12.6 Newton's Method

A *zero* of function f is x such that $f(x) = 0$. A numerical technique to determine zeroes is *Newton's method*:

$$c_{n+1} = c_n - \frac{f(c_n)}{f'(c_n)}$$

where it is assumed $f'(c_n) \neq 0$ and initial guess at c_1 is found in a closed interval $[a, b]$ where $f(a)$ and $f(b)$ are of opposite sign, on either side of the zero, $f(c) = 0$.

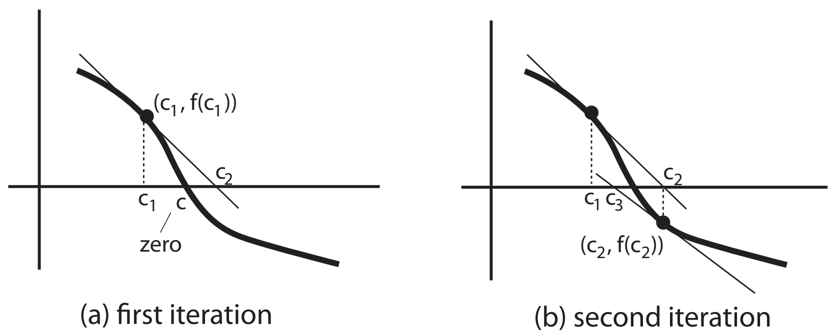

Figure 12.2 (Newton's approximation method for zeroes)

This method typically "zeroes" in on the zero where next c value equals last c value − "run" of slope:
$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = c_1 - f(c_1) \times \frac{1}{f'(c_1)} = c_1 -$ "rise" × ("run" ÷ "rise") $= c_1 +$ "run" (because "rise < 0")
and $c_3 = c_2 -$ "run" (because "rise > 0").

Exercise 12.6 (Newton's Method)

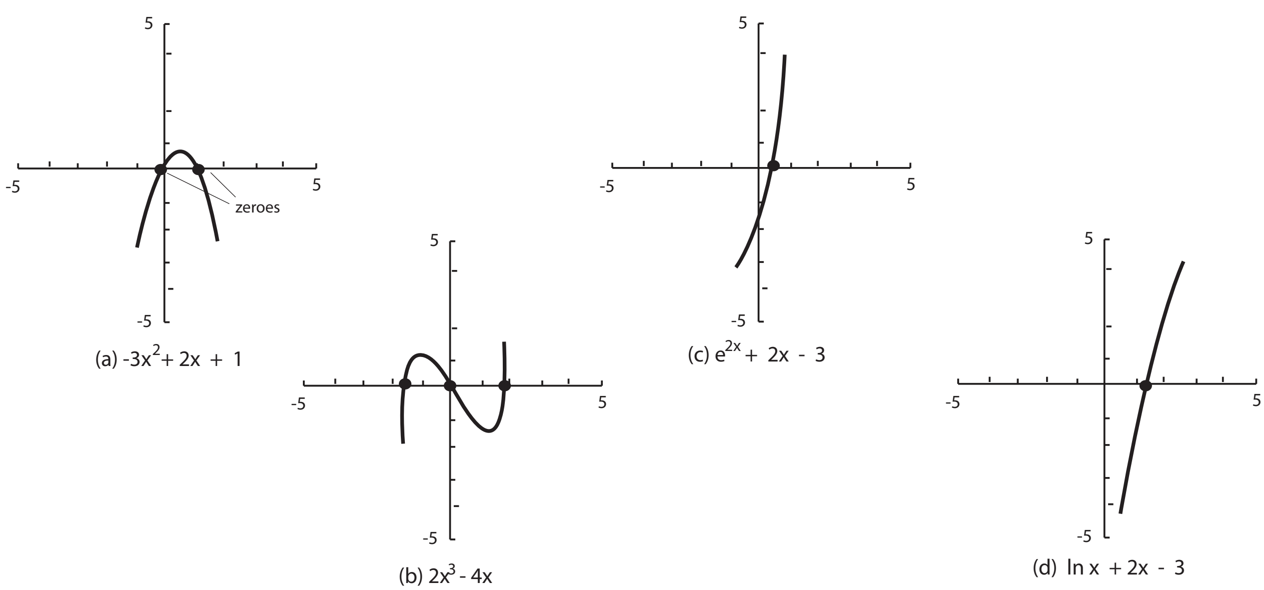

Figure 2.3 (Various functions and their zeroes)

1. *Find zeroes of functions in given interval(s) using Newton's method.*

   (a) $f(x) = -3x^2 + 2x + 1, \quad [-1, 0], \quad [0, 2]$
   Derivative is
   $$f'(x) = -3(2)x^{2-1} + 2(1)x^{1-1} =$$
   (i) $\mathbf{6x + 2}$ (ii) $\mathbf{6x^2 + 2x}$ (iii) $\mathbf{-6x + 2}$

   zero in $[0, 2]$*:* let initial guess be $c_1 = 2$, so
   $$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{-3(2)^2 + 2(2) + 1}{-6(2) + 2} \approx$$
   (i) **1.1** (ii) **1.2** (iii) **1.3**
   $Y_1 = -3X^2 + 2X + 1$, $Y_2 = -6X + 2$, then $2 \to X$ and $X - Y_1/Y_2 \to X$

   $$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = 1.3 - \frac{f(1.3)}{f'(1.3)} = 1.3 - \frac{-3(1.3)^2 + 2(1.3) + 1}{-6(1.3) + 2} \approx$$
   (i) **1.047** (ii) **1.051** (iii) **1.058**
   $X - Y_1/Y_2 \to X$

   $$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)} \approx 1.047 - \frac{f(1.047)}{f'(1.047)} \approx 1.047 - \frac{-3(1.047)^2 + 2(1.047) + 1}{-6(1.047) + 2} \approx$$
   (i) **1.001** (ii) **1.004** (iii) **1.006**
   $X - Y_1/Y_2 \to X$

   zero in $[-1, 0]$*:* let initial guess be $c_1 = -1$, so
   $$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = -1 - \frac{f(-1)}{f'(-1)} = -1 - \frac{-3(-1)^2 + 2(-1) + 1}{-6(-1) + 2} \approx$$
   (i) $\mathbf{-0.3}$ (ii) $\mathbf{-0.4}$ (iii) $\mathbf{-0.5}$
   $-1 \to X$ and $X - Y_1/Y_2 \to X$

   $$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = -0.5 - \frac{f(-0.5)}{f'(-0.5)} = -0.5 - \frac{-3(1.3)^2 + 2(1.3) + 1}{-6(1.3) + 2} \approx$$
   (i) $\mathbf{-0.25}$ (ii) $\mathbf{-0.35}$ (iii) $\mathbf{-0.45}$
   $X - Y_1/Y_2 \to X$

$$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)} \approx -0.35 - \frac{f(-0.35)}{f'(-0.35)} \approx -0.35 - \frac{-3(-0.35)^2 + 2(-0.35) + 1}{-6(-0.35) + 2} \approx$$

(i) $\mathbf{-0.314}$ (ii) $\mathbf{-0.324}$ (iii) $\mathbf{-0.334}$

$X - Y_1/Y_2 \to X$

(b) $f(x) = 2x^3 - 4x, \quad [-2, -1], \quad [-1, 1], \quad [1, 2]$

Derivative is

$$f'(x) = 2(3)x^{3-1} - 4(1)x^{1-1} =$$

(i) $\mathbf{6x^2 - 4}$ (ii) $\mathbf{6x^2 + 4x}$ (iii) $\mathbf{-6x^2 + 2}$

zero in $[-2, -1]$*:* let initial guess be $c_1 = -2$, so

$$c_{-2} = c_1 - \frac{f(c_1)}{f'(c_1)} = -2 - \frac{f(-2)}{f'(-2)} = -2 - \frac{2(-2)^3 - 4(-2)}{6(-2)^2 - 4} \approx$$

(i) $\mathbf{-1.6}$ (ii) $\mathbf{-1.5}$ (iii) $\mathbf{-1.4}$

$Y_1 = 2X^3 - 4X$, $Y_2 = 6X^2 - 4$, then $-2 \to X$ and $X - Y_1/Y_2 \to X$

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = -1.6 - \frac{f(-1.6)}{f'(-1.6)} = -1.6 - \frac{2(-1.6)^3 - 4(-1.6)}{6(-1.6)^2 - 4} \approx$$

(i) $\mathbf{-1.432}$ (ii) $\mathbf{-1.442}$ (iii) $\mathbf{-1.452}$

$X - Y_1/Y_2 \to X$

$$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)} = -1.442 - \frac{f(-1.442)}{f'(-1.442)} = -1.442 - \frac{2(-1.442)^3 - 4(-1.442)}{6(-1.442)^2 - 4} \approx$$

(i) $\mathbf{-1.415}$ (ii) $\mathbf{-1.425}$ (iii) $\mathbf{-1.435}$

$X - Y_1/Y_2 \to X$

zero in $[-1, -1]$*:* let initial guess be $c_1 = -0.5$, so

$$c_{-2} = c_1 - \frac{f(c_1)}{f'(c_1)} = -0.5 - \frac{f(-0.5)}{f'(-0.5)} = -0.5 - \frac{2(-0.5)^3 - 4(-0.5)}{6(-0.5)^2 - 4} \approx$$

(i) $\mathbf{0.6}$ (ii) $\mathbf{0.4}$ (iii) $\mathbf{0.2}$

$-0.5 \to X$ and $X - Y_1/Y_2 \to X$

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = 0.2 - \frac{f(0.2)}{f'(0.2)} = 0.2 - \frac{2(0.2)^3 - 4(0.2)}{6(0.2)^2 - 4} \approx$$

(i) $\mathbf{-0.009}$ (ii) $\mathbf{-0.010}$ (iii) $\mathbf{-0.011}$

$X - Y_1/Y_2 \to X$

$$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)} = -0.009 - \frac{f(-0.009)}{f'(-0.009)} = -0.009 - \frac{2(-0.009)^3 - 4(-0.009)}{6(-0.009)^2 - 4} \approx$$

(i) **0.000** (ii) **0.001** (iii) **0.002**

$X - Y_1/Y_2 \to X$

zero in $[1, 2]$: let initial guess be $c_1 = 1$, so

$$c_{-2} = c_1 - \frac{f(c_1)}{f'(c_1)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{2(1)^3 - 4(1)}{6(1)^2 - 4} \approx$$

(i) **0** (ii) **1** (iii) **2**

$1 \to X$ and $X - Y_1/Y_2 \to X$

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{2(2)^3 - 4(2)}{6(2)^2 - 4} \approx$$

(i) **1.4** (ii) **1.6** (iii) **1.8**

$X - Y_1/Y_2 \to X$

$$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)} = 1.6 - \frac{f(1.6)}{f'(1.6)} = 1.6 - \frac{2(1.6)^3 - 4(1.6)}{6(1.6)^2 - 4} \approx$$

(i) **1.442** (ii) **1.452** (iii) **1.462**

$X - Y_1/Y_2 \to X$

(c) $f(x) = e^{2x} + 2x - 3, \quad [0, 1]$
Derivative is

$$f'(x) = 2e^{2x} + 2x^{1-1} =$$

(i) $\mathbf{2e^x + 2}$ (ii) $\mathbf{2e^{2x} + 1}$ (iii) $\mathbf{2e^{2x} + 2}$

let initial guess be $c_1 = 1$, so

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{e^{2(1)} + 2(1) - 3}{2e^{2(1)} + 2} \approx$$

(i) **0.15** (ii) **0.16** (iii) **0.17**

$Y_1 = e^{2X} + 2X - 3$, $Y_2 = 2e^{2X} + 2$, then $1 \to X$ and $X - Y_1/Y_2 \to X$

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = 0.17 - \frac{f(0.17)}{f'(0.17)} = 0.17 - \frac{e^{2(0.17)} + 2(0.17) - 3}{2e^{2(0.17)} + 2} \approx$$

(i) **0.134** (ii) **0.145** (iii) **0.155**

$X - Y_1/Y_2 \to X$

$$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)} = 0.134 - \frac{f(0.134)}{f'(0.134)} = 0.134 - \frac{e^{2(0.134)} + 2(0.134) - 3}{2e^{2(0.134)} - 2} \approx$$

(i) **0.149** (ii) **0.159** (iii) **0.169**

$X - Y_1/Y_2 \to X$

(d) $f(x) = \ln x + 2x - 3, \quad [1, 2]$
Derivative is

$$f'(x) = \frac{1}{x} + 2x^{1-1} =$$

(i) $\boldsymbol{\frac{2}{x} - 2}$ (ii) $\boldsymbol{\frac{1}{x} + 2}$ (iii) $\boldsymbol{\frac{2}{x} + x}$

let initial guess be $c_1 = 2$, so

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{\ln(2) + 2(2) - 3}{\frac{1}{2} + 2} \approx$$

(i) **1.123** (ii) **1.223** (iii) **1.323**

$Y_1 = \ln X + 2X - 3$, $Y_2 = \frac{1}{X} + 2$, then $2 \to X$ and $X - Y_1/Y_2 \to X$

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = 1.323 - \frac{f(1.323)}{f'(1.323)} = 1.323 - \frac{\ln(1.323) + 2(1.323) - 3}{\frac{1}{1.323} + 2} \approx$$

(i) **1.350** (ii) **1.360** (iii) **1.370**

$X - Y_1/Y_2 \to X$

$$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)} = 1.350 - \frac{f(1.350)}{f'(1.350)} = 1.350 - \frac{\ln(1.350) + 2(1.350) - 3}{\frac{1}{1.350} + 2} \approx$$

(i) **1.350** (ii) **1.360** (iii) **1.370**

$X - Y_1/Y_2 \to X$

2. *Find roots of functions using Newton's method.*

(a) $f(x) = \sqrt{23}$

$\sqrt{23}$ is solution to $x^2 - 23 = 0$, so let $f(x) = x^2 - 23$, so

$$f'(x) = 2x^{2-1} =$$

(i) $\boldsymbol{x}$ (ii) **2** (iii) $\boldsymbol{2x}$

since $4 < \sqrt{23} < 5$, let initial guess be $c_1 = 4$, so

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 4 - \frac{f(4)}{f'(4)} = 4 - \frac{(4)^2 - 23}{2(4)} \approx$$

(i) **4.675** (ii) **4.775** (iii) **4.875**

$Y_1 = X^2 - 23$, $Y_2 = 2X$, then $4 \to X$ and $X - Y_1/Y_2 \to X$

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = 4.875 - \frac{f(4.875)}{f'(4.875)} = 4.875 - \frac{(4.875)^2 - 23}{2(4.875)} \approx$$

(i) **4.596** (ii) **4.696** (iii) **4.796**

$X - Y_1/Y_2 \to X$

$$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)} = 4.796 - \frac{f(4.796)}{f'(4.796)} = 4.796 - \frac{(4.796)^2 - 23}{2(4.796)} \approx$$

(i) **4.596** (ii) **4.696** (iii) **4.796**

$X - Y_1/Y_2 \to X$

(b) $f(x) = \sqrt{32}$

$\sqrt{23}$ is solution to $x^2 - 32 = 0$, so let $f(x) = x^2 - 32$, so

$$f'(x) = 2x^{2-1} =$$

(i) $\boldsymbol{x}$ (ii) **2** (iii) $\boldsymbol{2x}$

since $5 < \sqrt{32} < 6$, let initial guess be $c_1 = 5$, so

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 5 - \frac{f(5)}{f'(5)} = 5 - \frac{(5)^2 - 32}{2(5)} \approx$$

(i) **5.7** (ii) **5.8** (iii) **5.9**

$Y_1 = X^2 - 32$, $Y_2 = 2X$, then $5 \to X$ and $X - Y_1/Y_2 \to X$

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = 5.7 - \frac{f(5.7)}{f'(5.7)} = 5.7 - \frac{(5.7)^2 - 32}{2(5.7)} \approx$$

(i) **5.657** (ii) **5.757** (iii) **5.857**

$X - Y_1/Y_2 \to X$

$$c_4 = c_3 - \frac{f(c_3)}{f'(c_3)} = 5.657 - \frac{f(5.657)}{f'(5.657)} = 5.657 - \frac{(5.657)^2 - 32}{2(5.657)} \approx$$

(i) **5.657** (ii) **5.757** (iii) **5.857**

$X - Y_1/Y_2 \to X$

3. *Find critical point(s) of functions using Newton's method:* $g(x) = x^2 + 4x - 21$

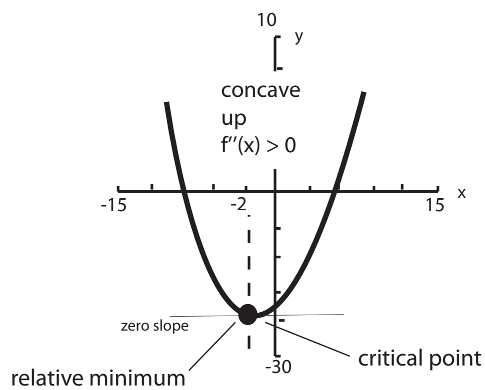

Figure 12.4 (Second derivative test: $g(x) = x^2 + 4x - 21$)

GRAPH using $Y_2 = x^2 + 4x - 21$, with WINDOW -15 15 1 -30 10 1 1

Recall, critical points occur when derivative zero:

$f'(x) = 2x^{2-1} + 4(1)x^{1-1} = 2x + 4 = 0$, so, clearly, critical point is at $2x + 4 = 0$ or $x = -2$

but, continuing with Newton's method, let $f(x) = 2x + 4$ and

$$f'(x) = 2x^{1-1} =$$

(i) $\boldsymbol{x}$ (ii) **2** (iii) $\boldsymbol{2x}$

let initial guess be $c_1 = 1$, so

$$c_2 = c_1 - \frac{f(c_1)}{f'(c_1)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{2(1) + 4}{2} =$$

(i) $\mathbf{-2}$ (ii) $\mathbf{-4}$ (iii) $\mathbf{-6}$

$Y_1 = 2x + 4$, $Y_2 = 2$, then $5 \to X$ and $X - Y_1/Y_2 \to X$

$$c_3 = c_2 - \frac{f(c_2)}{f'(c_2)} = -2 - \frac{f(-2)}{f'(-2)} = -2 - \frac{2(-2)+4}{2} =$$

(i) $\mathbf{-2}$ (ii) $\mathbf{-4}$ (iii) $\mathbf{-6}$

$X - Y_1/Y_2 \to X$

so there is a critical number at
$c =$ (i) $\mathbf{-2}$ (ii) $\mathbf{-4}$ (iii) $\mathbf{-6}$

12.7 L'Hospital's Rule

Let f and g be functions and a a real number such that

$$\lim_{x\to a} f(x) = 0, \quad \lim_{x\to a} g(x) = 0$$

or

$$\lim_{x\to a} f(x) = \pm\infty, \quad \lim_{x\to a} g(x) = \pm\infty$$

and both f and g are differentiable in an open interval containing a. Then

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = L \quad \Rightarrow \quad \lim_{x\to a} \frac{f(x)}{g(x)} = L$$

and also if

$$\lim_{x\to a} \frac{f'(x)}{g'(x)}$$

does not exist then

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

also does not exist. If necessary, L'Hospital's Rule can be applied more than once and can be applied to limits at infinity.

Exercise 12.7 (L'Hospital's Rule)

1. *Limit of* $\frac{0}{0}$: $\lim_{x\to 2} \frac{x^2-x-2}{x^2-2x}$
   Since

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \left(x^2 - x - 2\right) =$$

(i) $\mathbf{-1}$ (ii) $\mathbf{0}$ (iii) $\mathbf{1}$

and

$$\lim_{x \to 2} g(x) = \lim_{x \to 2} \left(x^2 - 2x\right) =$$

(i) $\mathbf{-1}$ (ii) $\mathbf{0}$ (iii) $\mathbf{1}$

then L'Hospital's Rule (i) **applies** (ii) **does not apply** and so

$$f'(x) =$$

(i) $\boldsymbol{2x-1}$ (ii) $\boldsymbol{2x-2}$ (iii) $\boldsymbol{x^2-x-3}$

and

$$g'(x) =$$

(i) $\boldsymbol{2x-1}$ (ii) $\boldsymbol{2x-2}$ (iii) $\boldsymbol{x^2-x-3}$

and so

$$\lim_{x \to 2} \frac{f(x)}{g(x)} = \lim_{x \to 2} \frac{f'(x)}{g'(x)} = \lim_{x \to 2} \frac{2x-1}{2x-2} =$$

(i) $\mathbf{0}$ (ii) $\mathbf{1}$ (iii) $\boldsymbol{\frac{3}{2}}$

2. *Limit of* $\frac{0}{0}$: $\lim_{x \to 1^+} \frac{\ln(2x-1)}{x-1}$
Since

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \ln(2x-1) =$$

(i) $\mathbf{-1}$ (ii) $\mathbf{0}$ (iii) $\mathbf{1}$

must converge to 1 from right for $\ln(2x-1)$ because undefined converging to $\ln(2x-1)$ from the left

and

$$\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} (x-1) =$$

(i) $\mathbf{-1}$ (ii) $\mathbf{0}$ (iii) $\mathbf{1}$

then L'Hospital's Rule (i) **applies** (ii) **does not apply** and so

$$f'(x) = \frac{1}{2x-1} \times 2 =$$

(i) $\boldsymbol{\frac{1}{x-1}}$ (ii) $\boldsymbol{\frac{1}{2x-1}}$ (iii) $\boldsymbol{\frac{2}{2x-1}}$

and

$$g'(x) =$$

(i) **1** (ii) $\boldsymbol{x}$ (iii) **2**

and so

$$\lim_{x \to 1^+} \frac{f(x)}{g(x)} = \lim_{x \to 1^+} \frac{f'(x)}{g'(x)} = \lim_{x \to 1^+} \frac{\frac{2}{2x-1}}{1} =$$

(i) **0** (ii) **2** (iii) $\boldsymbol{\frac{2}{3}}$

3. *Limit of* $\frac{0}{0}$: $\lim_{x \to 8} \frac{\sqrt[3]{x}-2}{x-8}$
   Since

$$\lim_{x \to 8} f(x) = \lim_{x \to 8} \left(\sqrt[3]{x} - 2 \right) =$$

(i) **−1** (ii) **0** (iii) **1**

and

$$\lim_{x \to 8} g(x) = \lim_{x \to 8} (x - 8) =$$

(i) **−1** (ii) **0** (iii) **1**

then L'Hospital's Rule (i) **applies** (ii) **does not apply** and so

$$f'(x) = \frac{1}{3} x^{\frac{1}{3}-1} =$$

(i) $\boldsymbol{\frac{1}{3x^2}}$ (ii) $\boldsymbol{\frac{1}{3x^3}}$ (iii) $\boldsymbol{\frac{1}{3x^{2/3}}}$

and

$$g'(x) =$$

(i) **1** (ii) $\boldsymbol{x}$ (iii) **2**

and so

$$\lim_{x \to 8} \frac{f(x)}{g(x)} = \lim_{x \to 8} \frac{f'(x)}{g'(x)} = \lim_{x \to 8} \frac{\frac{1}{3x^{2/3}}}{1} =$$

(i) $\boldsymbol{\frac{1}{3}}$ (ii) $\boldsymbol{\frac{1}{4}}$ (iii) $\boldsymbol{\frac{1}{12}}$

4. *Limit of* $\frac{0}{0}$: $\lim_{x \to 0} \frac{e^x-1}{x^2}$
   Since

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} (e^x - 1) =$$

(i) **−1** (ii) **0** (iii) **1**

and

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} x^2 =$$

(i) $\mathbf{-1}$ (ii) $\mathbf{0}$ (iii) $\mathbf{1}$

then L'Hospital's Rule (i) **applies** (ii) **does not apply** and so

$$f'(x) =$$

(i) $\boldsymbol{2e^{x}}$ (ii) $\boldsymbol{e^{x}}$ (iii) $\boldsymbol{e^{-x}}$

and

$$g'(x) = 2x^{2-1} =$$

(i) $\mathbf{1}$ (ii) $\boldsymbol{2x}$ (iii) $\mathbf{2}$

and so

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{e^x}{2x} =$$

(choose two) (i) $\mathbf{0}$ (ii) $\boldsymbol{\infty}$ (iii) **does not exist** L'Hospital's Rule does not always work

5. *Limit:* $\lim_{x\to 4} \frac{x^2-4}{\sqrt{x}}$
   Since

$$\lim_{x\to 4} f(x) = \lim_{x\to 4} \left(x^2 - 4\right) =$$

(i) $\mathbf{-1}$ (ii) $\mathbf{0}$ (iii) $\mathbf{12}$

and

$$\lim_{x\to 4} g(x) = \lim_{x\to 4} \sqrt{x} =$$

(i) $\mathbf{-1}$ (ii) $\mathbf{0}$ (iii) $\mathbf{2}$

then L'Hospital's Rule (i) **applies** (ii) **does not apply**, but

$$\lim_{x\to 4} \frac{x^2-4}{\sqrt{x}} = \frac{4^2-4}{\sqrt{4}} =$$

(i) $\mathbf{6}$ (ii) $\boldsymbol{\infty}$ (iii) **does not exist** L'Hospital's Rule does not always apply

6. *Limit of* $\frac{0}{0}$: $\lim_{x\to 0} \frac{2e^x-2x-2}{x^2}$
   Since

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} (2e^x - 2x - 2) =$$

(i) $\mathbf{-1}$ (ii) $\mathbf{0}$ (iii) $\mathbf{1}$

and

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} x^2 =$$

(i) $\mathbf{-1}$ (ii) $\mathbf{0}$ (iii) $\mathbf{1}$

then L'Hospital's Rule (i) **applies** (ii) **does not apply** and so

$$f'(x) = 2e^x - 2x^{1-1} =$$

(i) $\boldsymbol{2e^x}$ (ii) $\boldsymbol{2e^x - 2}$ (iii) $\boldsymbol{e^{-x}}$

and

$$g'(x) = 2x^{2-1} =$$

(i) $\mathbf{1}$ (ii) $\boldsymbol{2x}$ (iii) $\mathbf{2}$

but since

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0} (2e^x - 2) =$$

(i) $\mathbf{-1}$ (ii) $\mathbf{0}$ (iii) $\mathbf{1}$

and

$$\lim_{x\to 0} g'(x) = \lim_{x\to 0} 2x =$$

(i) $\mathbf{-1}$ (ii) $\mathbf{0}$ (iii) $\mathbf{1}$

and so, unfortunately, $\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \frac{0}{0}$;
however, L'Hospital's Rule (still) (i) **applies** (ii) **does not apply**, so

$$f''(x) =$$

(i) $\boldsymbol{2e^x}$ (ii) $\boldsymbol{2e^x - 2}$ (iii) $\boldsymbol{e^{-x}}$

and

$$g''(x) = 2x^{1-1} =$$

(i) $\mathbf{1}$ (ii) $\boldsymbol{2x}$ (iii) $\mathbf{2}$

and so

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{f''(x)}{g''(x)} = \lim_{x\to 0} \frac{2e^x}{2} =$$

(i) $\mathbf{1}$ (ii) $\boldsymbol{\infty}$ (iii) **does not exist**

7. *Limit of* $\frac{\infty}{\infty}$: $\lim_{x\to 0^+} (2x \ln x) = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{2x}}$
   Since

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \ln x =$$

(i) $\mathbf{-1}$ (ii) $\mathbf{-\infty}$ (iii) $\mathbf{\infty}$

and

$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} \frac{1}{2x} =$$

(i) $\mathbf{-1}$ (ii) $\mathbf{0}$ (iii) $\mathbf{\infty}$

then L'Hospital's Rule (i) **applies** (ii) **does not apply** and so

$$f'(x) =$$

(i) $\frac{1}{x}$ (ii) $-\frac{1}{2x}$ (iii) $-\frac{1}{2x^2}$

and

$$g'(x) = -\frac{1}{2}x^{-1-1} =$$

(i) $\frac{1}{x}$ (ii) $-\frac{1}{2x}$ (iii) $-\frac{1}{2x^2}$

and so

$$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{f'(x)}{g'(x)} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2x^2}} = \lim_{x\to 0^+} -2x =$$

(i) $\mathbf{0}$ (ii) $\mathbf{\infty}$ (iii) **does not exist**

8. *Limit of* $\frac{\infty}{\infty}$: $\lim_{x\to\infty} \frac{\ln(e^x+1)}{4x}$
Since

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \ln(e^x+1) =$$

(i) $\mathbf{-1}$ (ii) $\mathbf{-\infty}$ (iii) $\mathbf{\infty}$

and

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} 4x =$$

(i) $\mathbf{-1}$ (ii) $\mathbf{0}$ (iii) $\mathbf{\infty}$

then L'Hospital's Rule (i) **applies** (ii) **does not apply** and so

$$f'(x) = \frac{1}{e^x+1} \times e^x =$$

(i) $\frac{1}{e^x}$ (ii) $-\frac{1}{e^x+1}$ (iii) $\frac{e^x}{e^x+1}$

and

$$g'(x) = 4x^{1-1} =$$

(i) $\boldsymbol{\frac{4}{x}}$ (ii) **4** (iii) $\boldsymbol{4x}$

and so

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{f'(x)}{g'(x)} = \lim_{x\to\infty}\frac{\frac{e^x}{e^x+1}}{4} = \lim_{x\to\infty}\frac{e^x}{4e^x+4} =$$

(i) $\boldsymbol{\frac{1}{e^x}}$ (ii) $\boldsymbol{\frac{1}{4}}$ (iii) **does not exist**