Chapter 7

Integration

The "reverse" of differentiation is called *integration*. If $F'(x) = f(x)$, then $F(x)$ is the *antiderivative* of $f(x)$; or

$$\int f(x)\,dx = F(x) + C,$$

where $\int$ in the *integral sign*, $f(x)$ is the *integrand* and $\int f(x)\,dx$ is the *indefinite integral*. Whereas differentiation determines the *slope* of a tangent line to a curve, integration determines the *area* under a curve.

7.1 Antiderivatives

- *power rule* $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$
- *constant multiple rule* $\int k \cdot f(x)\,dx = k \int f(x)\,dx + C$
- *sum or difference rule* $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
- *exponential functions*
  1. $\int e^{kx}\,dx = \frac{e^{kx}}{k} + C$, $k \neq 0$
  2. $\int a^{kx}\,dx = \frac{a^{kx}}{k(\ln a)} + C$, $a > 0, a \neq 1$
- $\int \frac{1}{x}\,dx = \int x^{-1}\,dx = \int \frac{dx}{x} = \ln|x| + C$

Use boundary conditions to determine the constant of integration, C.

Exercise 7.1 (Antiderivatives)

1. *Antiderivatives, derivatives and constants of integration.*

(a) (i) **True** (ii) **False**
If $F(x) = 5x$, then $F'(x) = 5(1)x^{1-1} = 5x^0 = 5$ is the derivative and so the original function, $F(x) = 5x$, is an *anti*derivative.

(b) An antiderivative of $F'(x) = 5$ is $F(x) =$ (i) $\boldsymbol{5x^2}$ (ii) $\boldsymbol{5x}$ (iii) $\boldsymbol{5}$
Check if $F(x) = 5x$ is the antiderivative of $F'(x) = 5$: $\frac{d}{dx}F(x) = \frac{d}{dx}(5x) = 5$, so, yes, it is

(c) An antiderivative of $F'(x) = -3$ is $F(x) =$ (i) $\boldsymbol{-3x^2}$ (ii) $\boldsymbol{-3}$ (iii) $\boldsymbol{-3x}$
$F(x) = -3x$ is the antiderivative of $F'(x) = -3$ because $\frac{d}{dx}F(x) = \frac{d}{dx}(-3x) = -3$

(d) An antiderivative of $F'(x) = x$ is $F(x) =$ (i) $\boldsymbol{\frac{3}{2}x^2}$ (ii) $\boldsymbol{\frac{1}{2}x^2}$ (iii) $\boldsymbol{\frac{1}{2}}$
$F(x) = \frac{1}{2}x^2$ is the antiderivative of $F'(x) = x$ because $\frac{d}{dx}F(x) = \frac{d}{dx}\left(\frac{1}{2}x^2\right) = \frac{1}{2}(2)x^{2-1} = x$

(e) An antiderivative of $F'(x) = x$ is $F(x) =$ (i) $\boldsymbol{\frac{1}{2}x^2 + 7}$ (ii) $\boldsymbol{\frac{3}{2}x^2}$ (iii) $\boldsymbol{\frac{1}{2}}$
$F(x) = \frac{1}{2}x^2 + 7$ is the antiderivative of $F'(x) = x$ because $\frac{d}{dx}\left(\frac{1}{2}x^2 + 7\right) = \frac{1}{2}(2)x^{2-1} + 0 = x$

(f) An antiderivative of $F'(x) = x$ is (i) $\boldsymbol{\frac{1}{2}x^2 - 39}$ (ii) $\boldsymbol{\frac{3}{2}x^2}$ (iii) $\boldsymbol{-39}$
$F(x) = \frac{1}{2}x^2 - 39$ is the antiderivative of $F'(x) = x$ because $\frac{d}{dx}F(x) = \frac{d}{dx}\left(\frac{1}{2}x^2 - 39\right) = x$

(g) The antiderivative of $F'(x) = x$ is (i) $\boldsymbol{\frac{3}{2}x + C}$ (ii) $\boldsymbol{\frac{1}{2}x^2 + C}$ (iii) $\boldsymbol{\frac{1}{2}}$
$F(x) = \frac{1}{2}x^2$ antiderivative of $F'(x) = x + C$ because $\frac{d}{dx}F(x) = \frac{d}{dx}\left(\frac{1}{2}x^2 + C\right) = x$, *if* C is a *constant*

(h) The antiderivative of $F'(x) = 5x$ is (i) $\boldsymbol{\frac{5}{2}x^2 + C}$ (ii) $\boldsymbol{C}$ (iii) $\boldsymbol{\frac{5}{2}x^2}$
$F(x) = 5$ antiderivative of $F'(x) = 5x$ because $\frac{d}{dx}F(x) = \frac{d}{dx}\left(\frac{5}{2}x^2 + C\right) = 5$, where C constant

2. *Power rule, constant multiple rule and notation.*

(a) The antiderivative of $F'(x) = 5 = 5x^0$, or, equivalently, the *integral* of $5x^0$

$$F(x) = \int f(x)\,dx = \int 5x^0\,dx = 5\int x^0\,dx = 5\left(\frac{1}{0+1}x^{0+1} + C\right) =$$

(i) $\boldsymbol{5C}$ (ii) $\boldsymbol{5 + 5C}$ (iii) $\boldsymbol{5x + C}$
since C is *any* constant, $5 \times C$ is also any constant, so just keep calling it C;
also, $F(x) = 5x + C$ integral of 5 because $\frac{d}{dx}F(x) = \frac{d}{dx}(5x + C) = 5$, where C constant

(b) The integral of $4 = 4x^0$

$$F(x) = \int 4x^0\,dx = 4\int x^0\,dx = 4\left(\frac{1}{0+1}x^{0+1} + C\right) =$$

(i) $\boldsymbol{C}$ (ii) $\boldsymbol{4+C}$ (iii) $\boldsymbol{4x+C}$

This is an example of the *power rule* listed above

(c) The integral of k, k a constant,

$$\int k\,dx = \int kx^0\,dx = k\int x^0\,dx = k\left(\frac{1}{0+1}x^{0+1}+C\right) =$$

(i) $\boldsymbol{C}$ (ii) $\boldsymbol{k+C}$ (iii) $\boldsymbol{kx+C}$

This is an example of the power rule and also *multiple constant rule* listed above.

(d) The integral of $f(x) = x = x^1$,

$$\int x^1\,dx = \frac{1}{1+1}x^{1+1} + C =$$

(i) $\boldsymbol{C}$ (ii) $\boldsymbol{\frac{1}{2}x^2}$ (iii) $\boldsymbol{\frac{1}{2}x^2+C}$

$F(x) = \frac{1}{2}x^2 + C$, where C constant, integral of x^2 because $\frac{d}{dx}\left(\frac{1}{2}x^2+C\right) = \frac{1}{2}(2)x^{1-1} + 0 = x$

(e) The integral of $f(x) = x^2$,

$$\int x^2\,dx = \frac{1}{2+1}x^{2+1} + C =$$

(i) $\boldsymbol{C}$ (ii) $\boldsymbol{\frac{1}{3}x^3}$ (iii) $\boldsymbol{\frac{1}{3}x^3+C}$

A word on notation: *Both* the integral sign "$\int$" and the differential "dx" are necessary components to say you want to integrate the function enclosed between them, in this case, x^2. Neither "$\int$" nor "dx" are part of the function. They do not have to be "solved" or "calculated" or "determined" in any sense. They are simply the *notation* used to say the function is to be integrated.

(f) Integral of $f = x^{10}$,

$$\int x^{10}\,dx = \frac{1}{10+1}x^{10+1} + C =$$

(i) $\boldsymbol{\frac{1}{11}x^5}$ (ii) $\boldsymbol{\frac{1}{5}x^{11}+11}$ (iii) $\boldsymbol{\frac{1}{11}x^{11}+C}$

(g) Integral of $f = x^{-5}$,

$$\int x^{-5}\,dx = \frac{1}{-5+1}x^{-5+1} + C =$$

(i) $\boldsymbol{-\frac{1}{4}x^{-4}+C}$ (ii) $\boldsymbol{-\frac{1}{6}x^{-5}+C}$ (iii) $\boldsymbol{-\frac{1}{5}x^{-6}+C}$

(h) Integral of $f = x^{-7}$,

$$\int x^{-7}\,dx = \frac{1}{-7+1}x^{-7+1} + C =$$

(i) $\boldsymbol{-\frac{1}{6}x^{-8} + C}$ (ii) $\boldsymbol{-\frac{1}{8}x^{-8} + C}$ (iii) $\boldsymbol{-\frac{1}{6}x^{-6} + C}$

(i) Integral of $f = 3x^{10}$,

$$\int 3x^{10}\,dx = 3\int x^{10}\,dx = 3\left(\frac{1}{10+1}x^{10+1} + C\right) =$$

(i) $\boldsymbol{\frac{3}{11}x^{3} + C}$ (ii) $\boldsymbol{\frac{1}{3}x^{11} + C}$ (iii) $\boldsymbol{\frac{3}{11}x^{11} + C}$

(j) Integral of $f = -3x^{10}$,

$$\int \left(-3x^{10}\right)\,dx = -3\int x^{10}\,dx = -3\left(\frac{1}{10+1}x^{10+1} + C\right) =$$

(i) $\boldsymbol{-\frac{3}{11}x^{3} + C}$ (ii) $\boldsymbol{\frac{1}{3}x^{11} - 3C}$ (iii) $\boldsymbol{-\frac{3}{11}x^{11} + C}$

(k) Integral of $f = kx$, k a constant,

$$\int kx\,dx = \int kx^{1}\,dx = k\left(\frac{1}{1+1}x^{1+1} + C\right) =$$

(i) $\boldsymbol{C}$ (ii) $\boldsymbol{k + C}$ (iii) $\boldsymbol{\frac{k}{2}x^{2} + C}$

(l) Integral of $f = \sqrt[4]{x}$,

$$\int \sqrt[4]{x}\,dx = \int x^{\frac{1}{4}}\,dx = \frac{1}{\frac{1}{4}+1}x^{\frac{1}{4}+1} + C = \frac{1}{\frac{1}{4}+\frac{4}{4}}x^{\frac{1}{4}+\frac{4}{4}} + C =$$

(i) $\boldsymbol{C}$ (ii) $\boldsymbol{\frac{5}{4}x^{\frac{5}{4}} + C}$ (iii) $\boldsymbol{\frac{4}{5}x^{\frac{5}{4}} + C}$

(m) Integral of $f = 6\sqrt[5]{x}$,

$$\int 6\sqrt[5]{x}\,dx = \int 6x^{\frac{1}{5}}\,dx = 6\left(\frac{1}{\frac{1}{5}+1}x^{\frac{1}{5}+1} + C\right) = 6\left(\frac{1}{\frac{6}{5}}x^{\frac{6}{5}} + C\right) =$$

(i) $\boldsymbol{x^{\frac{6}{5}} + C}$ (ii) $\boldsymbol{6x^{\frac{6}{5}} + C}$ (iii) $\boldsymbol{5x^{\frac{6}{5}} + C}$

(n) Integral of $f = 6\sqrt[5]{x}$,

$$\int 6\sqrt[5]{x}\,dx = \int 6x^{\frac{1}{5}}\,dx =$$

(i) $\boldsymbol{5x^{\frac{6}{5}} + m}$ (ii) $\boldsymbol{5x^{\frac{6}{5}} + k}$ (iii) $\boldsymbol{5x^{\frac{6}{5}} + C}$

All are correct as long as k, m and C are all constants.

3. *Integration for exponential, logarithm and other functions.*

(a) Integral of $f = \frac{1}{x} = x^{-1}$,

$$\int x^{-1}\,dx =$$

(i) $\boldsymbol{-2x^{-2} + C}$ (ii) $\boldsymbol{\ln|x| + C}$ (iii) $\boldsymbol{3\ln|x| + C}$

This is the $\frac{1}{x}$ integration rule above;
but *not* the power rule because $\int x^{-1}\,dx = \frac{x^{-1+1}}{-1+1} = \frac{x^0}{0}$ which does not exist.

(b) Integral of $f = \frac{3}{x} = 3x^{-1}$,

$$\int 3x^{-1}\,dx =$$

(i) $\boldsymbol{-6x^{-2} + C}$ (ii) $\boldsymbol{\ln|x| + C}$ (iii) $\boldsymbol{3\ln|x| + C}$

(c) Integral of $f = e^x$,

$$\int e^x\,dx =$$

(i) $\boldsymbol{C}$ (ii) $\boldsymbol{e^x + C}$ (iii) $\boldsymbol{e^x}$

This is one of the exponential function integration rules above

(d) Integral of $f = 5e^{3x}$,

$$\int 5e^{3x}\,dx = 5\int e^{3x}\,dx = 5\left(\frac{1}{3}e^{3x} + C\right) =$$

(i) $\boldsymbol{\frac{1}{3}e^{3x} + C}$ (ii) $\boldsymbol{\frac{1}{3}x^{11} + C}$ (iii) $\boldsymbol{\frac{5}{3}e^{3x} + C}$

This is another one of the exponential function integration rules above

(e) Integral of $f = 5e^{-3x}$,

$$\int e^{3x}\,dx = 5\left(\frac{1}{-3}e^{-3x} + C\right) =$$

(i) $\boldsymbol{\frac{1}{3}e^{3x} + C}$ (ii) $\boldsymbol{-\frac{5}{3}e^{x} + C}$ (iii) $\boldsymbol{-5e^{3x} + C}$

(f) Integral of $5x - 7x^3$,

$$\int \left(5x - 7x^3\right)\, dx = 5\int x\, dx - 7\int x^3\, dx = 5\left(\frac{1}{1+1}x^{1+1}\right) - 7\left(\frac{1}{3+1}x^{3+1}\right) + C =$$

(i) $\mathbf{5\left(\frac{x^3}{3}\right) - 7\left(\frac{x^4}{4}\right) + C}$ (ii) $\mathbf{5\left(\frac{x^2}{2}\right) + 7\left(\frac{x^4}{4}\right) + C}$ (iii) $\mathbf{5\left(\frac{x^2}{2}\right) - 7\left(\frac{x^4}{4}\right) + C}$

(g) Integral of $3x^{-1} + 1$,

$$\int \left(3x^{-1} + 1\right)\, dx = 3\int x^{-1}\, dx + \int x^0\, dx = 3\ln|x| + \left(\frac{1}{0+1}x^{0+1}\right) + C =$$

(i) $\mathbf{3\ln|x| + x^2 + C}$ (ii) $\mathbf{3\ln|x| + x + C}$ (iii) $\mathbf{3e^x + x^2 + C}$

(h) Integral of $6e^{3x} + \sqrt[3]{x}$,

$$\int \left(6e^{3x} + \sqrt[3]{x}\right)\, dx = 6\int e^{3x}\, dx + \int x^{\frac{1}{3}}\, dx = 6\left(\frac{1}{3}e^{3x}\right) + \frac{1}{\frac{1}{3}+1}x^{\frac{1}{3}+1} + C =$$

(i) $\mathbf{2e^{3x} + \frac{3}{4}e^{\frac{4}{3}} + C}$ (ii) $\mathbf{e^{3x} + \frac{3}{4}e^{\frac{4}{3}} + C}$ (iii) $\mathbf{2e^{3x} + \frac{4}{3}e^{\frac{4}{3}} + C}$

4. *Integration with boundary or initial conditions: determining C.*

(a) Integrate $f'(x) = 5x^2$, where $f(-1) = 5$.

Since $f(x) = \int \left(5x^2\right)\, dx = 5\left(\frac{1}{2+1}x^{2+1}\right) + C =$
(i) $\mathbf{\frac{1}{3}e^{3x} + C}$ (ii) $\mathbf{\frac{5}{3}x^3 + C}$ (iii) $\mathbf{5e^{3x} + C}$

and since $f(-1) = 5$, then $f(-1) = \frac{5}{3}(-1)^3 + C = 5$, or
$C =$ (i) $\mathbf{5}$ (ii) $\mathbf{\frac{20}{3}}$ (iii) $\mathbf{5e^{3x}}$

and so $f(x) = \frac{5}{3}x^3 + C =$
(i) $\mathbf{\frac{5}{3}x^3 + C}$ (ii) $\mathbf{\frac{5}{3}x^3 + \frac{15}{3}}$ (iii) $\mathbf{\frac{5}{3}x^3 + \frac{20}{3}}$

(b) Integrate $f'(x) = 5x^2$, where $f(-1) = 6$.

Since $f(x) = \int \left(5x^2\right)\, dx = 5\left(\frac{1}{2+1}x^{2+1}\right) + C =$
(i) $\mathbf{\frac{1}{3}e^{3x} + C}$ (ii) $\mathbf{\frac{5}{3}x^3 + C}$ (iii) $\mathbf{5e^{3x} + C}$

and since $f(-1) = 6$, then $f(-1) = \frac{5}{3}(-1)^3 + C = 6$, or
$C =$ (i) $\mathbf{6}$ (ii) $\mathbf{\frac{23}{3}}$ (iii) $\mathbf{\frac{21}{3}}$

and so $f(x) = \frac{5}{3}x^3 + C =$
(i) $\boldsymbol{\frac{5}{3}x^3 + C}$ (ii) $\boldsymbol{\frac{5}{3}x^3 + \frac{21}{3}}$ (iii) $\boldsymbol{\frac{5}{3}x^3 + \frac{23}{3}}$

(c) Integrate $f'(x) = 6x^{-1}$, where $f(2) = 4$.

Since $f(x) = \int (6x^{-1})\, dx = 6(\ln|x|) + C =$
(i) $\boldsymbol{6x^{-1} + C}$ (ii) $\boldsymbol{-6\ln x + C}$ (iii) $\boldsymbol{6\ln|x| + C}$

and since $f(2) = 4$, then $f(2) = 6\ln|2| + C = 4$, or
$C =$ (i) $\mathbf{4 + 6(2)}$ (ii) $\mathbf{4 - 6\ln 2}$ (iii) $\mathbf{4 + 6\ln 2}$

and so $f(x) = 6\ln|2| + C =$
(i) $\boldsymbol{-\frac{6}{2}x^{-2} + \frac{19}{4}}$ (ii) $\boldsymbol{6\ln|x| + 4 - 6\ln(2)}$ (iii) $\boldsymbol{-\frac{6}{2}x^{-2} + \frac{21}{4}}$

5. *Application: economics.* Find total cost function, $C(x)$, such that marginal cost is $C'(x) = x^2 - 2x$ and where fixed costs are \$45 (in other words, $C(0) = 45$).

(a) Since $C(x) = \int C'(x)\, dx = \int (x^2 - 2x)\, dx = \frac{1}{2+1}x^{2+1} - \frac{2}{1+1}x^{1+1} + k =$
(i) $\boldsymbol{\frac{1}{3}e^{3x} + k}$ (ii) $\boldsymbol{\frac{1}{3}x^3 - x^2 + k}$ (iii) $\boldsymbol{5e^{3x} + k}$
Let's use constant k instead of C, to avoid confusion with cost C.

(b) and $C(0) = 45$, then $C(0) = \frac{1}{3}(0)^3 - (0)^2 + k = 45$, or,
$k =$ (i) $\mathbf{5 + \frac{5}{3}}$ (ii) $\mathbf{\frac{20}{3}}$ (iii) $\mathbf{45}$

(c) and so $C(x) = \int C'(x)\, dx = \frac{1}{3}x^3 - x^2 + k =$
(i) $\boldsymbol{\frac{5}{3}x^3 + C}$ (ii) $\boldsymbol{\frac{1}{3}x^3 - x^2}$ (iii) $\boldsymbol{\frac{1}{3}x^3 - x^2 + 45}$

6. *Application: physics.* Find position function $s(t)$ of a rolling ball such that velocity function is $v(t) = s'(t) = 6t^3$ and where the ball is 9 meters from the start position at time zero ($s(0) = 9$).

(a) Since $s(t) = \int s'(t)\, dt = \int (6t^3)\, dt = \frac{6}{3+1}t^{3+1} + C =$
(i) $\boldsymbol{\frac{1}{3}e^{3t} + C}$ (ii) $\boldsymbol{\frac{3}{2}t^4 + C}$ (iii) $\boldsymbol{5e^{3t} + C}$

(b) and $s(0) = 9$, then $s(0) = \frac{3}{2}(0)^4 + C = 9$, or,
$C =$ (i) $\mathbf{9}$ (ii) $\mathbf{\frac{9}{3}}$ (iii) $\boldsymbol{e^9}$

(c) and so $s(t) = \int s'(t)\, dx = \frac{3}{2}t^4 + C =$
(i) $\boldsymbol{\frac{3}{2}t^4 - 5}$ (ii) $\boldsymbol{\frac{3}{2}t^4 + 0}$ (iii) $\boldsymbol{\frac{3}{2}t^4 + 9}$

7. *Application: more physics.* Find velocity function $v(t)$ of a rolling ball such that acceleration function is $a(t) = v'(t) = -7t^4$ and where the ball has velocity -3 meters per second at time 2 ($v(2) = -3$).

   (a) Since $v(t) = \int v'(t)\, dt = \int (-7t^4)\ dt = \frac{-7}{4+1}t^{4+1} + C =$
   (i) $\boldsymbol{\frac{1}{3}e^{3t} + C}$ (ii) $\boldsymbol{-\frac{7}{5}t^5 + C}$ (iii) $\boldsymbol{5e^{3t} + C}$

   (b) and $v(2) = -3$, then $v(2) = -\frac{7}{5}(2)^5 + C = -3$, or,
   $C =$ (i) $\mathbf{9}$ (ii) $\boldsymbol{-\frac{120}{5}}$ (iii) $\boldsymbol{\frac{209}{5}}$

   (c) and so $v(t) = \int v'(t)\, dx = -\frac{7}{5}t^5 + C =$
   (i) (i) $\boldsymbol{-\frac{7}{5}t^5 - \frac{120}{5}}$ (ii) $\boldsymbol{-\frac{7}{5}t^5}$ (iii) $\boldsymbol{-\frac{7}{5}t^5 + \frac{209}{5}}$

7.2 Substitution

We look at an integration technique called *substitution*, which often simplifies a complicated integration. Roughly, the substitution integration technique is the reverse of the chain rule differentiation technique. We use the following formulas as a basis for the substitution technique, after substituting $u = f(x)$ (and so $du = f'(x)dx$).

- $\int [f(x)]^n f'(x)\, dx$ becomes $\int u^n\, du = \frac{u^{n+1}}{n+1} + C$, $n \neq 1$
- $\int e^{f(x)} f'(x)\, dx$ becomes $\int e^u\, du = e^u + C$
- $\int \frac{f'(x)}{f(x)}\, dx$ becomes $\int \frac{1}{u}\, du = \int u^{-1}\, du = \ln|u| + C$

Substitution method typically concerned with three cases; chose substitution u to be

- quantity under root or raised to a power
- quantity in denominator
- exponent of e

and allow for constants. We also look at how to deal with fractions in integration.

Exercise 7.2 (Substitution)

1. *Power Function and Integral Substitution Technique.*

(a) Find $\int f(x)\,dx = \int \frac{3}{2}\sqrt{3x+x^2}(3+2x)\,dx = \frac{3}{2}\int (3x+x^2)^{\frac{1}{2}}(3+2x)\,dx$.

guess $u = 3x + x^2$
then $\frac{du}{dx} = 3(1)x^{1-1} + 2x^{2-1} = 3 + 2x$ or $du = (3+2x)\,dx$
substituting u and du into $\int f(x)\,dx$,

$$\frac{3}{2}\int \left(3x+x^2\right)^{\frac{1}{2}}(3+2x)\,dx = \frac{3}{2}\int u^{\frac{1}{2}}\,du = \frac{3}{2}\left(\frac{1}{\frac{1}{2}+1}u^{\frac{1}{2}+1} + C\right) =$$

(i) $\boldsymbol{\frac{3}{2} + C}$ (ii) $\boldsymbol{\frac{2}{3}u^{\frac{1}{2}} + C}$ (iii) $\boldsymbol{u^{\frac{3}{2}} + C}$

but $u = 3x + x^2$, so

$$\int f(x)\,dx = u^{\frac{3}{2}} + C =$$

(i) $\boldsymbol{(3+2x)^{\frac{3}{2}} + C}$ (ii) $\boldsymbol{(3x+x^2)^{\frac{3}{2}} + C}$ (iii) $\boldsymbol{(3x^2+x^3)^{\frac{3}{2}} + C}$

(b) Find $\int f(x)\,dx = \int \frac{5}{2}(3x+x^2)^{\frac{3}{2}}(3+2x)\,dx$.

guess $u = 3x + x^2$
then $\frac{du}{dx} = 3(1)x^{1-1} + 2x^{2-1} = 3 + 2x$ or $du = (3+2x)\,dx$
substituting u and du into $\int f(x)\,dx$,

$$\frac{5}{2}\int \left(3x+x^2\right)^{\frac{3}{2}}(3+2x)\,dx = \frac{5}{2}\int u^{\frac{3}{2}}\,du = \frac{5}{2}\left(\frac{1}{\frac{3}{2}+1}u^{\frac{3}{2}+1} + C\right) =$$

(i) $\boldsymbol{\frac{5}{2} + C}$ (ii) $\boldsymbol{\frac{2}{5}u^{\frac{3}{2}} + C}$ (iii) $\boldsymbol{u^{\frac{5}{2}} + C}$

but $u = 3x + x^2$, so

$$\int f(x)\,dx = u^{\frac{5}{2}} + C =$$

(i) $\boldsymbol{(3+2x)^{\frac{3}{2}} + C}$ (ii) $\boldsymbol{(3x+x^2)^{\frac{3}{2}} + C}$ (iii) $\boldsymbol{(3x+x^3)^{\frac{5}{2}} + C}$

(c) Find $\int f(x)\,dx = \int \frac{1}{2}\frac{3+2x}{\sqrt{3x+x^2}}\,dx = \frac{1}{2}\int (3x+x^2)^{-\frac{1}{2}}(3+2x)\,dx$.

guess $u =$ (i) $\boldsymbol{3x + x^2}$ (ii) $\boldsymbol{3 + 2x}$ (iii) $\boldsymbol{\sqrt{3x + x^2}}$

then $\frac{du}{dx} = 3(1)x^{1-1} + 2x^{2-1} = 3 + 2x$ or $du = (3+2x)\,dx$
substituting u and du into $\int f(x)\,dx$,

$$\frac{1}{2}\int \left(3x+x^2\right)^{-\frac{1}{2}}(3+2x)\,dx = \frac{1}{2}\int u^{-\frac{1}{2}}\,du = \frac{1}{2}\left(\frac{1}{-\frac{1}{2}+1}u^{-\frac{1}{2}+1} + C\right) =$$

(i) $\boldsymbol{\frac{3}{2} + C}$ (ii) $\boldsymbol{\frac{1}{2}u^{\frac{3}{2}} + C}$ (iii) $\boldsymbol{u^{\frac{1}{2}} + C}$

but $u = 3x + x^2$, so

$$\int f(x)\,dx = u^{\frac{1}{2}} + C =$$

(i) $\boldsymbol{(3 + 2x)^{\frac{3}{2}} + C}$ (ii) $\boldsymbol{(3x + x^2)^{\frac{1}{2}} + C}$ (iii) $\boldsymbol{(3x + x^3)^{\frac{5}{2}} + C}$

(d) Find $\int \frac{9+6x}{\sqrt{3x+x^2}}\,dx = \int (3x + x^2)^{-\frac{1}{2}}(9 + 6x)\,dx$.

guess $u =$ (i) $\boldsymbol{3x + x^2}$ (ii) $\boldsymbol{3 + 2x}$ (iii) $\boldsymbol{\sqrt{3x + x^2}}$

then $\frac{du}{dx} = 3(1)x^{1-1} + 2x^{2-1} = 3 + 2x$ or $du = (3 + 2x)\,dx$
substituting u and du into $\int f(x)\,dx$,

$$\begin{aligned}\int \left(3x + x^2\right)^{-\frac{1}{2}}(9 + 6x)\,dx &= \int \left(3x + x^2\right)^{-\frac{1}{2}}(3)(3 + 2x)\,dx \\ &= \int u^{-\frac{1}{2}}\,(3)du \;=\; 3\int u^{-\frac{1}{2}}\,du \\ &= 3\left(\frac{1}{-\frac{1}{2}+1}u^{-\frac{1}{2}+1} + C\right) =\end{aligned}$$

(i) $\boldsymbol{\frac{3}{2} + C}$ (ii) $\boldsymbol{\frac{3}{2}u^{\frac{3}{2}} + C}$ (iii) $\boldsymbol{6u^{\frac{1}{2}} + C}$

but $u = 3x + x^2$, so

$$\int f(x)\,dx = 6u^{\frac{1}{2}} + C =$$

(i) $\boldsymbol{\frac{3}{2}(3 + 2x)^{\frac{3}{2}} + C}$ (ii) $\boldsymbol{6(3x + x^2)^{\frac{1}{2}} + C}$ (iii) $\boldsymbol{(3x + x^3)^{\frac{5}{2}} + C}$

(e) Find $\int 5\left(-2x^4 + 7x\right)^4\left(-8x^3 + 7\right)dx$.

guess $u =$ (i) $\boldsymbol{(-2x^4 + 7x)^4}$ (ii) $\boldsymbol{-8x^3 + 7}$ (iii) $\boldsymbol{-2x^4 + 7x}$

then $\frac{du}{dx} = -2(4)x^{4-1} + 7(1)x^{1-1} = -8x^3 + 7$ or $du = (-8x^3 + 7)\,dx$
substituting u and du into $\int f(x)\,dx$,

$$5\int \left(-2x^4 + 7x\right)^4(-8x^3 + 7)\,dx = 5\int u^4\,du = 5\left(\frac{1}{4+1}u^{4+1} + C\right) =$$

(i) $\boldsymbol{5 + C}$ (ii) $\boldsymbol{5u^5 + C}$ (iii) $\boldsymbol{u^5 + C}$

but $u = -2x^4 + 7x$, so

$$\int f(x)\,dx = u^5 + C =$$

(i) $\boldsymbol{(-2x^4+7x)^5+C}$ (ii) $\boldsymbol{(-2x^4+7x)^6+C}$ (iii) $\boldsymbol{5(-2x^4+7x)^5+C}$

(f) Find $\int f(x)\,dx = \int (3x^3+2x^2-4x)^6\,(9x^2+4x-4)\;dx$.

guess $u =$ (i) $\boldsymbol{3x^3+2x^2-4x}$ (ii) $\boldsymbol{9x^2+4x-4}$

then $du = (3(3)x^{3-1}+2(2)x^{2-1}-4(1)x^{1-1})\,dx = (9x^2+4x+4)\,dx$
substituting u and du into $\int f(x)\,dx$,

$$\int \left(3x^3+2x^2-4x\right)^6\left(9x^2+4x-4\right)\,dx = \int u^6\,du = \left(\frac{1}{6+1}u^{6+1}+C\right) =$$

(i) $\boldsymbol{\frac{1}{7}u^7+C}$ (ii) $\boldsymbol{\frac{1}{7}u^6+C}$ (iii) $\boldsymbol{u^7+C}$

but $u = 3x^3+2x^2-4x$, so

$$\int f(x)\,dx = \frac{1}{7}u^7+C =$$

(i) $\boldsymbol{\frac{1}{7}\left(3x^3+2x^2-4x\right)^7+C}$
(ii) $\boldsymbol{\left(3x^3+2x^2-4x\right)^7+C}$
(iii) $\boldsymbol{7\left(3x^3+2x^2-4x\right)^7+C}$

(g) Find $\int (1+4x^2)^6\,(15x)\;dx$.

guess $u =$ (i) $\boldsymbol{1+4x^2}$ (ii) $\boldsymbol{15x}$

then $du = (0+4(2)x^{2-1})\,dx =$ (i) $\boldsymbol{(8x)\,dx}$ (ii) $\boldsymbol{(1+8x)\,dx}$

substituting u and du into $\int f(x)\,dx$,

$$\int \left(1+4x^2\right)^6(15x)\;dx = \int \left(1+4x^2\right)^6\left(\frac{15}{8}\right)(8x)\;dx = \frac{15}{8}\int u^6\,du = \frac{15}{8}\left(\frac{1}{6+1}u^{6+1}+C\right) =$$

(i) $\boldsymbol{\frac{15}{8}u^7+C}$ (ii) $\boldsymbol{\frac{15}{56}u^7+C}$ (iii) $\boldsymbol{u^7+C}$

but $u = 1+4x^2$, so

$$\int f(x)\,dx = \frac{15}{56}u^7+C =$$

(i) $\boldsymbol{\frac{1}{56}\left(1+4x^2\right)^7+C}$
(ii) $\boldsymbol{\frac{15}{56}\left(1+4x^2\right)^7+C}$
(iii) $\boldsymbol{15\left(1+4x^2\right)^7+C}$

(h) Find $\int f(x)\,dx = \int x\sqrt{x+5}\,dx = \int x(x+5)^{\frac{1}{2}}\,dx$.

guess $u =$ (i) $\boldsymbol{\sqrt{x+5}}$ (ii) $\boldsymbol{x+5}$

then $du = dx$ and also $x = u - 5$

substituting u, du and x into $\int f(x)\,dx$,

$$\begin{aligned}\int x(x+5)^{\frac{1}{2}}\,dx &= \int (u-5)(u)^{\frac{1}{2}}\,du \\ &= \int \left(u^{\frac{3}{2}} - 5u^{\frac{1}{2}}\right)\,du \\ &= \int u^{\frac{3}{2}}\,du - 5\int u^{\frac{1}{2}}\,du \\ &= \frac{1}{\frac{3}{2}+1}u^{\frac{3}{2}+1} - 5\left(\frac{1}{\frac{1}{2}+1}u^{\frac{1}{2}+1}\right) + C =\end{aligned}$$

(i) $\boldsymbol{\frac{2}{5}u^{\frac{5}{2}} + C}$ (ii) $\boldsymbol{\frac{2}{5}u^{\frac{5}{2}} - \frac{10}{3}u^{\frac{3}{2}} + C}$ (iii) $\boldsymbol{10u^{\frac{3}{2}} + C}$

but $u = x + 5$, so

$$\int f(x)\,dx = \frac{2}{5}u^{\frac{5}{2}} - \frac{10}{3}u^{\frac{3}{2}} + C =$$

(i) $\boldsymbol{\frac{2}{5}(x+5)^{\frac{5}{2}} - \frac{10}{3}(x+5)^{\frac{3}{2}} + C}$
(ii) $\boldsymbol{-\frac{10}{3}(x+5)^{\frac{3}{2}} + C}$
(iii) $\boldsymbol{\frac{2}{5}(x+5)^{\frac{5}{2}} + C}$

2. *Exponential Function and Substitution Technique.*

(a) Find $\int e^{\left(7+x^3\right)}\left(3x^2\right)\,dx$.

guess $u =$ (i) $\boldsymbol{7+x^3}$ (ii) $\boldsymbol{3x^2}$

then $du = (0 + 3x^{3-1})\,dx =$ (i) $\boldsymbol{(1+3x^2)\,dx}$ (ii) $\boldsymbol{(3x^2)\,dx}$

substituting u and du into $\int f(x)\,dx$,

$$\int e^{\left(7+x^3\right)}\left(3x^2\right)\,dx = \int e^u\,du = e^u + C$$

but $u = 7 + x^3$, so

$$\int f(x)\,dx = e^u + C =$$

(i) $\boldsymbol{x^3e^{7+x^3} + C}$ (ii) $\boldsymbol{7e^{7+x^3} + C}$ (iii) $\boldsymbol{e^{7+x^3} + C}$

(b) Find $\int f(x)\,dx = \int -\frac{7}{2}\left[\frac{e^{(-7\sqrt{x})}}{\sqrt{x}}\right]\,dx = \int e^{\left(-7x^{\frac{1}{2}}\right)}\left(-\frac{7}{2}x^{-\frac{1}{2}}\right)\,dx.$

guess $u =$ (i) $\boldsymbol{-7x^{\frac{1}{2}}}$ (ii) $\boldsymbol{-\frac{7}{2}x^{-\frac{1}{2}}}$

then $du = \left(-7\left(\frac{1}{2}\right)x^{\frac{1}{2}-1}\right)\,dx =$ (i) $\boldsymbol{\left(-\frac{7}{2}x^{-\frac{1}{2}}\right)\,dx}$ (ii) $\boldsymbol{\left(\frac{7}{2}x^{-\frac{1}{2}}\right)\,dx}$

substituting u and du into $\int f(x)\,dx$,

$$\int e^{\left(-7x^{\frac{1}{2}}\right)}\left(-\frac{7}{2}x^{-\frac{1}{2}}\right)\,dx = \int e^u\,du = e^u + C$$

but $u = -7x^{\frac{1}{2}} = -7\sqrt{x}$, so

$$\int f(x)\,dx = e^u + C =$$

(i) $\boldsymbol{\sqrt{x}e^{-7\sqrt{x}} + C}$ (ii) $\boldsymbol{-7e^{-7\sqrt{x}} + C}$ (iii) $\boldsymbol{e^{-7\sqrt{x}} + C}$

(c) Find $\int f(x)\,dx = \int \frac{e^{(-7\sqrt{x})}}{\sqrt{x}}\,dx = \int e^{\left(-7x^{\frac{1}{2}}\right)}\left(x^{-\frac{1}{2}}\right)\,dx.$

guess $u =$ (i) $\boldsymbol{-7x^{\frac{1}{2}}}$ (ii) $\boldsymbol{x^{-\frac{1}{2}}}$

then $du = \left(-7\left(\frac{1}{2}\right)x^{\frac{1}{2}-1}\right)\,dx =$ (i) $\boldsymbol{\left(-\frac{7}{2}x^{-\frac{1}{2}}\right)\,dx}$ (ii) $\boldsymbol{\left(-\frac{1}{2}x^{-\frac{3}{2}}\right)\,dx}$

substituting u and du into $\int f(x)\,dx$,

$$\int e^{\left(-7x^{\frac{1}{2}}\right)}\left(x^{-\frac{1}{2}}\right)\,dx = \int e^{\left(-7x^{\frac{1}{2}}\right)}\left(-\frac{2}{7}\right)\left(-\frac{7}{2}x^{-\frac{1}{2}}\right)\,dx = \int e^u\left(-\frac{2}{7}\right)\,du =$$

(i) $\boldsymbol{-\frac{2}{7}e^u + C}$ (ii) $\boldsymbol{-7e^u + C}$ (iii) $\boldsymbol{e^u + C}$

but $u = -7x^{\frac{1}{2}} = -7\sqrt{x}$, so

$$\int f(x)\,dx = -\frac{2}{7}e^u + C =$$

(i) $\boldsymbol{-\frac{2}{7}e^{-7\sqrt{x}} + C}$ (ii) $\boldsymbol{-7e^{-7\sqrt{x}} + C}$ (iii) $\boldsymbol{e^{-7\sqrt{x}} + C}$

(d) Find $\int f(x)\,dx = \int xe^{x^2}\,dx = \int e^{x^2}\,(x)\,dx.$

guess $u =$ (i) $\boldsymbol{e^x}$ (ii) $\boldsymbol{x^2}$

then $du = (2x^{2-1})\,dx =$ (i) $\boldsymbol{(2x)\,dx}$ (ii) $\boldsymbol{(e^x)\,dx}$

substituting u and du into $\int f(x)\,dx$,

$$\int e^{x^2}\,(x)\;dx = \int e^{x^2}\left(\frac{1}{2}\right)(2x)\;dx = \int e^{u}\left(\frac{1}{2}\right)\,du =$$

(i) $\boldsymbol{-\frac{1}{2}e^u + C}$ (ii) $\boldsymbol{\frac{1}{2}e^u + C}$ (iii) $\boldsymbol{e^u + C}$

but $u = x^2$, so

$$\int f(x)\,dx = \frac{1}{2}e^u + C =$$

(i) $\boldsymbol{-\frac{1}{2}e^{x^2} + C}$ (ii) $\boldsymbol{\frac{1}{2}e^{x^2} + C}$ (iii) $\boldsymbol{e^{x^2} + C}$

(e) Find $\int f(x)\,dx = \int 2x7^{x^2}\,dx = \int 7^{x^2}\,(2x)\;dx$.

notice if $y = 7^{x^2}$, then $\ln y = \ln 7^{x^2} = x^2 \ln 7$ or $y = e^{x^2 \ln 7}$, so $\int f(x)\,dx = \int 7^{x^2}\,(2x)\;dx = \int e^{x^2 \ln 7}\,(2x)\;dx$

guess $u =$ (i) $\boldsymbol{x^2 \ln 7}$ (ii) $\boldsymbol{x^2}$

then $du = (2x^{2-1} \ln 7)\;dx =$ (i) $\boldsymbol{(2x \ln 7)\;dx}$ (ii) $\boldsymbol{(e^{\ln 7})\;dx}$

substituting u and du into $\int f(x)\,dx$,

$$\int e^{x^2 \ln 7}\,(2x)\;dx = \int e^{x^2 \ln 7}\frac{1}{\ln 7}\,(2x \ln 7)\;dx = \int e^{u}\left(\frac{1}{\ln 7}\right)\,du =$$

(i) $\boldsymbol{-\frac{1}{\ln 7}e^u + C}$ (ii) $\boldsymbol{\frac{1}{\ln 7}e^u + C}$ (iii) $\boldsymbol{e^u + C}$

but $u = x^2 \ln 7$, so

$$\int f(x)\,dx = \frac{1}{\ln 7}e^u + C =$$

(i) $\boldsymbol{-\frac{1}{\ln 7}e^{x^2 \ln 7} + C}$ (ii) $\boldsymbol{\frac{1}{\ln 7}e^{x^2 \ln 7} + C}$ (iii) $\boldsymbol{e^{x^2 \ln 7} + C}$
which is $\frac{1}{\ln 7}7^{x^2} + C$

3. *Logarithmic Function and Substitution Technique.*

(a) Find $\int \frac{2+2x}{2x+x^2}\,dx = \int (2x + x^2)^{-1}(2 + 2x)\,dx$.

guess $u =$ (i) $\boldsymbol{2 + 2x}$ (ii) $\boldsymbol{2x + x^2}$

then $du = (2(1)x^{1-1} + 2x^{2-1})\;dx =$ (i) $\boldsymbol{(2 + 2x^2)\;dx}$ (ii) $\boldsymbol{(2 + 2x)\;dx}$

substituting u and du into $\int f(x)\,dx$,

$$\int \left(2x+x^2\right)^{-1}(2+2x)\,dx = \int u^{-1}\,du = \ln|u| + C$$

but $u = 2x + x^2$, so

$$\int f(x)\,dx = \ln|u| + C =$$

(i) $\boldsymbol{2\ln|2x+x^2|+C}$ (ii) $\boldsymbol{\ln|2x+x^2|+C}$ (iii) $\boldsymbol{x\ln|2x+x^2|+C}$

(b) Find $\int \frac{4+4x}{2x+x^2}\,dx = \int (2x+x^2)^{-1}(4+4x)\,dx$.

guess $u =$ (i) $\boldsymbol{4+4x}$ (ii) $\boldsymbol{2x+x^2}$

then $du = \left(2(1)x^{1-1} + 2x^{2-1}\right)\,dx =$ (i) $\boldsymbol{(2+2x^2)\;dx}$ (ii) $\boldsymbol{(2+2x)\;dx}$

substituting u and du into $\int f(x)\,dx$,

$$\int \left(2x+x^2\right)^{-1}(4+4x)\,dx = \int \left(2x+x^2\right)^{-1}(2)(2+2x)\,dx = \int u^{-1}(2)\,du =$$

(i) $\boldsymbol{-\ln|u|+C}$ (ii) $\boldsymbol{\ln|u|+C}$ (iii) $\boldsymbol{2\ln|u|+C}$

but $u = 2x + x^2$, so

$$\int f(x)\,dx = 2\ln|u| + C =$$

(i) $\boldsymbol{2\ln|2x+x^2|+C}$ (ii) $\boldsymbol{\ln|2x+x^2|+C}$ (iii) $\boldsymbol{x\ln|2x+x^2|+C}$

(c) Find $\int \frac{\ln x}{x}\,dx = \int \ln x\,(x^{-1})\;dx$.

guess $u =$ (i) $\boldsymbol{\ln x}$ (ii) $\boldsymbol{x^{-1}}$

then $du = \left(x^{-1}\right)\;dx =$ (i) $\boldsymbol{(x^{-1})\;dx}$ (ii) $\boldsymbol{(-x^{-2})\;dx}$

substituting u and du into $\int f(x)\,dx$,

$$\int \ln x\left(x^{-1}\right)\;dx =$$

(i) $\boldsymbol{\int u\,du = \frac{1}{1+1}u^{1+1} + C = \frac{u^2}{2} + C}$
(ii) $\boldsymbol{\int u^{-1}\,du = \ln|u| + C}$
(iii) $\boldsymbol{2\int u\,du = 2\left(\frac{1}{1+1}u^{1+1} + C\right) = u^2 + C}$

but $u = \ln x$, so

$$\int f(x)\,dx = \frac{u^2}{2} + C =$$

(i) $\boldsymbol{\frac{1}{2}(\ln x)^2 + C}$ (ii) $\boldsymbol{\ln|\ln x| + C}$ (iii) $\boldsymbol{(\ln x)^2 + C}$

(d) Find $\int \frac{\ln 8x}{x}\,dx = \int \ln 8x\,(x^{-1})\,dx$.

guess $u =$ (i) $\boldsymbol{\ln 8x}$ (ii) $\boldsymbol{x^{-1}}$

recall if $u = f[g(x)] = \ln 8x$ and $g(x) = 8x$ and $f(x) = \ln x$

and $g'(x) =$ (i) $\boldsymbol{8x}$ (ii) $\boldsymbol{8}$ (iii) $\boldsymbol{\ln 8x}$
and $f'(x) =$ (i) $\boldsymbol{\frac{1}{x^2}}$ (ii) $\boldsymbol{\frac{1}{x}}$ (iii) $\boldsymbol{\frac{1}{8x}}$
and so by chain rule

$$\frac{du}{dx} = f'[g(x)] \cdot g'(x) = f'[8x] \cdot (2) = \frac{1}{8x}(8) =$$

(i) $\boldsymbol{\frac{1}{x} = x^{-1}}$ (ii) $\boldsymbol{\frac{2}{x} = 2x^{-1}}$ (iii) $\boldsymbol{\frac{1}{8x}}$

in other words $du =$ (i) $\boldsymbol{(x^{-1})\,dx}$ (ii) $\boldsymbol{(-x^{-2})\,dx}$

substituting u and du into $\int f(x)\,dx$,

$$\int \ln 8x \left(x^{-1}\right)\,dx =$$

(i) $\boldsymbol{\int u\,du = \frac{1}{1+1}u^{1+1} + C = \frac{u^2}{2} + C}$
(ii) $\boldsymbol{\int u^{-1}\,du = \ln|u| + C}$
(iii) $\boldsymbol{2\int u\,du = 2\left(\frac{1}{1+1}u^{1+1} + C\right) = u^2 + C}$

but $u = \ln 8x$, so

$$\int f(x)\,dx = \frac{u^2}{2} + C =$$

(i) $\boldsymbol{\frac{1}{2}(\ln 8x)^2 + C}$ (ii) $\boldsymbol{\ln|\ln 8x| + C}$ (iii) $\boldsymbol{(\ln 8x)^2 + C}$

(e) Find $\int \frac{x-5}{x-4}\,dx$.

notice

$$\frac{x-5}{x-4} = A + \frac{B}{x-4}$$

$$= \frac{A(x-4)+B}{x-4}$$
$$= \frac{Ax+(B-4A)}{x-4}$$

and so $A = 1$ and $B - 4A = -5$
and so $B = -5 + 4A = -5 + 4(1) = -1$
and so $\frac{x-5}{x-4} = A + \frac{B}{x-4} =$ (i) $\mathbf{1 + \frac{1}{x-4}}$ (ii) $\mathbf{1 - \frac{1}{x-4}}$ (iii) $\mathbf{1 + \frac{1}{x+4}}$

in other words,

$$\begin{aligned}
\int \frac{x-5}{x-4}\,dx &= \int \left(1 - \frac{1}{x-4}\right) dx \\
&= \int 1\,dx - \int \frac{1}{x-4}\,dx \\
&= \int x^0\,dx - \int \frac{1}{x-4}\,dx \\
&= \frac{1}{0+1}x^{\frac{1}{0+1}} - \int \frac{1}{x-4}\,dx \\
&= x - \int \frac{1}{x-4}\,dx + C
\end{aligned}$$

where, for the second integral, guess $u =$ (i) $\boldsymbol{x - 4}$ (ii) $\boldsymbol{x}$

so $du =$ (i) $\boldsymbol{(x^{-1})\,dx}$ (ii) $\boldsymbol{dx}$

substituting u and du into second integral,

$$x - \int \frac{1}{x-4}\,dx + C =$$

(i) $\boldsymbol{x - \int u^{-1}\,du = x - \ln u + C}$
(ii) $\boldsymbol{\int u^{-1}\,du = \ln|u| + C}$
(iii) $\boldsymbol{2\int u\,du = 2\left(\frac{1}{1+1}u^{1+1} + C\right) = u^2 + C}$

but $u = x - 4$, so

$$\int f(x)\,dx = x - \ln u + C =$$

(i) $\boldsymbol{\frac{1}{2}(x-4)^2 + C}$ (ii) $\boldsymbol{x - \ln(x-4) + C}$ (iii) $\boldsymbol{(x-4)^2 + C}$

7.3 Area and the Definite Integral

Not covered.