 \times 5$

iii. 210

(g) *By definition* (in other words, accept as true that), $0! = 1$, and so $0! =$ (i) $\mathbf{1!}$ (ii) $\mathbf{2!}$ (iii) $\mathbf{3!}$.

2. *Approximation and exact values for $f(x) = e^x$ near $x = 0$.*

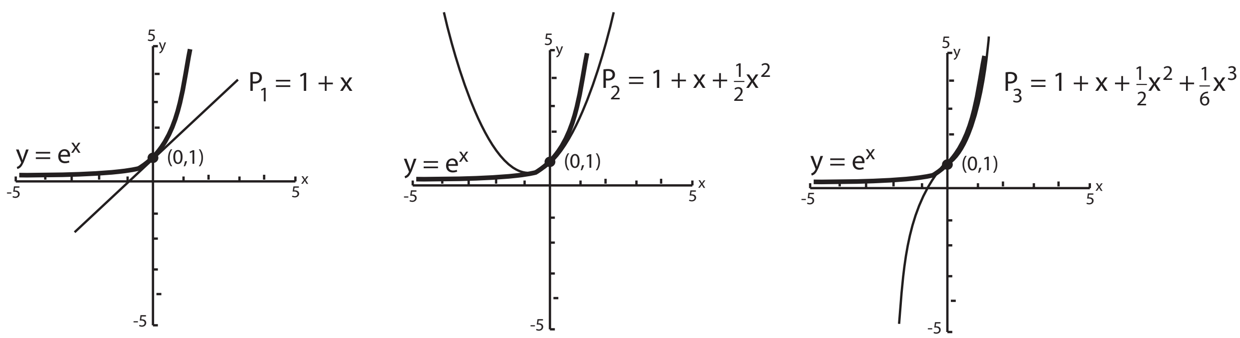

Figure 12.1 (Taylor polynomial approximations to exponential function)

Use WINDOW -5 5 1 -5 5 1 1 define $Y_1 = 1 + X$, $Y_2 = Y_1 + \frac{X^2}{2}$, $Y_3 = Y_2 + \frac{X^3}{6}$, $Y_4 = e^X$

since $f(0) = e^0 =$ (i) $\mathbf{0}$ (ii) $\mathbf{1}$ (iii) $\boldsymbol{x}$

and $f^{(1)}(0) = f'(x) = e^x$, so $f^{(1)}(0) = e^0 =$ (i) $\mathbf{0}$ (ii) $\mathbf{1}$ (iii) $\boldsymbol{x}$

then *Taylor polynomial of degree 1* for $f(x) = e^x$ at $x = 0$ is

$$P_1(x) = f(0) + \frac{f^{(1)}(0)}{1!}x = 1 + \frac{1}{1!}x =$$

(i) $\boldsymbol{1 + x}$ (ii) $\boldsymbol{1 + x + \frac{1}{2}x^2}$ (iii) $\boldsymbol{1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3}$

and since $f^{(2)}(0) = e^x$, so $f^{(2)}(0) = e^0 =$ (i) $\mathbf{0}$ (ii) $\mathbf{1}$ (iii) $\boldsymbol{x}$

and *Taylor polynomial of degree 2* for $f(x) = e^x$ at $x = 0$ is

$$P_2(x) = f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 =$$

(i) $\boldsymbol{1 + x}$ (ii) $\boldsymbol{1 + x + \frac{1}{2}x^2}$ (iii) $\boldsymbol{1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3}$

and so since $f^{(3)}(0) = e^x$, so $f^{(3)}(0) = e^0 =$ (i) **0** (ii) **1** (iii) $\boldsymbol{x}$

and *Taylor polynomial of degree 3* for $f(x) = e^x$ at $x = 0$ is

$$P_3(x) = f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 =$$

(i) $\mathbf{1 + x}$ (ii) $\mathbf{1 + x + \frac{1}{2}x^2}$ (iii) $\mathbf{1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3}$

Functions $P_1(x), P_2(x), P_3(x)$ and $f(x)$ evaluated at various values of x:

x	$P_1(x)$ Approximation $1 + x$	$P_2(x)$ Approximation $1 + x + \frac{1}{2}x^2$	$P_3(x)$ Approximation $1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$	Exact $f(x) = e^x$
-1	0	0.5	0.33333	0.36788
-0.1	0.9	0.905	0.90483	0.90484
-0.01	0.99	0.99005	0.99005	0.99005
-0.001	0.999	0.9990005	0.9990005	0.9990005
0	1	1	1	1
0.001	1.001	1.001	1.001001	1.001001
0.01	1.01	1.01005	1.010050	1.010050
0.1	1.1	1.105	1.105167	1.10517
1	2	2.5	2.666667	2.71828

Define $Y_1 = 1 + X$, $Y_2 = Y_1 + \frac{X^2}{2}$, $Y_3 = Y_2 + \frac{X^3}{6}$, $Y_4 = e^X$
2nd TBLSET -1 1 Ask Auto 2nd TABLE, type -1 -0.1 -0.01 ...

From the table, the Taylor polynomial $P_n(x)$ is closer to $f(x) = e^x$
for x (i) **closer to** (ii) **farther from** 0
and for (i) **smaller** (ii) **larger** n

3. *Approximation and exact values for $f(x) = e^{4x}$ near $x = 0$.*

since $f(0) = e^0 =$ (i) **0** (ii) **1** (iii) $\boldsymbol{x}$

and

$$f^{(1)}(x) = e^{4x} \times 4 \cdot 1x^{1-1} =$$

(i) $\mathbf{4e^{4x}}$ (ii) $\mathbf{e^{4x}}$ (iii) $\mathbf{4x}$
by chain rule

then Taylor polynomial of degree 1 for $f(x) = e^{4x}$ at $x = 0$ is

$$P_1(x) = f(0) + \frac{f^{(1)}(0)}{1!}x = 1 + \frac{4e^{4(0)}}{1!}x =$$

(i) $\mathbf{1+4x}$ (ii) $\mathbf{1+4x+8x^2}$ (iii) $\mathbf{1+4x+8x^2+\frac{32}{3}x^3}$

and

$$f^{(2)}(x) = 4e^{4x} \times 4 \cdot 1x^{1-1} =$$

(i) $\mathbf{4e^{4x}}$ (ii) $\mathbf{16e^{4x}}$ (iii) $\mathbf{4x}$

by chain rule and product rule

then Taylor polynomial of degree 2 for $f(x) = e^{4x}$ at $x = 0$ is

$$P_2(x) = f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 = 1 + \frac{4e^{4(0)}}{1!}x + \frac{16e^{4(0)}}{2!}x^2 =$$

(i) $\mathbf{1+4x}$ (ii) $\mathbf{1+4x+8x^2}$ (iii) $\mathbf{1+4x+8x^2+\frac{32}{3}x^3}$

and

$$f^{(3)}(x) = 16e^{4x} \times 4 \cdot 1x^{1-1} =$$

(i) $\mathbf{4e^{4x}}$ (ii) $\mathbf{16e^{4x}}$ (iii) $\mathbf{64e^{4x}}$

by chain rule and product rule

and Taylor polynomial of degree 3 for $f(x) = e^{4x}$ at $x = 0$ is

$$P_3(x) = f(0)+\frac{f^{(1)}(0)}{1!}x+\frac{f^{(2)}(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3 = 1+\frac{4e^{4(0)}}{1!}x+\frac{16e^{4(0)}}{2!}x^2+\frac{64e^{4(0)}}{3!}x^3 =$$

(i) $\mathbf{1+4x}$ (ii) $\mathbf{1+4x+8x^2}$ (iii) $\mathbf{1+4x+8x^2+\frac{32}{3}x^3}$

and

$$f^{(4)}(x) = 64e^{4x} \times 4 \cdot 1x^{1-1} =$$

(i) $\mathbf{16e^{4x}}$ (ii) $\mathbf{64e^{4x}}$ (iii) $\mathbf{256e^{4x}}$

by chain rule and product rule

and Taylor polynomial of degree 4 for $f(x) = e^{4x}$ at $x = 0$ is

$$\begin{aligned} P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\ &= 1 + \frac{4e^{4(0)}}{1!}x + \frac{16e^{4(0)}}{2!}x^2 + \frac{64e^{4(0)}}{3!}x^3 + \frac{256e^{4(0)}}{4!}x^4 = \end{aligned}$$

(i) $\mathbf{1+4x}$ (ii) $\mathbf{1+4x+8x^2}$ (iii) $\mathbf{1+4x+8x^2+\frac{32}{3}x^3+\frac{32}{3}x^4}$

Functions $P_4(x)$ and $f(x)$ evaluated at various values of x:

x	$P_4(x)$ Approximation $1+4x+8x^2+\frac{32}{3}x^3+\frac{32}{3}x^4$	Exact $f(x)=e^{4x}$
-0.1	0.6704	0.67032
-0.01	0.96079	0.96079
0	1	1
0.01	______	1.04081
0.1	______	1.49182

Define $Y_1 = 1+4X$, $Y_2 = Y_1 + 8X^2$, $Y_3 = Y_2 + \frac{32X^3}{3}$, $Y_3 = Y_3 + \frac{32X^4}{3}$, $Y_4 = e^{4X}$
2nd TBLSET -1 1 Ask Auto 2nd TABLE, type -0.1 -0.01 0 0.01 0.1

4. *Approximation and exact values for* $f(x) = \frac{1}{1-3x} = (1-3x)^{-1}$ *near* $x = 0$.

since $f(0) = \frac{1}{1-3(0)} =$ (i) $\mathbf{0}$ (ii) $\mathbf{1}$ (iii) $\boldsymbol{x}$

and

$$f^{(1)}(x) = -(1-3x)^{-2} \times -3 \cdot 1x^{1-1} =$$

(i) $\frac{\mathbf{1}}{\mathbf{(1-3x)^2}}$ (ii) $\frac{\mathbf{3}}{\mathbf{(1-3x)}}$ (iii) $\frac{\mathbf{3}}{\mathbf{(1-3x)^2}}$

then Taylor polynomial of degree 1 for $f(x) = \frac{1}{1-3x}$ at $x = 0$ is

$$P_1(x) = f(0) + \frac{f^{(1)}(0)}{1!}x = 1 + \frac{\frac{3}{(1-3(0))^2}}{1!}x =$$

(i) $\mathbf{1+3x}$ (ii) $\mathbf{1+3x+9x^2}$ (iii) $\mathbf{1+3x+9x^2+27x^3}$

and

$$f^{(2)}(x) = 3 \cdot -2(1-3x)^{-3} \times -3 \cdot 1x^{1-1} =$$

(i) $\frac{\mathbf{3}}{\mathbf{(1-3x)^2}}$ (ii) $\frac{\mathbf{6}}{\mathbf{(1-3x)}}$ (iii) $\frac{\mathbf{18}}{\mathbf{(1-3x)^3}}$

then Taylor polynomial of degree 2 for $f(x) = \frac{1}{1-3x}$ at $x = 0$ is

$$P_2(x) = f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 = 1 + \frac{\frac{3}{(1-3(0))^2}}{1!}x + \frac{\frac{18}{(1-3(0))^3}}{2!}x^2 =$$

(i) $\mathbf{1+3x}$ (ii) $\mathbf{1+3x+9x^2}$ (iii) $\mathbf{1+3x+9x^2+27x^3}$

and

$$f^{(3)}(x) = 18 \cdot -3(1-3x)^{-4} \times -3 \cdot 1x^{1-1} =$$

(i) $\frac{\mathbf{3}}{\mathbf{(1-3x)^2}}$ (ii) $\frac{\mathbf{162}}{\mathbf{(1-3x)^4}}$ (iii) $\frac{\mathbf{162}}{\mathbf{(1-3x)^3}}$

and Taylor polynomial of degree 3 for $f(x) = \frac{1}{1-3x}$ at $x = 0$ is

$$P_3(x) = f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 = 1 + \frac{\frac{3}{(1-3(0))^2}}{1!}x + \frac{\frac{18}{(1-3(0))^3}}{2!}x^2 + \frac{\frac{162}{(1-3x)^4}}{3!}x^3 =$$

(i) $\mathbf{1+3x}$ (ii) $\mathbf{1+3x+9x^2}$ (iii) $\mathbf{1+3x+9x^2+27x^3}$

and

$$f^{(4)}(x) = 162 \cdot -4(1-3x)^{-5} \times -3 \cdot 1x^{1-1} =$$

(i) $\mathbf{\frac{1944}{(1-3x)^5}}$ (ii) $\mathbf{\frac{648}{(1-3x)^4}}$ (iii) $\mathbf{\frac{162}{(1-3x)^3}}$

and Taylor polynomial of degree 4 for $f(x) = \frac{1}{1-3x}$ at $x = 0$ is

$$\begin{aligned} P_4(x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\ &= 1 + \frac{\frac{3}{(1-3(0))^2}}{1!}x + \frac{\frac{18}{(1-3(0))^3}}{2!}x^2 + \frac{\frac{162}{(1-3x)^4}}{3!}x^3 + \frac{\frac{1944}{(1-3(0))^5}}{4!}x^4 = \end{aligned}$$

(i) $\mathbf{1+3x+9x^2}$ (ii) $\mathbf{1+3x+9x^2+27x^3}$ (iii) $\mathbf{1+3x+9x^2+27x^3+81x^4}$

Functions $P_4(x)$ and $f(x)$ evaluated at various values of x:

x	$P_4(x)$ Approximation $1+3x+9x^2+27x^3+81x^4$	Exact $f(x) = \frac{1}{1-3x}$
-0.1	______	0.76923
-0.01	______	0.97087
0	1	1
0.01	______	1.0309
0.1	______	1.0309

Define $Y_1 = 1+3X$, $Y_2 = Y_1 + 9X^2$, $Y_3 = Y_2 + 27X^3$, $Y_3 = Y_3 + 81X^4$, $Y_4 = \frac{1}{1-3x}$
2nd TBLSET -1 1 Ask Auto 2nd TABLE, type -0.1 -0.01 0 0.01 0.1

5. *Application.* Electric potential, V, at a distance z from disk R is

$$V = k_1\left(\sqrt{z^2+R^2} - z\right).$$

(a) If z is much larger than R, both $\frac{R}{z}$ and $\frac{R^2}{z^2}$ are small; so $x = \frac{R^2}{z^2}$ is close to zero and

$$\sqrt{z^2+R^2} = \sqrt{z^2\left(1+\frac{R^2}{z^2}\right)} = z\sqrt{1+\frac{R^2}{z^2}} = z\sqrt{1+x}$$

and a Taylor polynomial of degree 1 for $f(x) = \sqrt{1+x}$ at $x = 0$ is

$$P_1(x) = f(0) + \frac{f^{(1)}(0)}{1!}x = \sqrt{1+0} + \frac{\frac{1}{2}(1+0)^{-\frac{1}{2}}}{1!}x =$$

(i) $\boldsymbol{1+\frac{1}{2}x}$ (ii) $\boldsymbol{1+\frac{1}{2}x-\frac{1}{8}x^2}$ (iii) $\boldsymbol{1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{6}x^3}$

so

$$z\sqrt{1+x} \approx$$

(i) $\boldsymbol{z\left(1+\frac{1}{2}x\right)}$ (ii) $\boldsymbol{z\left(1+\frac{1}{2}x-\frac{1}{8}x^2\right)}$ (iii) $\boldsymbol{\left(1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{6}x^3\right)}$

and so

$$\begin{aligned} V &= k_1\left(\sqrt{z^2+R^2}-z\right) \\ &= k_1\left(z\sqrt{1+x}-z\right) \quad \text{remember } x=\tfrac{R^2}{z^2} \\ &\approx k_1\left(z\left(1+\frac{1}{2}x\right)-z\right) \\ &= k_1\left(z+\frac{z}{2}x-z\right) = \end{aligned}$$

(i) $\boldsymbol{\frac{k_1R}{2z}}$ (ii) $\boldsymbol{k_1\left(R+\frac{z^2}{2R}-z\right)}$ (iii) $\boldsymbol{\frac{k_1R^2}{2z}}$
remember $x=\frac{R^2}{z^2}$

(b) If z is much *smaller* than R, both $\frac{z}{R}$ and $\frac{z^2}{R^2}$ are small;
so $x=\frac{z^2}{R^2}$ is close to zero and

$$\sqrt{z^2+R^2}=\sqrt{R^2\left(\frac{z^2}{R^2}+1\right)}=R\sqrt{1+\frac{z^2}{R^2}}=z\sqrt{1+x}$$

and a Taylor polynomial of degree 1 for $f(x)=\sqrt{1+x}$ at $x=0$ is

$$P_1(x)=f(0)+\frac{f^{(1)}(0)}{1!}x=\sqrt{1+0}+\frac{\frac{1}{2}(1+0)^{-\frac{1}{2}}}{1!}x=$$

(i) $\boldsymbol{1+\frac{1}{2}x}$ (ii) $\boldsymbol{1+\frac{1}{2}x-\frac{1}{8}x^2}$ (iii) $\boldsymbol{1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{6}x^3}$

so

$$z\sqrt{1+x} \approx$$

(i) $\boldsymbol{z\left(1+\frac{1}{2}x\right)}$ (ii) $\boldsymbol{z\left(1+\frac{1}{2}x-\frac{1}{8}x^2\right)}$ (iii) $\boldsymbol{\left(1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{6}x^3\right)}$

and so

$$\begin{aligned} V &= k_1\left(\sqrt{z^2+R^2}-z\right) \\ &= k_1\left(R\sqrt{1+x}-z\right) \quad \text{remember } x=\tfrac{R^2}{z^2} \end{aligned}$$

$$\approx \quad k_1 \left(R \left(1 + \frac{1}{2}x \right) - z \right)$$
$$= \quad k_1 \left(R + \frac{R}{2}x - z \right) =$$

(i) $\boldsymbol{\frac{k_1 R}{2z}}$ (ii) $\boldsymbol{k_1 \left(R + \frac{z^2}{2R} - z \right)}$ (iii) $\boldsymbol{\frac{k_1 R^2}{2z}}$
remember $x = \frac{z^2}{R^2}$

12.4 Infinite Series

An *infinite series* is

$$a_1 + a_2 + a_3 + \cdots + a_n \cdots = \sum_{i=1}^{\infty} a_i$$

and let $S_n = a_1 + a_2 + a_3 + \cdots + a_n$ be the nth partial sum and suppose

$$\lim_{n \to \infty} S_n = L$$

for some real number L. Then L is the *sum of the infinite series* and the infinite series *converges*. If L does not exist, the series *diverges*. In particular, the *sum of the geometric series* is

$$\sum_{i=1}^{\infty} ar^{i-1} = a + ar + ar^2 + ar^3 + \cdots$$

which *converges* if r is in $(-1, 1)$ and has sum

$$\frac{a}{1-r}$$

and *diverges* if r is outside of $(-1, 1)$.

Exercise 12.4 (Infinite Series)

1. *Identify if geometric series converges or not; give sum of convergent series.*

   (a) geometric sequence

   $$16, 8, 4, 2, \ldots$$

   has common ratio $r =$ (i) $\boldsymbol{\frac{1}{2}}$ (ii) $\boldsymbol{\frac{1}{3}}$ (iii) $\boldsymbol{\frac{1}{4}}$ which is inside $(-1, 1)$
   notice $r = \frac{8}{16} = \frac{4}{8} = \frac{2}{4} = 0.5$

   and so (i) **is** (ii) **is not** a *convergent* geometric series

   and so

   $$\lim_{n \to \infty} S_n = \frac{a}{1-r} = \frac{16}{1 - \frac{1}{2}} =$$

(i) **30** (ii) **31** (iii) **32** (iv) **does not exist**

(b) geometric sequence

$$2, 6, 18, 54, \ldots$$

has common ratio $r =$ (i) $\mathbf{\frac{1}{3}}$ (ii) **2** (iii) **3** which is outside $(-1, 1)$

notice $r = \frac{6}{2} = \frac{18}{6} = \frac{54}{18}$

and so (i) **is** (ii) **is not** a convergent geometric series

and so

$$\lim_{n\to\infty} S_n = \frac{a}{1-r} = \frac{2}{1-3} =$$

(i) $\mathbf{-16}$ (ii) $\mathbf{-32}$ (iii) **0** (iv) **does not exist**

(c) geometric sequence

$$-3, -1, -\frac{1}{3}, -\frac{1}{9}, \ldots$$

has common ratio $r =$ (i) $\mathbf{\frac{1}{3}}$ (ii) **2** (iii) **3** which is inside $(-1, 1)$

notice $r = \frac{-1}{-3} = \frac{1}{3}$

and so (i) **is** (ii) **is not** a convergent geometric series

where

$$\lim_{n\to\infty} S_n = \frac{a}{1-r} = \frac{-3}{1-\frac{1}{3}} =$$

(i) $\mathbf{-\frac{7}{2}}$ (ii) $\mathbf{-4}$ (iii) $\mathbf{-\frac{9}{2}}$ (iv) **does not exist**

(d) geometric sequence

$$1, \frac{1}{2.02}, \frac{1}{2.02^2}, \frac{1}{2.02^3} \ldots$$

has common ratio $r =$ (i) $\mathbf{\frac{1}{2.01}}$ (ii) $\mathbf{\frac{1}{2.02}}$ (iii) $\mathbf{\frac{1}{2.03}}$ which is inside $(-1, 1)$

notice $r = \frac{\frac{1}{2.02}}{1} = \frac{1}{2.02}$

and so (i) **is** (ii) **is not** a convergent geometric series

and so

$$\lim_{n\to\infty} S_n = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2.02}} =$$

(i) $\mathbf{\frac{101}{51}}$ (ii) $\mathbf{\frac{202}{51}}$ (iii) $\mathbf{\frac{303}{51}}$ (iv) **does not exist**

(e) geometric sequence

$$\pi, 1, \frac{1}{\pi}, \frac{1}{\pi^2}, \ldots$$

has common ratio $r =$ (i) $\mathbf{\frac{1}{\pi}}$ (ii) $\mathbf{\frac{1}{\pi^2}}$ (iii) $\mathbf{\frac{1}{\pi^3}}$ which is inside $(-1, 1)$

and so (i) **is** (ii) **is not** a convergent geometric series

and so

$$\lim_{n\to\infty} S_n = \frac{a}{1-r} = \frac{\pi}{1-\frac{1}{\pi}} \approx$$

(i) **4.609** (ii) **4.709** (iii) **4.809** (iv) **does not exist**

(f) geometric sequence

$$0.197, 0.197\left(\frac{1}{1000}\right), 0.197\left(\frac{1}{1000}\right)^2, 0.197\left(\frac{1}{1000}\right)^3, \ldots$$

has common ratio $r =$ (i) $\mathbf{\frac{1}{10}}$ (ii) $\mathbf{\frac{1}{100}}$ (iii) $\mathbf{\frac{1}{1000}}$ which is inside $(-1, 1)$

and so (i) **is** (ii) **is not** a convergent geometric series

and so

$$\lim_{n\to\infty} S_n = \frac{a}{1-r} = \frac{0.197}{1-\frac{1}{1000}} \approx$$

(i) **0.197199201... =** $\mathbf{\frac{197}{199}}$
(ii) **0.197197197... =** $\mathbf{\frac{197}{199}}$
(iii) **0.197198199... =** $\mathbf{\frac{197}{199}}$
(iv) **does not exist**

2. *Given nth term of (non-geometric) sequence, find four terms and partial sums.*

(a) $a_n = \frac{1}{2n-1}$

sequence is

$$a_1, a_2, a_3, a_4 =$$

(i) $\mathbf{1, \frac{1}{4}, \frac{1}{5}, \frac{1}{7}}$ (ii) $\mathbf{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}}$ (iii) $\mathbf{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}}$

2nd LIST OPS seq$\left(\frac{1}{2X-1}, X, 1, 4\right)$, then MATH ENTER for fractions;
notice non-geometric sequence because $\frac{\frac{1}{3}}{1} \neq \frac{\frac{1}{5}}{\frac{1}{3}}$

and series is

$$S_1, S_2, S_3, S_4 =$$

(i) $\mathbf{1, \frac{4}{3}, \frac{24}{15}, \frac{176}{105}}$ (ii) $\mathbf{1, \frac{4}{3}, \frac{23}{15}, \frac{176}{105}}$ (iii) $\mathbf{1, \frac{4}{3}, \frac{23}{15}, \frac{177}{105}}$
$S_1 = a_1 = 1, S_2 = S_1 + a_2 = 1 + \frac{1}{3} = \frac{4}{3}, \ldots$

(b) $a_n = \frac{1}{(n+1)^2}$

sequence is

$$a_1, a_2, a_3, a_4 =$$

(i) $\mathbf{\frac{1}{2}, \frac{1}{9}, \frac{1}{16}, \frac{1}{23}}$ (ii) $\mathbf{\frac{1}{3}, \frac{1}{9}, \frac{1}{16}, \frac{1}{24}}$ (iii) $\mathbf{\frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}}$
2nd LIST OPS seq$\left(\frac{1}{(X+1)^2}, X, 1, 4\right)$, then MATH ENTER for fractions;
notice non-geometric sequence because $\frac{\frac{1}{9}}{\frac{1}{4}} \neq \frac{\frac{1}{16}}{\frac{1}{9}}$

and series is

$$S_1, S_2, S_3, S_4 =$$

(i) $\mathbf{\frac{1}{4}, \frac{13}{36}, \frac{97}{144}, \frac{2567}{3600}}$ (ii) $\mathbf{\frac{1}{4}, \frac{13}{36}, \frac{97}{144}, \frac{2568}{3600}}$ (iii) $\mathbf{\frac{1}{4}, \frac{13}{36}, \frac{61}{144}, \frac{1669}{3600}}$
$S_1 = a_1 = \frac{1}{4}, S_2 = S_1 + a_2 = \frac{1}{4} + \frac{1}{9} = \frac{13}{36}, \ldots$

3. *Application: Bouncing ball.* A ball dropped from a height of 20 meters bounces to $\frac{3}{8}$ths of its previous height. How far does the ball travel before coming to a rest?

   This geometric sequence has common ratio
   $r =$ (i) $\mathbf{\frac{2}{8}}$ (ii) $\mathbf{\frac{3}{8}}$ (iii) $\mathbf{\frac{4}{8}}$ which is inside $(-1, 1)$

   and so (i) **is** (ii) **is not** a convergent geometric series

   and so

$$\lim_{n\to\infty} S_n = \frac{a}{1-r} = \frac{20}{1-\frac{3}{8}} =$$

   (i) **30** (ii) **31** (iii) **32** (iv) **does not exist** meters

12.5 Taylor Series

The *Taylor series* for differentiable function f at $x = 0$ is

$$f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots.$$

Some analytic functions $f(x)$, Taylor series and interval of convergences are:

- $f(x) = e^x$, $1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots$, $(-\infty, \infty)$
- $f(x) = \ln(1+x)$, $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^n x^{n+1}}{n+1} + \cdots$, $(-1, 1]$
- $f(x) = \frac{1}{1-x}$, $1 + x + x^2 + x^3 + \cdots + x^n + \cdots$, $(-1, 1)$

Let f and g be functions with Taylor series

$$\begin{aligned} f(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots \\ g(x) &= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots + b_n x^n + \cdots \end{aligned}$$

and so Taylor series of

- $f + g$ is

$$(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3 + \cdots + (a_n + b_n)x^n + \cdots$$

- $c \cdot f(x)$ is

$$c \cdot a_0 + c \cdot a_1 x + c \cdot a_2 x^2 + c \cdot a_3 x^3 + \cdots + c \cdot a_n x^n + \cdots$$

- $x^k \cdot f(x)$ is

$$\begin{aligned} a_0 x^k &+ a_1 x^k \cdot x + a_2 x^k \cdot x^2 + a_3 x^k \cdot x^3 + \cdots + a_n x^k \cdot x^n + \cdots \\ &= a_0 x^k + a_1 x^{k+1} + a_2 x^{k+2} + a_3 x^{k+3} + \cdots + a_n x^{k+n} + \cdots \end{aligned}$$

- composition $f[g(x)]$, where $g(x) = cx^k$, is

$$a_0 + a_1[g(x)] + a_2[g(x)]^2 + a_3[g(x)]^3 + \cdots + a_n[g(x)]^n + \cdots$$

Taylor series of function is *limit* of Taylor polynomials of function; Taylor polynomial is an finite number of initial terms of Taylor series.

Exercise 12.5 (Taylor Series)

1. *Determine Taylor series for $f(x)$ and interval of convergence.*

   (a) $f(x) = \frac{3}{1-x}$

   Since Taylor series of $\frac{1}{1-x}$ is

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$

then Taylor series of

$$\begin{aligned} f(x) &= 3 \cdot \frac{1}{1-x} \\ &= 3\left(1 + x + x^2 + x^3 + \cdots + x^n + \cdots\right) = \end{aligned}$$

(i) $\mathbf{3 + 3x + 3x^2 + 3x^3 + \cdots + 3x^n + \cdots}$
(ii) $\mathbf{1 + 3x + 3x^2 + 3x^3 + \cdots + 3x^n + \cdots}$
(iii) $\mathbf{1 + x + 2x^2 + 3x^3 + \cdots + nx^n + \cdots}$

where since

$$-1 < x < 1,$$

the interval of convergence for x is
(i) $\mathbf{(-\infty, \infty)}$ (ii) $\mathbf{(-1, 1)}$ (iii) $\mathbf{(-3, 3)}$

(b) $f(x) = \frac{3}{1-x^2}$

Since Taylor series of $\frac{1}{1-x}$ is

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$

then Taylor series of

$$\begin{aligned} f(x) &= 3 \cdot \frac{1}{1-x^2} \\ &= 3\left(1 + x^2 + \left(x^2\right)^2 + \left(x^2\right)^3 + \cdots + \left(x^2\right)^n + \cdots\right) = \end{aligned}$$

(i) $\mathbf{3 + 3x + 3x^2 + 3x^3 + \cdots + 3x^n + \cdots}$
(ii) $\mathbf{3 + 3x^2 + 3x^4 + 3x^6 + \cdots + 3x^{2n} + \cdots}$
(iii) $\mathbf{1 + x^3 + 2x^4 + 3x^6 + \cdots + nx^{2n} + \cdots}$

where since

$$-1 < x^2 < 1 \quad \equiv \quad 0 < x^2 \leq 1,$$

the interval of convergence for x (not x^2) is
(i) $\mathbf{(-\infty, \infty)}$ (ii) $\mathbf{(-1, 1)}$ (iii) $\mathbf{(-\sqrt{2}, \sqrt{2})}$

(c) $f(x) = e^{3x^2}$

Since Taylor series of e^x is

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots$$

then Taylor series of

$$\begin{aligned} f(x) &= e^{3x^2} \\ &= 1 + 3x^2 + \frac{1}{2!}\left(3x^2\right)^2 + \frac{1}{3!}\left(3x^2\right)^3 + \cdots + \frac{1}{n!}\left(3x^2\right)^n + \cdots = \end{aligned}$$

(i) $\boldsymbol{\frac{3^2}{2!}x^4 + \frac{3^3}{3!}x^6 + \frac{3^4}{4!}x^6 + \cdots + \frac{3^n}{n!}x^{2n} + \cdots}$
(ii) $\boldsymbol{1 + \frac{3^2}{2!}x^2 + \frac{3^3}{3!}x^6 + \frac{3^4}{4!}x^6 + \cdots + \frac{3^n}{n!}x^{2n} + \cdots}$
(iii) $\boldsymbol{1 + 3x^2 + \frac{3^2}{2!}x^4 + \frac{3^3}{3!}x^6 + \frac{3^4}{4!}x^6 + \cdots + \frac{3^n}{n!}x^{2n} + \cdots}$

where since

$$-\infty < 3x^2 < \infty,$$

the interval of convergence for x is
(i) $\boldsymbol{(-\infty, \infty)}$ (ii) $\boldsymbol{(-1, 1)}$ (iii) $\boldsymbol{(-\sqrt{2}, \sqrt{2})}$

(d) $f(x) = \frac{e^{2x}+e^{-2x}}{2} = \frac{1}{2}e^{2x} + \frac{1}{2}e^{-2x}$

Since Taylor series of e^x is

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots$$

then Taylor series of

$$\begin{aligned} f(x) &= \frac{1}{2}e^{2x} + \frac{1}{2}e^{-2x} \\ &= \frac{1}{2}\left(1 + 2x + \frac{1}{2!}(2x)^2 + \frac{1}{3!}(2x)^3 + \cdots + \frac{1}{n!}(2x)^n + \cdots\right) \\ &\quad + \frac{1}{2}\left(1 - 2x + \frac{1}{2!}(-2x)^2 + \frac{1}{3!}(-2x)^3 + \cdots + \frac{1}{n!}(-2x)^n + \cdots\right) \\ &= \left(\frac{1}{2} + \frac{1}{2}\right) + \frac{1}{2}(2x - 2x) + \frac{1}{2 \cdot 2!}\left(2^2x^2 + (-2)^2x^2\right) \\ &\quad + \frac{1}{2 \cdot 3!}\left(2^3x^3 + (-2)^3x^3\right) + \frac{1}{2 \cdot 4!}\left(2^4x^4 + (-2)^4x^4\right) + \cdots \end{aligned}$$

(i) $\boldsymbol{1 + \frac{2^2}{2!}x^2 + \frac{2^4}{4!}x^4 + \cdots + \frac{2^n}{n!}x^{2n} + \cdots}$
(ii) $\boldsymbol{1 + x + \frac{2^2}{2!}x^2 + \frac{2^3}{3!}x^3 + \frac{2^4}{4!}x^4 + \cdots + \frac{2^n}{n!}x^n + \cdots}$
(iii) $\boldsymbol{1 + \frac{3^2}{2!}x^4 + \frac{3^3}{3!}x^6 + \frac{3^4}{4!}x^6 + \cdots + \frac{3^n}{n!}x^{2n} + \cdots}$

where since

$$-\infty < 2x < \infty, \quad -\infty < -2x < \infty,$$

the interval of convergence for x is
(i) $\boldsymbol{(-\infty, \infty)}$ (ii) $\boldsymbol{(-1, 1)}$ (iii) $\boldsymbol{(-\sqrt{2}, \sqrt{2})}$

(e) $f(x) = \ln(1+3x^2)$

Since Taylor series of $\ln(1+x)$ is

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^n x^{n+1}}{n+1} + \cdots$$

then Taylor series of

$$\begin{aligned} f(x) &= \ln(1+3x^2) \\ &= 3x^2 - \frac{(3x^2)^2}{2} + \frac{(3x^2)^3}{3} - \frac{(3x^2)^4}{4} + \cdots + \frac{(-1)^n (3x^2)^{n+1}}{n+1} + \cdots = \end{aligned}$$

(i) $\mathbf{3x^2 - \frac{3x^2}{2} + \frac{(3x^2)^3}{3} - \frac{(3x^2)^4}{4} + \cdots + \frac{(-1)^n (3x^2)^{n+1}}{n+1} + \cdots}$
(ii) $\mathbf{3x^2 - \frac{9x^4}{2} + \frac{27x^6}{3} - \frac{81x^8}{4} + \cdots + \frac{(-1)^n 3^{n+1} x^{2n+2}}{n+1} + \cdots}$
(iii) $\mathbf{3x - \frac{(3x^2)^2}{2} + \frac{(3x^2)^3}{3} - \frac{(3x^2)^4}{4} + \cdots + \frac{(-1)^n (3x^2)^{n+1}}{n+1} + \cdots}$

where since

$$-1 < 3x^2 \leq 1 \quad \equiv \quad -\frac{1}{3} < x^2 \leq \frac{1}{3} \quad \equiv \quad 0 < x^2 \leq \frac{1}{3},$$

the interval of convergence for x is
(i) $\left[-\frac{1}{3}, \frac{1}{3}\right]$ (ii) $\left[\sqrt{-\frac{1}{3}}, \sqrt{\frac{1}{3}}\right]$ (iii) $\left[-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}\right]$

2. *Application: normal density.* Density of IQ scores for 16 year olds, x:

$$f(x) = \frac{1}{16\sqrt{2\pi}} e^{-(1/2)[(x-100)/16]^2}.$$

Use four terms of Taylor series to determine $P(84 < X < 100)$

(a) *Taylor series.*

Since Taylor series of e^x is

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots$$

then Taylor series of

$$\begin{aligned} f(x) &= \frac{1}{16\sqrt{2\pi}} e^{-(1/2)[(x-100)/16]^2} \\ &= \frac{1}{16\sqrt{2\pi}} \left(1 - \frac{1}{2}\left(\frac{x-100}{16}\right)^2 + \frac{1}{2!}\left(-\frac{1}{2}\left(\frac{x-100}{16}\right)^2\right)^2 + \frac{1}{3!}\left(-\frac{1}{2}\left(\frac{x-100}{16}\right)^2\right)^3 + \cdots\right) = \end{aligned}$$

(i) $\mathbf{1 - \frac{1}{2}\left(\frac{x-100}{16}\right)^2 + \left(\frac{1}{2}\left(\frac{x-100}{16}\right)^2\right)^2 + \frac{1}{3!}\left(\frac{1}{2}\left(\frac{x-100}{16}\right)^2\right)^3 + \cdots}$

(ii) $\mathbf{\frac{1}{16\sqrt{2\pi}}\left(1 - \frac{1}{2}\left(\frac{x-100}{16}\right)^2 + \left(\frac{1}{2}\left(\frac{x-100}{16}\right)^2\right)^2 + \frac{1}{3!}\left(\frac{1}{2}\left(\frac{x-100}{16}\right)^2\right)^3 + \cdots\right)}$

(iii) $\mathbf{\frac{1}{16\sqrt{2\pi}}\left(1 - \frac{1}{2}\left(\frac{x-100}{16}\right)^2 + \frac{1}{8}\left(\frac{x-100}{16}\right)^4 - \frac{1}{48}\left(\frac{x-100}{16}\right)^6 + \cdots\right)}$

(b) $P(84 < X < 100)$ *using Taylor series*

$$\begin{aligned}
P(84 < X < 100) &= \frac{1}{16\sqrt{2\pi}} \int_{84}^{100} e^{-(1/2)[(x-100)/16]^2}\, dx \\
&\approx \frac{1}{16\sqrt{2\pi}} \int_{84}^{100} \left(1 - \frac{1}{2}\left(\frac{x-100}{16}\right)^2 + \frac{1}{8}\left(\frac{x-100}{16}\right)^4 - \frac{1}{48}\left(\frac{x-100}{16}\right)^6\right) dx \\
&= \frac{1}{16\sqrt{2\pi}} \int_{84}^{100} \left(1 - \frac{1}{2\cdot 16^2}(x-100)^2 + \frac{1}{8\cdot 16^4}(x-100)^4 - \frac{1}{48\cdot 16^6}(x-100)^6\right) dx \\
&= \frac{1}{16\sqrt{2\pi}} \left(x - \frac{1}{2\cdot 16^2\cdot 3}(x-100)^3 + \frac{1}{8\cdot 16^4\cdot 5}(x-100)^5 - \frac{1}{48\cdot 16^6\cdot 7}(x-100)^7\right)_{x=84}^{x=100} \\
&= \frac{1}{16\sqrt{2\pi}} \left(100 - \frac{1}{2\cdot 16^2\cdot 3}(100-100)^3 + \frac{1}{8\cdot 16^4\cdot 5}(100-100)^5 - \frac{1}{48\cdot 16^6\cdot 7}(100-100)^7\right) \\
&\quad - \frac{1}{16\sqrt{2\pi}} \left(84 - \frac{1}{2\cdot 16^2\cdot 3}(84-100)^3 + \frac{1}{8\cdot 16^4\cdot 5}(84-100)^5 - \frac{1}{48\cdot 16^6\cdot 7}(84-100)^7\right) \\
&= \frac{1}{16\sqrt{2\pi}} \left(16 + \frac{1}{2\cdot 16^2\cdot 3}(84-100)^3 - \frac{1}{8\cdot 16^4\cdot 5}(84-100)^5 + \frac{1}{48\cdot 16^6\cdot 7}(84-100)^7\right) \\
&= \frac{1}{16\sqrt{2\pi}} \left(16 - \frac{16^3}{2\cdot 16^2\cdot 3} + \frac{16^5}{8\cdot 16^4\cdot 5} - \frac{16^7}{48\cdot 16^6\cdot 7}\right) \\
&= \frac{1}{\sqrt{2\pi}} \left(1 - \frac{1}{2\cdot 3} + \frac{1}{8\cdot 5} - \frac{1}{48\cdot 7}\right)
\end{aligned}$$

(i) **0.3411** (ii) **0.3412** (iii) **0.3413** (iv) **0.5413**.

(c) $P(84 < X < 100)$ *using calculator*

$$P(84 < X < 100) =$$

(i) **0.3411** (ii) **0.3412** (iii) **0.3413** (iv) **0.5413**.

(2nd DISTR 2:normalcdf(84, 100, 200, 16).