Chapter 13

The Trigonometric Functions

13.3 Integrals Of The Trigonometric Functions

Integrals of trigonometric functions include:

$$\int \sin x\, dx = -\cos x + C \qquad \int \cos x\, dx = \sin x + C$$
$$\int \sec^2 x\, dx = \tan x + C \qquad \int \csc^2 x\, dx = -\cot x + C$$
$$\int \sec x \tan x\, dx = \sec x + C \qquad \int \csc x \cot x\, dx = -\csc x + C$$
$$\int \tan x\, dx = -\ln|\cos x| + C \qquad \int \cot x\, dx = \ln|\sin x| + C$$

Also, recall *derivatives* of trigonometric functions include:

$$D_x[\sin x] = \cos x \qquad D_x[\csc x] = -\cot x \csc x$$
$$D_x[\cos x] = -\sin x \qquad D_x[\sec x] = \tan x \sec x$$
$$D_x[\tan x] = \sec^2 x \qquad D_x[\cot x] = -\csc^2 x$$

Exercise 13.3 (Integrals of Trigonometric Functions)

1. Find $\int \cos x\, dx$.

   From list above,
   $$\int \cos x\, dx =$$
   (i) $\boldsymbol{-\ln|\cos x| + C}$ (ii) $\boldsymbol{\sin x + C}$ (iii) $\boldsymbol{\sec x + C}$

2. Find $\int \sec x \tan x\, dx$.

From list above,

$$\int \sec x \tan x \, dx =$$

(i) $\boldsymbol{-\ln|\cos x| + C}$ (ii) $\boldsymbol{\sin x + C}$ (iii) $\boldsymbol{\sec x + C}$

3. Find $\int \tan x \, dx$.

   From list above,

   $$\int \tan x \, dx =$$

   (i) $\boldsymbol{-\ln|\cos x| + C}$ (ii) $\boldsymbol{\sin x + C}$ (iii) $\boldsymbol{\sec x + C}$

4. Find $\int \sin^3 x \cos x \, dx$.

   guess $u =$ (i) $\boldsymbol{\sin x}$ (ii) $\boldsymbol{\sin^3 x}$ (ii) $\boldsymbol{2x}$

   then $du =$ (i) $\boldsymbol{\cos x \, dx}$ (ii) $\boldsymbol{2 \sin x \cos x \, dx}$ (iii) $\boldsymbol{2 \, dx}$
   recall, we are looking for the *derivative* here, use the trigonometric derivatives above

   substituting u and du into $\int f(x) \, dx$,

   $$\int \sin^3 x \cos x \, dx = \int u^3 \, du = \frac{1}{3+1} u^{3+1} + C =$$

   (i) $\boldsymbol{\int u^3 + C}$ (ii) $\boldsymbol{\frac{1}{4} u^4 + C}$ (iii) $\boldsymbol{\frac{1}{2} u^2 + C}$

   but $u = \sin x$, so

   $$\int f(x) \, dx = \frac{1}{4} u^4 + C =$$

   (i) $\boldsymbol{\frac{1}{2} \sin^4 x + C}$ (ii) $\boldsymbol{\frac{1}{4} \sin^4 x + C}$ (iii) $\boldsymbol{\frac{1}{3} \sin^4 x + C}$

5. Find $\int \tan^3 x \sec^2 x \, dx$.

   guess $u =$ (i) $\boldsymbol{\tan x}$ (ii) $\boldsymbol{\sec^2 x}$ (ii) $\boldsymbol{2x}$

   then $du =$ (i) $\boldsymbol{\sec^2 x \, dx}$ (ii) $\boldsymbol{2 \tan x \sec^2 x \, dx}$ (iii) $\boldsymbol{2 \, dx}$
   again, we are looking for the *derivative* here, use the trigonometric derivatives above

   substituting u and du into $\int f(x) \, dx$,

   $$\int \tan^3 x \sec^2 x \, dx = \int u^3 \, du = \frac{1}{3+1} u^{3+1} + C =$$

(i) $\int \boldsymbol{u^3 + C}$ (ii) $\boldsymbol{\frac{1}{4}u^4 + C}$ (iii) $\boldsymbol{\frac{1}{2}u^2 + C}$

but $u = \tan x$, so

$$\int f(x)\, dx = \frac{1}{4}u^4 + C =$$

(i) $\boldsymbol{\frac{1}{4}\tan^2 x + C}$ (ii) $\boldsymbol{\frac{1}{4}\tan^3 x + C}$ (iii) $\boldsymbol{\frac{1}{4}\tan^4 x + C}$

6. Find $\int \sec^2 5x\, dx$.

   guess $u =$ (i) $\boldsymbol{\tan x}$ (ii) $\boldsymbol{\sec^2 x}$ (iii) $\boldsymbol{5x}$

   then $du = (5x^{2-1})\, dx =$ (i) $\boldsymbol{\cos x\, dx}$ (ii) $\boldsymbol{\sec^2 x\, dx}$ (iii) $\boldsymbol{5\, dx}$
   again, remember, derivative here

   substituting u and du into $\int f(x)\, dx$,

$$\int \sec^2 5x\, dx = \int \sec^2 5x \left(\frac{1}{5}\right)(5x)\ dx = \int \sec^2 u \left(\frac{1}{5}\right)\ du =$$

   (i) $\boldsymbol{-\frac{1}{5}\tan u + C}$ (ii) $\boldsymbol{\frac{1}{5}\csc u + C}$ (iii) $\boldsymbol{\frac{1}{5}\tan u + C}$

   but $u = 5x$, so

$$\int f(x)\, dx =$$

   (i) $\boldsymbol{\frac{1}{5}\tan 5x + C}$ (ii) $\boldsymbol{\tan 5x + C}$ (iii) $\boldsymbol{-\frac{1}{5}\tan 5x + C}$

7. Find $\int (\sin x + x^2)^{-1}(\cos x + 2x)\, dx$.

   guess $u =$ (i) $\boldsymbol{\sin x + x^2}$ (ii) $\boldsymbol{\cos x + 2x}$

   then $du = (\cos x + 2x^{2-1})\ dx =$
   (i) $\boldsymbol{\ln x\, dx}$ (ii) $\boldsymbol{(\tan x \sec x + 2x)\, dx}$ (iii) $\boldsymbol{\cos x + 2x\, dx}$
   again, remember, derivative here

   substituting u and du into $\int f(x)\, dx$,

$$\int \left(\sin x + x^2\right)^{-1}(\cos x + 2x)\, dx = \int u^{-1}\, du =$$

   (i) $\boldsymbol{2\ln|u| + C}$ (ii) $\boldsymbol{\ln|u| + C}$ (iii) $\boldsymbol{3\ln|u| + C}$

   but $u = \sin x + x^2$, so

$$\int f(x)\, dx =$$

(i) $\mathbf{2\ln|\sin x + x^2| + C}$ (ii) $\mathbf{\ln|\sin x + x^2| + C}$ (iii) $\mathbf{3\ln|\sin x + x^2| + C}$

8. Find $\int \sin^3(2x+x^2)(2+2x)\cos(2x+x^2)\,dx$.

   guess $u =$ (i) $\mathbf{\sin(2x+x^2)}$ (ii) $\mathbf{(2x+x^2)}$ (iii) $\mathbf{2x}$

   recall if $u = f[g(x)] = \sin(2x+x^2)$ and $g(x) = 2x+x^2$ and $f(x) = \sin x$

   and $g'(x) =$ (i) $\mathbf{2+2x}$ (ii) $\mathbf{2x^2+3x^3}$ (iii) $\mathbf{1+x}$
   and $f'(x) =$ (i) $\mathbf{\cos x}$ (ii) $\mathbf{-\cos x}$ (iii) $\mathbf{\ln x}$
   and so by chain rule

   $$\frac{du}{dx} = f'[g(x)] \cdot g'(x) = f'\left[2x+x^2\right] \cdot (2+2x) = \cos(2x+x^2)\,(2+2x) =$$

   (i) $\mathbf{\cos(2x+x^2)}$ (ii) $\mathbf{\cos(2x+x^2)\,(2+2x)}$ (iii) $\mathbf{(2+2x)}$

   in other words $du =$ (i) $\mathbf{\cos(2x+x^2)\,(2+2x)\ dx}$ (ii) $\mathbf{(2+2x)\ dx}$

   substituting u and du into $\int f(x)\,dx$,

   $$\int \sin^3\left(2x+x^2\right)(2+2x)\cos\left(2x+x^2\right)dx =$$

   (i) $\mathbf{\int u^3\,du = \frac{1}{3+1}u^{3+1} + C = \frac{u^4}{4} + C}$
   (ii) $\mathbf{\int u^2\,du = \frac{1}{2+1}u^{2+1} + C = \frac{u^3}{3} + C}$
   (iii) $\mathbf{\int u\,du = \frac{1}{1+1}u^{1+1} + C = \frac{u^2}{2} + C}$

   but $u = \sin(2x+x^2)$, so

   $$\int f(x)\,dx = \frac{u^4}{4} + C =$$

   (i) $\mathbf{\frac{1}{4}(\sin(2x+x^2))^4 + C}$
   (ii) $\mathbf{\frac{1}{3}(\sin(2x+x^2))^3 + C}$
   (iii) $\mathbf{\frac{1}{2}(\sin(2x+x^2))^2 + C}$