Chapter 2

Nonlinear Functions

We look at different types of *nonlinear* functions, including quadratic functions, polynomials and rational, exponential and logarithmic functions, as well as some applications such as growth and decay and financial functions.

2.1 Properties of Functions

A *function* is a rule which assigns to each element in one set one and only element from another set. A *non* linear function is *not* a straight line. A function be described in different ways, such as using a set diagram, table or by equation. The *domain* is the set of all possible values of the independent variable of a function x; the *range* is the set of all possible values of the dependent variable of a function $y = f(x)$. Two special types of functions are discussed, including:

- *even function*: $f(-x) = f(x)$, a function symmetric about the y-axis,
- *odd function*: $f(-x) = -f(x)$, a function symmetric about the origin.

Step functions are also discussed.

Exercise 2.1 (Properties of Functions)

1. *Graphs of functions.* Which graphs are functions? (Hint: Use vertical line test.)

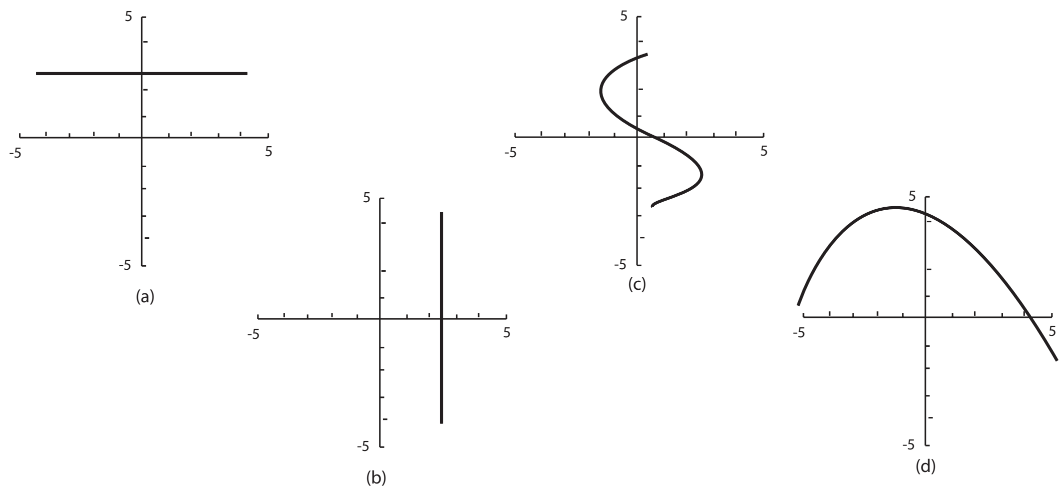

Figure 2.1 (Function or not?)

   (a) *Figure (a)* (i) **function** (ii) **not a function**
   (b) *Figure (b)* (i) **function** (ii) **not a function**
   (c) *Figure (c)* (i) **function** (ii) **not a function**
   (d) *Figure (d)* (i) **function** (ii) **not a function**

2. *Set diagrams and functions.* Are correspondences between x and y in following set diagrams functions or not?

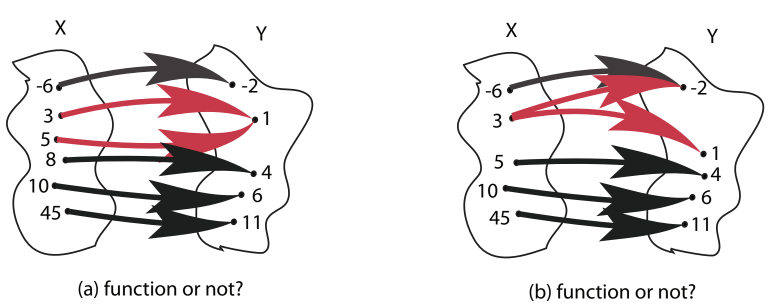

Figure 2.2 (Function or not?)

   (a) *Figure (a)* (i) **function** (ii) **not a function**
   (b) *Figure (b)* (i) **function** (ii) **not a function**

   There is more than one value of y for one value of x; in particular, when $x = 3$, $y = -2, 1$.

3. *Tables and functions.* Are correspondences between x and y in following tables functions or not?

(a) (i) **function** (ii) **not a function**

domain, x	-6	3	5	8	10	45
range, y	-2	1	1	4	6	11

(b) (i) **function** (ii) **not a function**

domain, x	-6	3	3	5	10	45
range, y	-2	-2	1	4	6	11

There is more than one value of y for one value of x; in particular, when $x = 3$, $y = -2, 1$.

(c) (i) **function** (ii) **not a function**

domain, x	-1	2	3.3	4
range, y	6	12	31.37	46

(d) (i) **function** (ii) **not a function**

domain, x	-1	-1	3.3	4
range, y	6	12	31.37	46

There is more than one value of y for one value of x; in particular, when $x = -1$, $y = 6, 12$.

(e) (i) **function** (ii) **not a function**

domain, x	6	12	31.37	46
range, y	-1	-1	3.3	4

4. *Function or not?*

(a) $y = x^2 - 3$ (i) **function** (ii) **not a function**

(b) $x = y^2 - 3$ (i) **function** (ii) **not a function**

There is more than one value of y for one value of x; for example, when $x = 1$, $y = -2, 2$.

(c) $y = x^3 - 3$ (i) **function** (ii) **not a function**

(d) $x = y^3 - 3$ (i) **function** (ii) **not a function**

There is only *one* value of y for one value of x; when $x = 5$, $y = 2$ and when $x = -11$, $y = -2$, say.

(e) $y = \sqrt{x}$ (i) **function** (ii) **not a function**

(f) $y = |x|$ (i) **function** (ii) **not a function**

(g) $|y| = x$ (i) **function** (ii) **not a function**

There is more than one value of y for one value of x; for example, when $x = 2$, $y = -2, 2$.

5. *Range.* If the domain is $\{-1, 0, 1, 2\}$, determine the range.

(a) $y = 2x - 3$
(i) $\mathbf{\{-5, -3, -1, 1\}}$
(ii) $\mathbf{\{-7, -3, -1, 1\}}$
(iii) $\mathbf{\{-5, -3, -1, 2\}}$

Calculator: type domain into L_1 (STAT ENTER), define $L_2 = 2L_1 - 3$ ENTER.

(b) $y = (x+1)(x+3)$
(i) $\mathbf{\{0, 3, 8, 15\}}$
(ii) $\mathbf{\{-1, 3, 7, 14\}}$
(iii) $\mathbf{\{2, 3, 4, 7\}}$
Calculator: define $L_2 = (L_1 + 1)(L_1 + 3)$ ENTER.

6. *Domain.* Determine the domain.

(a) $y = 2x - 3$
(i) $\mathbf{(-\infty, 1) \cup (1, \infty)}$ (ii) $\mathbf{(-1, 1)}$ (iii) $\mathbf{(-\infty, \infty)}$

(b) $y = (x+1)(x+3)$
(i) $\mathbf{(-\infty, \infty)}$ (ii) $\mathbf{(-3, -1)}$ (iii) $\mathbf{(-\infty, -3) \cup (-1, \infty)}$

(c) $y = \sqrt{x-5}$
(i) $\mathbf{[5, \infty)}$ (ii) $\mathbf{(-5, \infty)}$ (iii) $\mathbf{(-\infty, -5) \cup (5, \infty)}$
$y = \sqrt{x-5}$ defined when $(x-5) \geq 0$; that is, when $x \geq 5$.

(d) $y = \sqrt{(x+1)(x+3)}$
(i) $\mathbf{(-\infty, -3] \cup [-1, \infty)}$ (iii) $\mathbf{[-3, \infty)}$ (ii) $\mathbf{(-1, \infty)}$
$y = \sqrt{(x+1)(x+3)}$ defined when $(x+1)(x+3) \geq 0$; that is, either when $x \leq -3$ or $x \geq -1$.

(e) $y = -\frac{8}{x+1}$
(i) $\mathbf{(-\infty, -1) \cup (-1, \infty)}$ (iii) $\mathbf{[-1, \infty)}$ (ii) $\mathbf{(-1, \infty)}$
Equation $y = -\frac{8}{x+1}$ exists when $x + 1 \neq 0$; that is, when $x \neq -1$.

(f) Consider

$$f(x) = \begin{cases} -\frac{8}{x+1} & \text{if } x \neq -1 \\ -8 & \text{if } x = -1 \end{cases}$$

(i) $\mathbf{[-1, \infty)}$ (ii) $\mathbf{(-\infty, -1) \cup (-1, \infty)}$ (iii) $\mathbf{(-\infty, \infty)}$

7. *Evaluating functions.* Determine $f(1), f(k), f(1/k)$ and x such that $f(x) = 1$.

(a) $y = 2x - 3$

i. -1, $2k - 3$, $\frac{2-3k}{k}$, 2
ii. -1, $2k - 3$, $\frac{2}{k} - 3$, 2
iii. -1, $2k - 3$, $\frac{2}{k} - 3$, 3

If $y = 2x - 3 = 1$ then $2x = 4$, so $x = 2$.

(b) $y = -\frac{8}{x+1}$

i. -4, $\frac{8}{k+1}$, $-\frac{8k}{1+k}$, -9
ii. -4, $-\frac{8}{k+1}$, $-\frac{8k}{1+k}$, -9
iii. -4, $-\frac{8}{k+1}$, $\frac{8k}{1+k}$, -9

Notice $-\frac{8}{\frac{1}{k}+1} = -\frac{8}{\frac{1}{k}+\frac{k}{k}} = -\frac{8}{\frac{1+k}{k}} = -\frac{8k}{1+k}$.

8. *Evaluating functions.* Determine $f(x+h), f(x+h)-f(x), \frac{f(x+h)-f(x)}{h}$.

   (a) $y = 2x - 3$

      i. $2x + 2h - 3$, $2h$, 2
      ii. $2x + 2h + 3$, $2h$, 2
      iii. $2x + 2h - 3$, $2h + x$, 2

      $f(x+h) = 2(x+h) - 3 = 2x + 2h - 3$
      $f(x+h) - f(x) = (2x + 2h - 3) - (2x - 3) = 2h$
      $\frac{f(x+h)-f(x)}{h} = \frac{2h}{h} = 2.$

   (b) $y = -\frac{8}{x+1}$

      i. $-\frac{8}{x+1}, \frac{8h}{(x+h+1)(x+1)}, \frac{8}{(x+h+1)(x+1)}$
      ii. $-\frac{8}{x+h+1}, \frac{8}{(x+h+1)(x+1)}, \frac{8}{(x+h+1)(x+1)}$
      iii. $-\frac{8}{x+h+1}, \frac{8h}{(x+h+1)(x+1)}, \frac{8}{(x+h+1)(x+1)}$

      $f(x+h) = -\frac{8}{(x+h)+1} = -\frac{8}{x+h+1}$
      $f(x+h) - f(x) = -\frac{8}{x+h+1} - -\frac{8}{x+1} = -\frac{8(x+1)}{(x+h+1)(x+1)} + \frac{8(x+h+1)}{(x+h+1)(x+1)} = \frac{8h}{(x+h+1)(x+1)}$
      $\frac{f(x+h)-f(x)}{h} = [\frac{8h}{(x+h+1)(x+1)}] \div h = \frac{8h}{(x+h+1)(x+1)} \times \frac{1}{h}.$

9. *Even, odd or neither functions.* Classify functions as even, odd or neither.

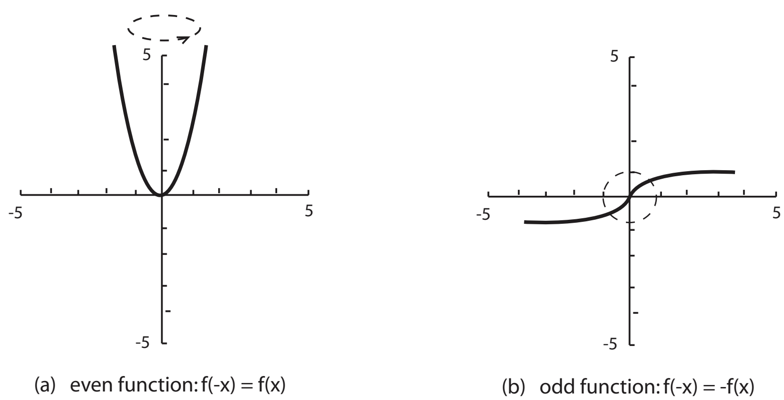

Figure 2.3 (Even and odd functions)

   (a) Since $f(-x) = f(x)$, figure (a) is (i) **Even** (ii) **Odd** (iii) **Neither**
   (b) Since $f(-x) = -f(x)$, figure (b) is (i) **Even** (ii) **Odd** (iii) **Neither**
   (c) $y = x^2 - 3$ (i) **Even** (ii) **Odd** (iii) **Neither**

      Notice $f(-x) = (-x)^2 - 3 = f(x) = x^2 - 3$.

   (d) $y = x^3 - 3$ (i) **Even** (ii) **Odd** (iii) **Neither**

      Notice $f(-x) = (-x)^3 - 3 = -x^3 - 3 \neq f(x) = x^3 - 3$
      and also $f(-x) = -x^3 - 3 \neq -f(x) = -(x^3 - 3) = -x^3 + 3$.

(e) $y = x^3$ (i) **Even** (ii) **Odd** (iii) **Neither**
Notice $f(-x) = (-x)^3 = -x^3 \neq f(x) = x^3$
but $f(-x) = -x^3 = -f(x) = -x^3$.

(f) $y = \sqrt{x}$ (i) **Even** (ii) **Odd** (iii) **Neither**
Notice $f(-x) = \sqrt{-x}$ is undefined.

(g) $y = |x|$ (i) **Even** (ii) **Odd** (iii) **Neither**
Notice $f(-x) = |-x| = x = f(x) = |x| = x$.

(h) $y = \sqrt{(x+1)(x+3)}$ (i) **Even** (ii) **Odd** (iii) **Neither**
Notice $f(-x) = (-x+1)(-x+3) \neq f(x) = (x+1)(x+3)$
and also $f(-x) = (-x+1)(-x+3) \neq -f(x) = -(x+1)(x+3)$.

(i) $y = -\frac{8}{x+1}$ (i) **Even** (ii) **Odd** (iii) **Neither**
Notice $f(-x) = -\frac{8}{-x+1} \neq f(x) = -\frac{8}{x+1}$
and also $f(-x) = -\frac{8}{-x+1} \neq -f(x) = \frac{8}{x+1}$.

10. *Step functions: lawyer fees.* Lawyer fees per hour, $f(x)$, versus number of years of experience, x, are given in graph below.

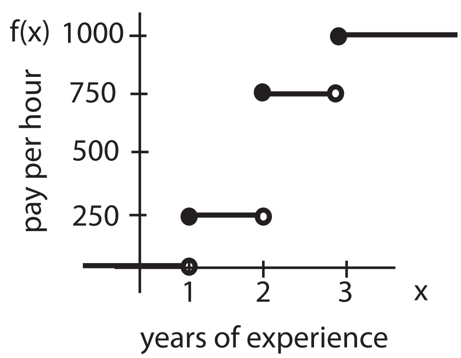

Figure 2.4 (Step function: lawyer fees)

(a) After 1 year of experience, a lawyer makes $f(1) =$
(i) **\$0** (ii) **\$250** (iii) **\$750** (iv) **\$1000**

(b) After 2 years of experience, a lawyer makes $f(2) =$
(i) **\$0** (ii) **\$250** (iii) **\$750** (iv) **\$1000**

(c) After 1.5 years of experience, a lawyer makes $f(1.5) =$
(i) **\$0** (ii) **\$250** (iii) **\$750** (iv) **\$1000**

(d) After 0.5 years of experience, a lawyer makes $f(0.5) =$
(i) **\$0** (ii) **\$250** (iii) **\$750** (iv) **\$1000**

11. *Financial formula:* $A = P\left(1 + \frac{r}{k}\right)^{kt}$

(a) If \$700 is invested at 11% *yearly* interest compounded *monthly*, what will be its value after 8 years?
$A = P\left(1 + \frac{r}{k}\right)^{kt} = 700\left(1 + \frac{0.11}{12}\right)^{8(12)} =$
(i) **1580.88** (ii) **1680.88** (iii) **1780.88**

(b) If \$121 is invested at 3% annual interest compounded *daily* (assume 365 days per year), its value after 4 years is
$A = P\left(1 + \frac{r}{k}\right)^{kt} = 121\left(1 + \frac{0.03}{365}\right)^{4(365)} =$
(i) **116.43** (ii) **126.43** (iii) **136.43**

2.2 Quadratic Functions; Translation and Reflection

A *quadratic* function is defined as:

$$f(x) = ax^2 + bx + c$$

where a, b, c are real, $a \neq 0$. The graph of a quadratic is always a *parabola*. The maximum/minimum point (*vertex*) of quadratic/parabola is given at:

$$(h, k) = \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$$

Increasing c moves (*translates*) parabola upwards; decreasing c translates parabola downwards. *Negative* a flips (*reflects*) parabola downwards; *positive* a reflects parabola upwards. Increasing *magnitude* of a increases steepness of parabola. *Completing the square* of the quadratic, by factoring first two terms of quadratic then adding the square of one-half of the coefficient of x in the parentheses and subtracting outside, gives

$$y = a(x - h)^2 + k$$

where (h, k) is, again, the vertex of the parabola. For *any* function f and positive h and k,

- $y = f(x) + k$ is graph of $f(x)$ translated *up*wards by k
- $y = f(x) - k$ is graph of $f(x)$ translated *down*wards by k
- $y = f(x - h)$ is graph of $f(x)$ translated *right* (not left!) by h
- $y = f(x + h)$ is graph of $f(x)$ translated *left* by h
- $y = -f(x)$ is graph of $f(x)$ reflected upward down, vertically, across x-axis

- $y = f(-x)$ is graph of $f(x)$ reflected horizontally, across y-axis

Exercise 2.2 (Quadratic Functions; Translation and Reflection)

1. *Describing the quadratic.* Consider the following graphs of quadratic functions.

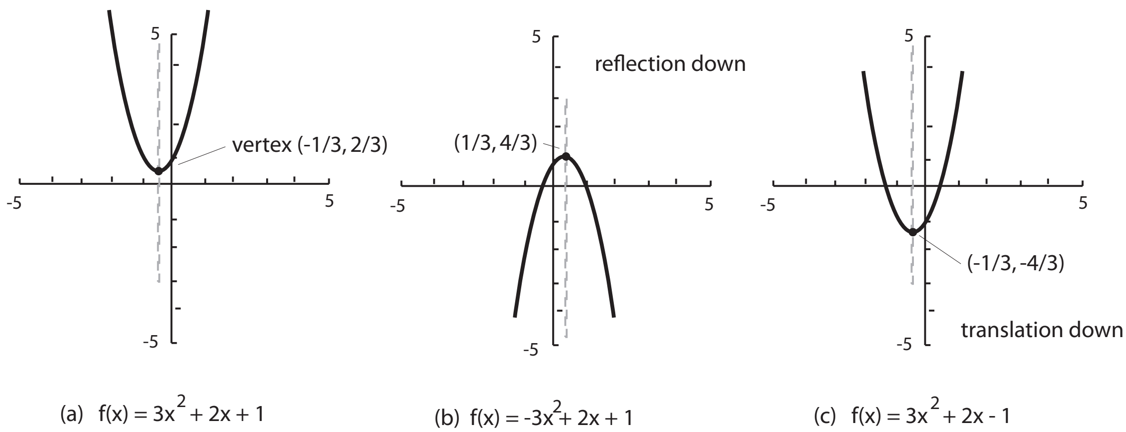

Figure 2.5 (Different Quadratic Functions)

Graph (a): Y=, then type $Y_1 = 3x^2 + 2x + 1$, 2nd QUIT, WINDOW -5, 5, 1, -5, 5, 1, 1, GRAPH
similar for graphs (b) and (c)

(a) *Function (a).* $f(x) = ax^2 + bx + c = 3x^2 + 2x + 1$.

Function $f(x)$ reflects (i) **up** (ii) **down** because $a = 3 > 0$

y-intercept: when $x = 0$, $f(0) =$ (i) $\mathbf{-1}$ (ii) $\mathbf{0}$ (iii) $\mathbf{1}$
x-intercept(s): when $y = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{2^2 - 4(3)(1)}}{2(3)} = \frac{-2 \pm \sqrt{-8}}{6} =$$

(i) **1** (ii) **does not exist** (iii) $\mathbf{-\frac{2}{3}}$ since $\sqrt{-8}$ is not a real number.

vertex. Since $h = -\frac{b}{2a} = -\frac{2}{2(3)} =$ (i) $\mathbf{-\frac{1}{3}}$ (ii) $\mathbf{-\frac{1}{6}}$ (iii) $\mathbf{\frac{1}{3}}$,
and $k = f\left(-\frac{b}{2a}\right) = f\left(-\frac{1}{3}\right) = 3\left(-\frac{1}{3}\right)^2 + 2\left(-\frac{1}{3}\right) + 1 = \frac{2}{3}$,
(Type $3x^2 + 2x + 1$ into Y=, 2nd QUIT, VARS Y–VARS ENTER ($-\frac{1}{3}$) ENTER, MATH ENTER.)
then $(h, k) =$ (i) $\mathbf{\left(-\frac{1}{3}, -\frac{1}{3}\right)}$ (ii) $\mathbf{\left(-\frac{1}{3}, \frac{2}{3}\right)}$ (iii) $\mathbf{\left(-\frac{1}{3}, \frac{4}{3}\right)}$
which is a (i) **minimum** (ii) **maximum**

(b) *Function (b).* $f(x) = ax^2 + bx + c = -3x^2 + 2x + 1$.

Function $f(x)$ reflects (i) **up** (ii) **down** because $a = -3 < 0$

y-intercept: when $x = 0$, $f(0) =$ (i) $\mathbf{-1}$ (ii) $\mathbf{0}$ (iii) $\mathbf{1}$
x-intercept(s): when $y = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{2^2 - 4(-3)(1)}}{2(-3)} = \frac{-2 \pm \sqrt{16}}{-6} =$$

(circle *two*!) (i) $\mathbf{1}$ (ii) $\mathbf{-\frac{1}{3}}$ (iii) $\mathbf{\frac{2}{3}}$

vertex. Since $h = -\frac{b}{2a} = -\frac{2}{2(-3)} =$ (i) $\mathbf{-\frac{1}{3}}$ (ii) $\mathbf{-\frac{1}{6}}$ (iii) $\mathbf{\frac{1}{3}}$,
and $k = f\left(-\frac{b}{2a}\right) = f\left(\frac{1}{3}\right) = -3\left(\frac{1}{3}\right)^2 + 2\left(\frac{1}{3}\right) + 1 = \frac{4}{3}$,
(Type $-3x^2 + 2x + 1$ into Y=, 2nd QUIT, VARS Y–VARS ENTER ($\frac{1}{3}$) ENTER, MATH ENTER.)
then $(h, k) =$ (i) $\mathbf{\left(\frac{1}{3}, \frac{4}{3}\right)}$ (ii) $\mathbf{\left(-\frac{1}{3}, \frac{2}{3}\right)}$ (iii) $\mathbf{\left(-\frac{1}{3}, \frac{4}{3}\right)}$
which is a (i) **minimum** (ii) **maximum**

(c) *Function (c).* $f(x) = ax^2 + bx + c = 3x^2 + 2x - 1$.

Function $f(x)$ reflects (i) **up** (ii) **down** because $a = 3 > 0$

Function (c) is translated (i) **up** (ii) **down** from function (a)
because $c = -1 < c = 1$

y-intercept: when $x = 0$, $f(0) =$ (i) $\mathbf{-1}$ (ii) $\mathbf{0}$ (iii) $\mathbf{1}$
x-intercept(s): when $y = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{2^2 - 4(3)(-1)}}{2(3)} = \frac{-2 \pm \sqrt{16}}{6} =$$

(circle *two*!) (i) $\mathbf{-1}$ (ii) $\mathbf{\frac{1}{3}}$ (iii) $\mathbf{\frac{2}{3}}$

vertex. Since $h = -\frac{b}{2a} = -\frac{2}{2(3)} =$ (i) $\mathbf{-\frac{1}{3}}$ (ii) $\mathbf{-\frac{1}{6}}$ (iii) $\mathbf{\frac{1}{3}}$,
and $k = f\left(-\frac{1}{3}\right) = 3\left(-\frac{1}{3}\right)^2 + 2\left(-\frac{1}{3}\right) - 1 =$ (i) $\mathbf{-\frac{4}{3}}$ (ii) $\mathbf{\frac{1}{3}}$ (iii) $\mathbf{\frac{2}{3}}$,
(Type $3x^2 + 2x - 1$ into Y=, 2nd QUIT, VARS Y–VARS ENTER ($-\frac{1}{3}$) ENTER, MATH ENTER.)
then $(h, k) =$ (i) $\mathbf{\left(\frac{1}{3}, \frac{4}{3}\right)}$ (ii) $\mathbf{\left(-\frac{1}{3}, \frac{2}{3}\right)}$ (iii) $\mathbf{\left(-\frac{1}{3}, -\frac{4}{3}\right)}$
which is a (i) **minimum** (ii) **maximum**

2. *Completing the square, $a(x-h)^2 + k$.*

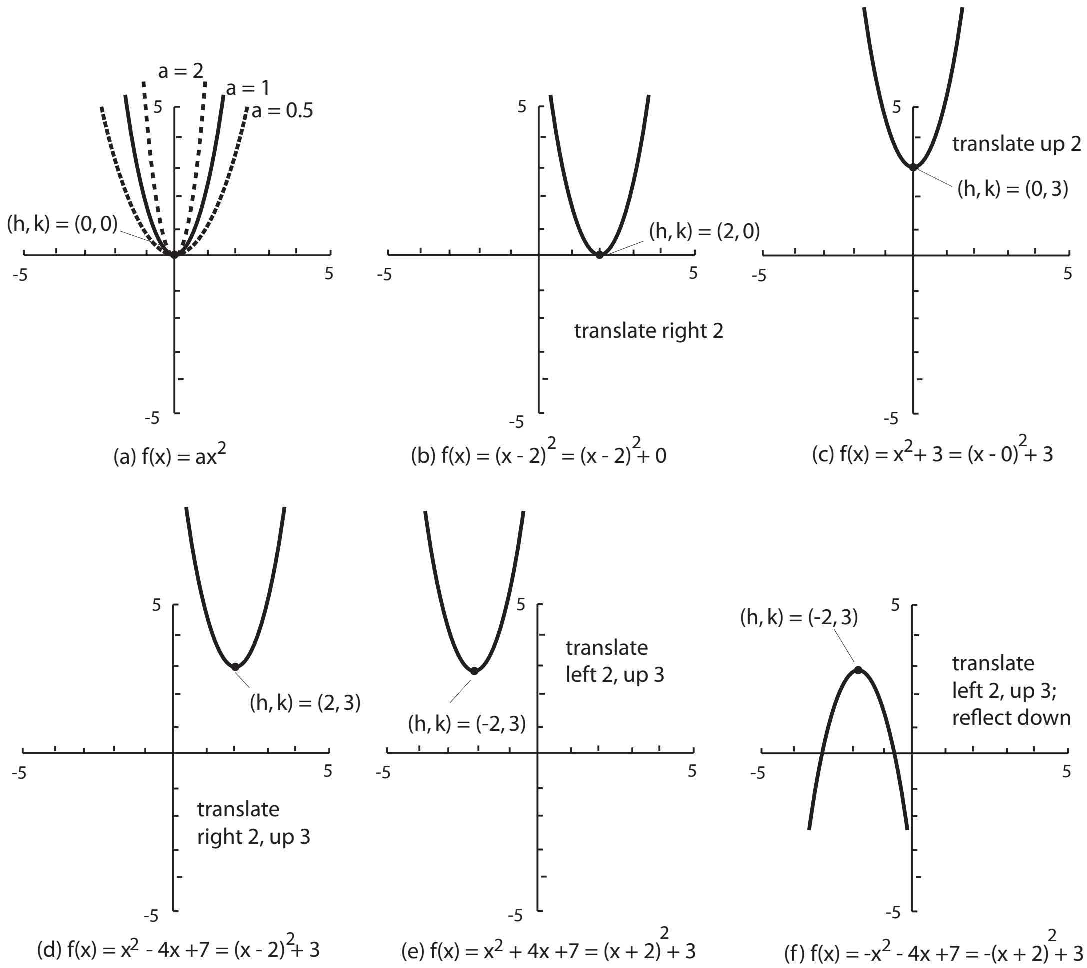

Figure 2.6 (Translating and reflecting $f(x) = x^2$)

(a) *Figure (a)* $f(x) = ax^2$, $a = \frac{1}{2}, 1, 2$.
Decreasing a from 2 to $\frac{1}{2}$ (i) **does** (ii) **does not** spread the parabola.

(b) *Figure (b)* $f(x) = x^2 - 4x + 4$.

$$\begin{aligned} x^2 - 4x + 4 &= 1\left(x^2 - 4x + 4\right) + 4 - 1\,(4) \quad \text{add/subtract } 1 \times \left(\tfrac{1}{2} \times -4\right)^2 = 1 \times 4 \\ &= 1\,(x-2)^2 + 0 \\ &= a(x-h)^2 + k \end{aligned}$$

so vertex $(h, k) =$ (i) $\mathbf{(-2, -3)}$ (ii) $\mathbf{(2, 0)}$ (iii) $\mathbf{(3, -2)}$
so $f(x) = x^2$ translated (i) **left** (ii) **right** 2 units because $h = 2$
and *not* translated up or down because $k = 0$
and reflected (i) **up** (ii) **down** because $a = 1 > 0$

(c) *Figure (c)* $f(x) = x^2 + 3$.

$$x^2 + 3 = 1\left(x^2 + 0\right) + 3 - 1\,(0) \quad \text{add/subtract } 1 \times \left(\tfrac{1}{2} \times 0\right)^2 = 1 \times 0$$

$$
\begin{aligned}
&= 1\,(x-0)^2 + 3 \\
&= a(x-h)^2 + k
\end{aligned}
$$

so vertex $(h, k) =$ (i) $\mathbf{(-2, -3)}$ (ii) $\mathbf{(0, 3)}$ (iii) $\mathbf{(3, -2)}$
so $f(x) = x^2$ *not* translated left or right because $h = 0$
and translated (i) **up** (ii) **down** 3 units because $k = 3$
and reflected (i) **up** (ii) **down** because $a = 1 > 0$

(d) *Figure (d)* $f(x) = x^2 - 4x + 7$.

$$
\begin{aligned}
x^2 - 4x + 7 &= 1\left(x^2 - 4x + 4\right) + 7 - 1\,(4) \quad \text{add/subtract } 1 \times \left(\tfrac{1}{2} \times -4\right)^2 = 1 \times 4 \\
&= 1\,(x-2)^2 + 3 \\
&= a(x-h)^2 + k
\end{aligned}
$$

so vertex $(h, k) =$ (i) $\mathbf{(-2, -3)}$ (ii) $\mathbf{(3, -2)}$ (iii) $\mathbf{(2, 3)}$
so $f(x) = x^2$ translated (i) **left** (ii) **right** 2 units because $h = 2$
and translated (i) **up** (ii) **down** 3 units because $k = 3$
and reflected (i) **up** (ii) **down** because $a = 1 > 0$

(e) *Figure (e)* $f(x) = x^2 + 4x + 7$.

$$
\begin{aligned}
x^2 + 4x + 7 &= 1\left(x^2 + 4x + 4\right) + 7 - 1\,(4) \quad \text{add/subtract } 1 \times \left(\tfrac{1}{2} \times 4\right)^2 = 1 \times 4 \\
&= 1\,(x+2)^2 + 3 \\
&= a(x-h)^2 + k
\end{aligned}
$$

so vertex $(h, k) =$ (i) $\mathbf{(-2, -3)}$ (ii) $\mathbf{(3, -2)}$ (iii) $\mathbf{(-2, 3)}$
so $f(x) = x^2$ translated (i) **left** (ii) **right** 2 units because $h = -2$
and translated (i) **up** (ii) **down** 3 units because $k = 3$
and reflected (i) **up** (ii) **down** because $a = 1 > 0$

(f) *Figure (f)* $f(x) = -x^2 - 4x - 1$.

$$
\begin{aligned}
-x^2 - 4x - 1 &= -1\left(x^2 + 4x + 4\right) - 1 + 1\,(4) \quad \text{add/subtract } -1 \times \left(\tfrac{1}{2} \times 4\right)^2 = -1 \times 4 \\
&= -1\,(x+2)^2 + 3 \\
&= a(x-h)^2 + k
\end{aligned}
$$

so vertex $(h, k) =$ (i) $\mathbf{(-2, 3)}$ (ii) $\mathbf{(-2, -3)}$ (iii) $\mathbf{(3, -2)}$
so $f(x) = x^2$ translated (i) **left** (ii) **right** 2 units because $h = -2$
and translated (i) **up** (ii) **down** 3 units because $k = 3$
and reflected (i) **up** (ii) **down** because $a = -1 < 0$

3. *More completing the square,* $a(x-h)^2 + k$.

(a) $f(x) = -3x^2 + 2x + 1$.

$$\begin{aligned}
-3x^2+2x+1 &= -3\left(x^2 - \frac{2}{3}x\right) + 1 \quad \text{factor out } -3 \\
&= -3\left(x^2 - \frac{2}{3}x + \frac{1}{9}\right) + 1 + 3\left(\frac{1}{9}\right) \quad \text{where } \pm -3 \times \left(\tfrac{1}{2} \times -\tfrac{2}{3}\right)^2 = -3 \times \tfrac{1}{9} \\
&= -3\left(x - \frac{1}{3}\right)^2 + \frac{4}{3} \\
&= a(x-h)^2 + k
\end{aligned}$$

so vertex $(h,k) =$ (i) $\left(\mathbf{\frac{1}{3}, \frac{4}{3}}\right)$ (ii) $\left(\mathbf{-\frac{1}{3}, \frac{2}{3}}\right)$ (iii) $\left(\mathbf{-\frac{1}{3}, \frac{4}{3}}\right)$

(b) $f(x) = 4x^2 + 2x - 5$.

$$\begin{aligned}
4x^2+2x-5 &= 4\left(x^2 + \frac{1}{2}x\right) - 5 \quad \text{factor out } 4 \\
&= 4\left(x^2 + \frac{1}{2}x + \frac{1}{16}\right) - 5 - 4\left(\frac{1}{16}\right) \quad \text{where } \pm\ 4 \times \left(\tfrac{1}{2} \times \tfrac{1}{2}\right)^2 = 4 \times \tfrac{1}{16} \\
&= 4\left(x + \frac{1}{4}\right)^2 - \frac{21}{4} \\
&= a(x-h)^2 + k
\end{aligned}$$

so vertex $(h,k) =$ (i) $\left(\mathbf{-\frac{1}{4}, \frac{21}{4}}\right)$ (ii) $\left(\mathbf{\frac{1}{4}, -\frac{21}{4}}\right)$ (iii) $\left(\mathbf{-\frac{1}{4}, -\frac{21}{4}}\right)$

4. *Quadratic example: throwing a ball.* A ball is thrown upwards with an initial velocity of 32 feet per second and from an initial height of 150 feet. A function relating height, $f(t)$, to time, f, when throwing this ball is:

$$f(t) = -12t^2 + 32t + 150$$

(a) Function $f(t)$ reflects (i) **up** (ii) **down** because $a = -12 < 0$, so the vertex is a (i) **minimum** (ii) **maximum**

(b) *Completing the square*

$$\begin{aligned}
-12t^2+32t+150 &= -12\left(t^2 - \frac{32}{12}t\right) + 150 \quad \text{factor out } -12 \\
&= -12\left(t^2 - \frac{8}{3}t + \frac{16}{9}\right) + 150 + 12\left(\frac{16}{9}\right); \quad -12 \times \left(\tfrac{1}{2} \times \tfrac{8}{3}\right)^2 = -12 \times \tfrac{16}{9} \\
&= -12\left(t - \frac{4}{3}\right)^2 + \frac{514}{3} \\
&= a(t-h)^2 + k
\end{aligned}$$

so vertex $(h,k) =$ (i) $\left(\mathbf{\frac{4}{3}, \frac{514}{3}}\right)$ (ii) $\left(\mathbf{-\frac{4}{3}, \frac{514}{3}}\right)$ (iii) $\left(\mathbf{-\frac{4}{3}, -\frac{514}{3}}\right)$

(c) Maximum height of ball is (i) $\mathbf{\frac{4}{3}}$ (ii) $\mathbf{\frac{514}{3}}$ (iii) **150** feet.

(d) Time until ball hits ground is t-intercept, when $f(t) = 0$,

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-32 \pm \sqrt{32^2 - 4(-12)(150)}}{2(-12)} = \frac{-32 \pm \sqrt{8224}}{-24} \approx$$

(i) **5.01** (ii) **5.11** (iii) **5.21** seconds.

ignore negative answer, choose positive answer

5. *Translation and reflection of $f(x)$, which are not necessarily quadratic.*

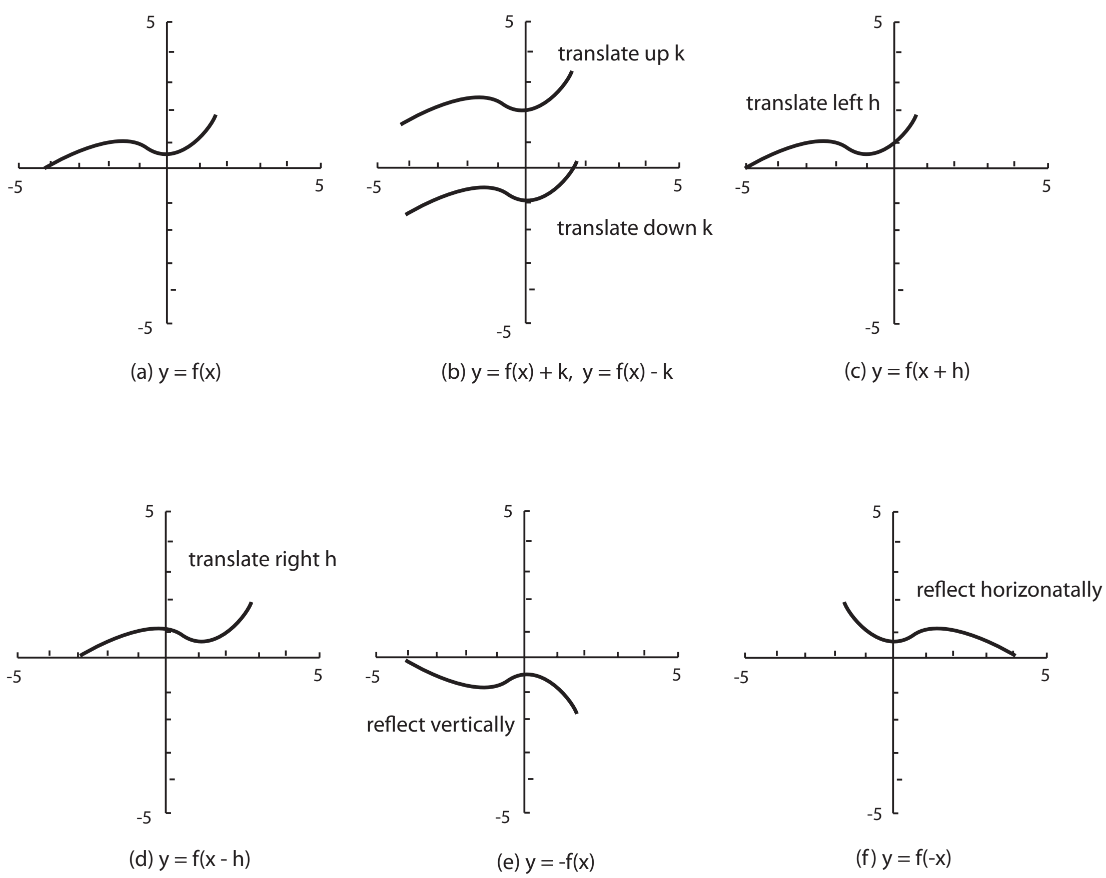

Figure 2.7 (Translations and reflections of $f(x)$)

True / **False** Various translations and reflections of a function $f(x)$ are correctly given in figure above.

6. *Translation and reflection of $f(x) = \sqrt{x}$.*

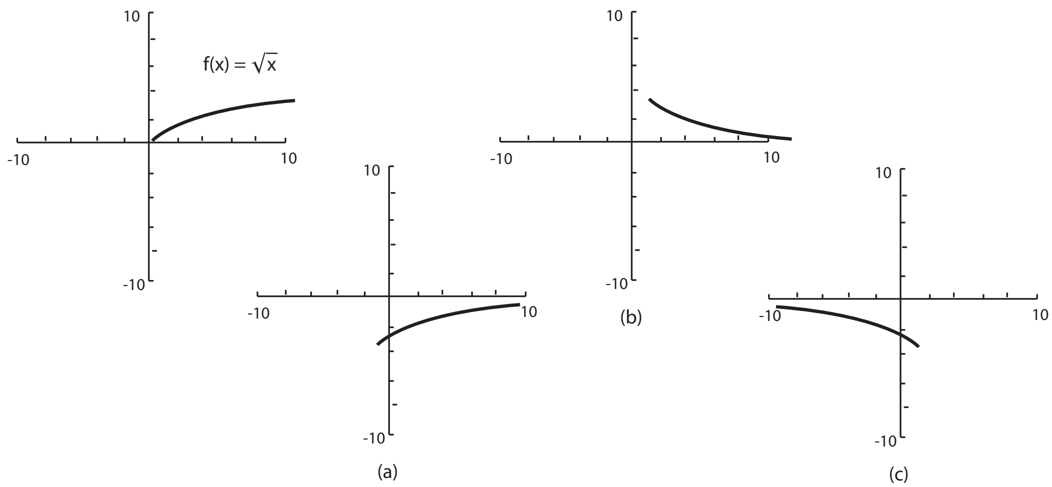

Figure 2.8 (Translations and reflections of $f(x) = \sqrt{x}$)

(a) $f_1(x) = \sqrt{x+1} - 4$
Since $f(x) = \sqrt{x}$,
translate left 1 unit, $f(x+1) = \sqrt{x+1}$,
and translate down 4 units, $f(x+1) - 4 = \sqrt{x+1} - 4$
So $f_1(x)$ corresponds to Figure (i) **(a)** (ii) **(b)** (iii) **(c)**
(Type $\sqrt{x+1} - 4$ into Y=, ZOOM ZStandard.)

(b) $f_2(x) = -\sqrt{x-1} + 4$
Since $f(x) = \sqrt{x}$,
translate right 1 unit, $f(x-1) = \sqrt{x-1}$,
reflect vertically (flip upsidedown), $-f(x-1) = -\sqrt{x-1}$,
and translate up 4 units, $-f(x-1) + 4 = -\sqrt{x-1} + 4$
So $f_2(x)$ corresponds to Figure (i) **(a)** (ii) **(b)** (iii) **(c)**
(Type $-\sqrt{x-1} + 4$ into Y=, ZOOM ZStandard.)

(c) $f_3(x) = \sqrt{1-x} - 4$
Since $f(x) = \sqrt{x}$,
reflect horizontally and translate right 1 unit, $f(-(x-1)) = \sqrt{-(x-1)}$,
and translate down 4 units, $f(-(x-1)) - 4 = \sqrt{-(x-1)} - 4$
So $f_3(x)$ corresponds to Figure (i) **(a)** (ii) **(b)** (iii) **(c)**
(Type $\sqrt{1-x} - 4$ into Y=, ZOOM ZStandard.)

2.3 Polynomial and Rational Functions

A *polynomial* function of *degree* n is defined by

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where leading coefficient $a_n \neq 0$, the a_i are real numbers and n is a nonnegative integer. Linear and quadratic functions are polynomials of degree 1 and 2, respectively; *cubic* and *quartic* polynomials are of degrees 3 and 4, respectively. Simple polynomials of the form $f(x) = x^n$ are called *power* functions. Some properties of polynomials:

- polynomials of degree n have *at most* $n-1$ *turning points* (or *relative extrema*); graphs of polynomials with n turning points are *at least* of degree $n+1$
- ends of a polynomial with *even* degree either both turn up or both turn down; one end of a polynomial with *odd* degree turns up and other turns down
- graph goes up as x becomes a large positive number if leading coefficient positive; goes down if leading coefficient negative

A *rational* function is

$$f(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$. Since

- if a function grows larger in magnitude as x approaches k, $x = k$ is a *vertical asymptote*,
- if a function approaches k as $|x|$ gets larger, $y = k$ is a *horizontal asymptote*,

then if both numerator $p(x)$ and denominator $q(x)$ of a rational function are zero at same $x = k$, graph has a hole (*removable discontinuity*) at k, but if only denominator $q(x)$ is zero at $x = k$, $x = k$ is a vertical asymptote.

Exercise 2.3 (Polynomial and Rational Functions)

1. *Properties of polynomial functions.*

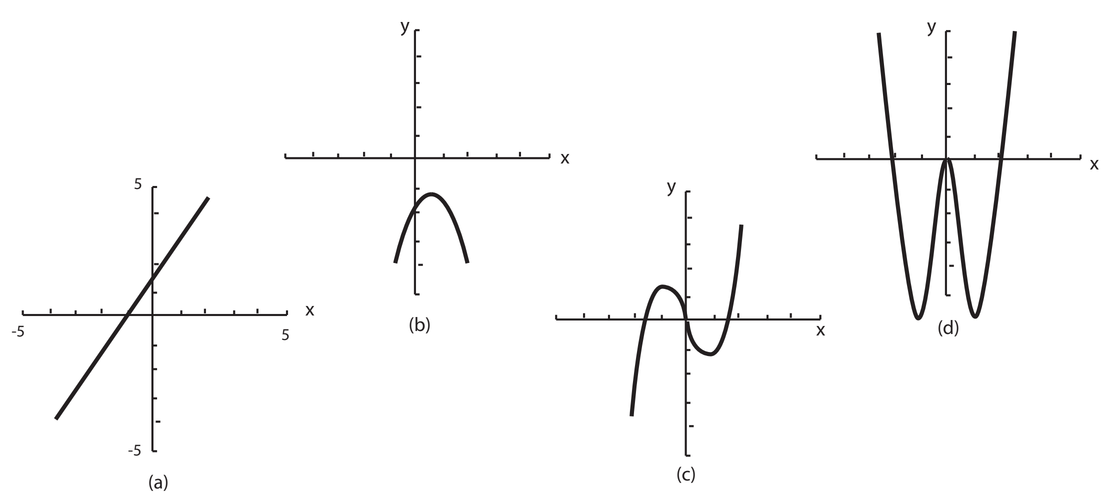

Figure 2.9 (Polynomial functions)

(To graph, set WINDOW to -5 5 1 -5 5 1 1 then Y= and type function $2x+3$ and then the others. Set graphs to "dot" display by typing Y=, then arrowing left twice and ENTERing until the dotted line appears.)

(a) Linear function $f(x) = 2x + 3$ corresponds to
graph (i) **(a)** (ii) **(b)** (iii) **(c)** (iv) **(d)**
with degree (i) **0** (ii) **1** (iii) **2** (iv) **3** (v) **4**
and so is a polynomial of (i) **even** (ii) **odd** degree
with (i) **0** (ii) **1** (iii) **2** (iv) **3** (v) **4** turning points
and since leading coefficient is (i) **positive** (ii) **negative**
graph moves (i) **up** (ii) **down** as x becomes large positive number

(b) Quadratic function $f(x) = -2x^2 + 2x - 2$ corresponds to
graph (i) **(a)** (ii) **(b)** (iii) **(c)** (iv) **(d)**
with degree (i) **0** (ii) **1** (iii) **2** (iv) **3** (v) **4**
and so is a polynomial of (i) **even** (ii) **odd** degree
with (i) **0** (ii) **1** (iii) **2** (iv) **3** (v) **4** turning points
and since leading coefficient is (i) **positive** (ii) **negative**
both ends turn (i) **up** (ii) **down**

(c) Cubic function $f(x) = 2x^3 - 4x$ corresponds to
graph (i) **(a)** (ii) **(b)** (iii) **(c)** (iv) **(d)**
with degree (i) **0** (ii) **1** (iii) **2** (iv) **3** (v) **4**
and so is a polynomial of (i) **even** (ii) **odd** degree
with (i) **0** (ii) **1** (iii) **2** (iv) **3** (v) **4** turning points
and since leading coefficient is (i) **positive** (ii) **negative**
graph moves (i) **up** (ii) **down** as x becomes large positive number

(d) Quartic function $f(x) = x^4 - 5x^2$ corresponds to
graph (i) **(a)** (ii) **(b)** (iii) **(c)** (iv) **(d)**
with degree (i) **0** (ii) **1** (iii) **2** (iv) **3** (v) **4**
and so is a polynomial of (i) **even** (ii) **odd** degree
with (i) **0** (ii) **1** (iii) **2** (iv) **3** (v) **4** turning points
and since leading coefficient is (i) **positive** (ii) **negative**
both ends turn (i) **up** (ii) **down**

2. *More properties of polynomial functions.*

(a) Polynomial function $f(x) = 3x^3 + 4x - 2$
(Set WINDOW to -2 2 1 1 -10 10 1 1 before graphing function.)
has degree (i) **0** (ii) **1** (iii) **2** (iv) **3** (v) **4**
and so is a polynomial of (i) **even** (ii) **odd** degree
with *at most* (i) **0** (ii) **1** (iii) **2** (iv) **3** (v) **4** turning points
and since leading coefficient is (i) **positive** (ii) **negative**
graph moves (i) **up** (ii) **down** as x becomes large positive number

(b) Polynomial function $f(x) = -2x^4 - 5x^3 + 3x - 2x + 1$
(Set WINDOW to -5 5 1 1 -5 5 1 1 before graphing function.)
has degree (i) **0** (ii) **1** (iii) **2** (iv) **3** (v) **4**
and so is a polynomial of (i) **even** (ii) **odd** degree
with *at most* (i) **0** (ii) **1** (iii) **2** (iv) **3** (v) **4** turning points
and since leading coefficient is (i) **positive** (ii) **negative**
both ends turn (i) **up** (ii) **down**
and (i) **is** (ii) **is not** a power function

3. *Properties of rational functions.*

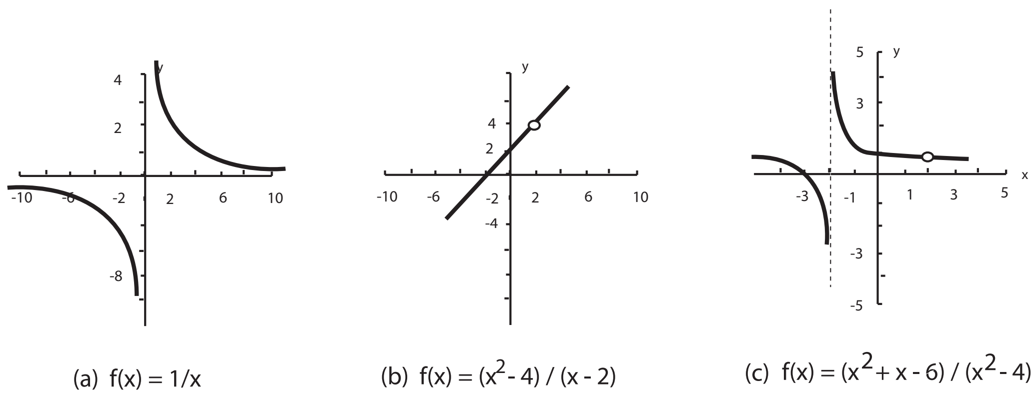

Figure 1.10 (Different Rational Functions)

(a) Rational function $f(x) = \frac{1}{x}$ corresponds to
graph (i) **(a)** (ii) **(b)** (iii) **(c)**
with vertical asymptote at $x =$ (i) **0** (ii) **1** (iii) **2**
because $f(x) = \frac{1}{x}$ grows larger as x approaches 0
and horizontal asymptote at $y =$ (i) **0** (ii) **1** (iii) **2**
because $f(x) = \frac{1}{x}$ approaches 0 as $|x|$ gets larger
and has no x-intercepts, y-intercepts or removable discontinuities (holes)

(b) Rational function $f(x) = \frac{x^2-4}{x-2}$ corresponds to
graph (i) **(a)** (ii) **(b)** (iii) **(c)**
has removable discontinuity (hole) at $x =$ (i) **0** (ii) **1** (iii) **2** because

$$f(x) = \frac{x^2 - 4}{x - 2} = \frac{(x+2)(x-2)}{x-2} = x + 2, \quad x \neq 2$$

and y-intercept $y =$ (i) **0** (ii) **1** (iii) **2**
because $y = f(0) = 0 + 2 = 2$
and x-intercept $x =$ (i) **0** (ii) **−1** (iii) **−2**
because $f(x) = x + 2 = 0$ when $x = -2$
and has no vertical or horizontal asymptotes

(c) Rational function $f(x) = \frac{x^2+x-6}{x^2-4}$ corresponds to
graph (i) **(a)** (ii) **(b)** (iii) **(c)**
has removable discontinuity (hole) at $x =$ (i) **0** (ii) **1** (iii) **2** because

$$f(x) = \frac{x^2+x-6}{x^2-4} = \frac{(x+3)(x-2)}{(x+2)(x-2)} = \frac{x+3}{x+2}, \quad x \neq 2$$

with vertical asymptote at $x =$ (i) **0** (ii) **−1** (iii) **−2**
because $f(x) = \frac{x+3}{x+2}$ grows larger as x approaches -2
and horizontal asymptote at $y =$ (i) **0** (ii) **1** (iii) **2**
because $f(x) = \frac{x+3}{x+2} \approx \frac{x}{x}$ approaches 1 (not 0!) as $|x|$ gets larger
and y-intercept $y =$ (i) **0** (ii) **1** (iii) **1.5**
because $y = f(0) = \frac{0+3}{0+2} = 1.5$
and x-intercept $x =$ (i) **−1** (ii) **−2** (iii) **−3**
because $f(x) = \frac{x+3}{x+2} = 0$ when $x = -3$

4. *More properties of rational functions.*

(a) Rational function $f(x) = \frac{7x}{3-x}$
(Set WINDOW to -10 10 1 1 -20 20 1 1 before graphing function.)
has vertical asymptote at $x =$ (i) **1** (ii) **2** (iii) **3**
because $f(x) = \frac{7x}{3-x}$ grows larger as x approaches 3
and horizontal asymptote at $y =$ (i) **−7** (ii) $\mathbf{\frac{7}{3}}$ (iii) **7**
because $f(x) = \frac{7x}{3-x} \approx \frac{7x}{-x}$ approaches -7 as $|x|$ gets larger
and y-intercept $y =$ (i) **0** (ii) **1** (iii) **1.5**
because $y = f(0) = \frac{7(0)}{3-0} = 0$
and x-intercept $x =$ (i) **0** (ii) **1** (iii) **2**
because $f(x) = \frac{7x}{3-x} = 0$ when $x = 0$
and no removable discontinuities

(b) Rational function $f(x) = \frac{x^2+4x+3}{x-3} = \frac{(x+1)(x+3)}{x-3}$
(Set WINDOW to -7 15 1 1 -20 40 1 1 before graphing function.)
has vertical asymptote at $x =$ (i) **1** (ii) **2** (iii) **3**
because $f(x) = \frac{(x+1)(x+3)}{x-3}$ grows larger as x approaches 3
and no horizontal asymptote because $f(x) = \frac{x^2+4x+3}{x-3} \approx \frac{x^2+4x}{x} \approx x$ and so
graph moves (i) **up** (ii) **down** as x becomes large positive number
and y-intercept $y =$ (i) **0** (ii) **−1** (iii) **−2**
because $y = f(0) = \frac{(0+1)(0+3)}{0-3} = -1$
and x-intercepts $x =$ (choose *two*!) (i) **−1** (ii) **−2** (iii) **−3**
because $f(x) = \frac{(x+1)(x+3)}{x-3} = 0$ when $x = -1, -3$
and no removable discontinuities

5. *Rational example: average cost.* Monthly fixed costs of using machine I are \$15,000 and marginal costs of manufacturing one widget using machine I is \$20.

Consequently, *average* costs are

$$\overline{C}(x) = \frac{20x + 15000}{x}, \quad x > 0$$

(Set WINDOW to 0 700 1 1 0 1000 1 1 before graphing function.)

(a) vertical asymptote at $x =$ (i) **0** (ii) **1** (iii) **2**
because $\overline{C} = \frac{20x+15000}{x}$ grows larger as x approaches 0

(b) horizontal asymptote at $\overline{C} =$ (i) **1** (ii) **20** (iii) **200**
because $\bar{C}(x) = \frac{20x+15000}{x} \approx \frac{20x}{x} \approx 20$

(c) (i) **True** (ii) **False** no y-intercept, x-intercept or removable discontinuities