7.4 The Fundamental Theorem of Calculus

The *fundamental theorem of calculus* says if f is a continuous function on $[a, b]$, and F is any antiderivative of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a) = \left. F(x)\right|_a^b .$$

This theorem essentially shows how antiderivatives and *definite* integrals are related to one another. Properties of definite integrals include:

- $\int_a^a f(x)\,dx = 0$
- $\int_a^b k \cdot f(x)\,dx = k \cdot \int_a^b f(x)\,dx$, for real k
- $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Not all integrals have antiderivatives; these integrals would have to be solved using *numerical* integration. Both limits and variable of integration (not just variable of integration) are changed when using method of substitution. Definite integrals can be negative and so some care must be taken when associating them to area, which (of course) is always positive.

Exercise 7.4 (Definite Integrals)

1. *Definite integrals,* $f(x) = 3$.

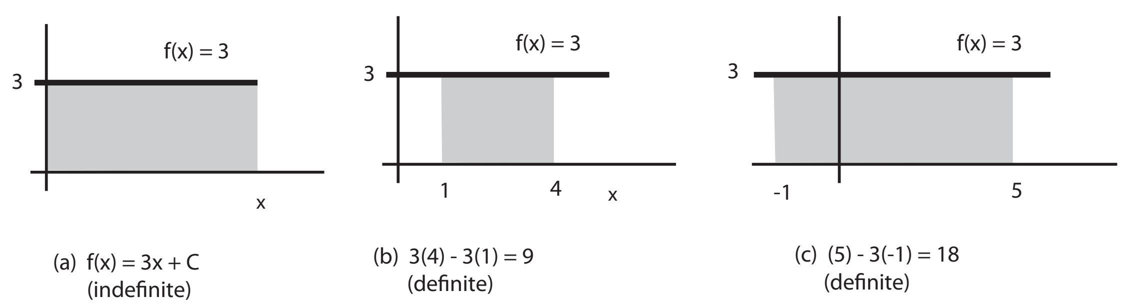


Figure 7.1 ($f(x) = 3$)

(a) $f(x) = 3$, *graph (a).* The *in*definite integral of 3,

$$\int 3\,dx = \int 3x^0\,dx = 3\int x^0\,dx = 3\left(\frac{1}{0+1}x^{0+1} + C\right) =$$

(i) $\boldsymbol{C}$ (ii) $\boldsymbol{3 + C}$ (iii) $\boldsymbol{3x + C}$

(b) $f(x) = 3$, *graph (b).* Area in interval [1,4] is *definite* integral,

$$\int_1^4 3\,dx = [3x + C]_{x=1}^{x=4}$$
$$= [3(4) + C] - [3(1) + C] =$$

(i) **8** (ii) **9** (iii) **10**

$Y_1 = 3$ WINDOW -2 5 1 -1 4 1 GRAPH 2nd CALC 7:$\int f(x)\,dx$ ENTER 1 ENTER 4 ENTER
OR MATH fnInt(Y_1,X,1,4)
also, notice, area of rectangle is $(4 - 1) \times 3 = 3 \times 3 = 9$

(c) $f(x) = 3$, *graph (c).* Area in interval $[-1, 5]$ is definite integral,

$$\int_{-1}^5 3\,dx = [3x + C]_{x=-1}^{x=5}$$
$$= [3(5) + C] - [3(-1) + C] =$$

(i) **18** (ii) **19** (iii) **20**

GRAPH 2nd CALC 7:$\int f(x)\,dx$ ENTER -1 ENTER 5 ENTER
also, notice, area of rectangle is $(5 - (-1)) \times 3 = 6 \times 3 = 18$

(d) (i) **True** (ii) **False**. Constant of integration, C, *always* disappears (is cancelled out) in a definite integration. In other words, a definite integration gives the same answer whether $C = 1$ or $C = -37$ or $C = 0$. It is for this reason that it is often easiest to let $C = 0$ in a definite integration:

$$\int_{-1}^5 3\,dx = [3x]_{x=-1}^{x=5} = [3(5) + 0] - [3(-1) + 0] = 3(5) - 3(-1) = 18$$

(e) (i) **True** (ii) **False**. The two integrals in figures (b) and (c) are both *positive* definite integrals because the two areas under the function $f(x) = 3$ are found *above* the x–axis.

2. *Definite integrals, $f(x) = 5x$.*

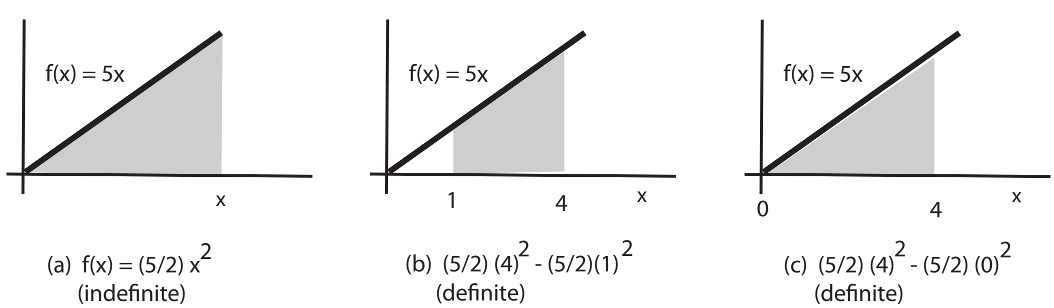


Figure 7.2 ($f(x) = 5x$)

(a) $f(x) = 5x$, *graph (a).*

$$\int 5x\,dx = 5\int x\,dx = 5\left(\frac{1}{1+1}x^{1+1} + C\right) =$$

(i) $\boldsymbol{C}$ (ii) $\mathbf{5}\boldsymbol{x} + \boldsymbol{C}$ (iii) $\boldsymbol{\frac{5}{2}x^2 + C}$

(b) $f(x) = 5x$, $[1,4]$, *graph (b).*

$$\int_1^4 5x\,dx = \left[\frac{5}{2}x\right]_{x=1}^{x=4} = \frac{5}{2}(4)^2 - \frac{5}{2}(1)^2 =$$

(i) **37.5** (ii) **52.0** (iii) **52.5**

$Y_2 = 5x$ WINDOW -2 5 1 -1 20 2 GRAPH 2nd CALC 7:$\int f(x)\,dx$ ENTER 1 ENTER 4 ENTER

OR MATH fnInt(Y_2,X,1,4) instead.

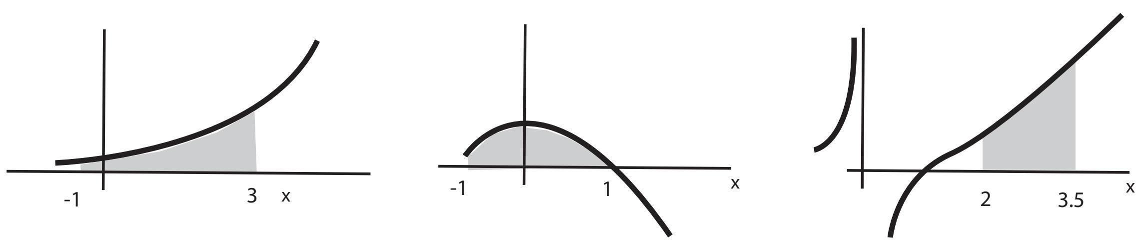
(c) $f(x) = 5x$, $[0,4]$, *graph (c).*

$$\int_0^4 5x\,dx = \left[\frac{5}{2}x\right]_{x=0}^{x=4} = \frac{5}{2}(4)^2 - \frac{5}{2}(0)^2 =$$

(i) **37.5** (ii) **40.0** (iii) **52.5**

GRAPH 2nd CALC 7:$\int f(x)\,dx$ ENTER 0 ENTER 4 ENTER

3. *More definite integrals.*



(a) $f(x) = e^x$ (b) $f(x) = e^x - 5x^2 - 2x + 5$ (c) $f(x) = x^2 - 2/x$

Figure 7.3 (More definite integrals)

(a) $f(x) = e^x$, $[-1,3]$, *graph (a).*

$$\int_{-1}^3 e^x\,dx = [e^x]_{x=-1}^{x=3} = e^3 - e^{-1} \approx$$

(i) **19.7** (ii) **52.0** (iii) **54.2**

$Y_3 = e^x$ WINDOW -1 3 1 -10 20 2

GRAPH 2nd CALC 7:$\int f(x)\,dx$ ENTER -1 ENTER 3 ENTER

OR MATH fnInt(Y_3,X,1,4) instead.

(b) $f(x) = e^x - 5x^2 - 2x + 5,\ [-1, 1]$, *graph (b).*

$$\begin{aligned}\int_{-1}^{1} f(x)\,dx &= \left[e^x - 5\cdot\frac{1}{2+1}x^{2+1} - 2\cdot\frac{1}{1+1}x^{1+1} + 5\cdot\frac{1}{0+1}x^{0+1}\right]_{x=-1}^{x=1} \\ &= \left[e^x - \frac{5}{3}x^3 - x^2 + 5x\right]_{x=-1}^{x=1} \\ &= \left[e^1 - \frac{5}{3}(1)^3 - (1)^2 + 5(1)\right] - \left[e^{-1} - \frac{5}{3}(-1)^3 - (-1)^2 + 5(-1)\right] \approx\end{aligned}$$

(i) **8.23** (ii) **9.02** (iii) **10.33**

$Y_4 = e^x - 5x^2 - 2x + 5$ WINDOW -2 5 1 -10 20 2

GRAPH 2nd CALC 7:$\int f(x)\,dx$ ENTER -1 ENTER 1 ENTER

OR MATH fnInt(Y_4,X,-1,1) instead.

(c) $f(x) = x^2 - \frac{2}{x},\ [2, 3.5]$, *graph (c).*

$$\begin{aligned}\int_{2}^{3.5} \left(x^2 - \frac{2}{x}\right) dx &= \left[\frac{1}{2+1}x^{2+1} - 2\ln x\right]_{x=2}^{x=3.5} \\ &= \left[\frac{1}{3}x^3 - 2\ln x\right]_{x=2}^{x=3.5} \\ &= \left[\frac{1}{3}(3.5)^3 - 2\ln(3.5)\right] - \left[\frac{1}{3}(2)^3 - 2\ln(2)\right] \approx\end{aligned}$$

(i) **8.23** (ii) **9.02** (iii) **10.51**

$Y_5 = x^2 - \frac{2}{x}$ WINDOW -2 5 1 -10 20 2

GRAPH 2nd CALC 7:$\int f(x)\,dx$ ENTER 2 ENTER 3.5 ENTER

OR MATH fnInt(Y_5,X,2,3.5) instead.

4. *Area and definite integrals,* $f(x) = 5x$.

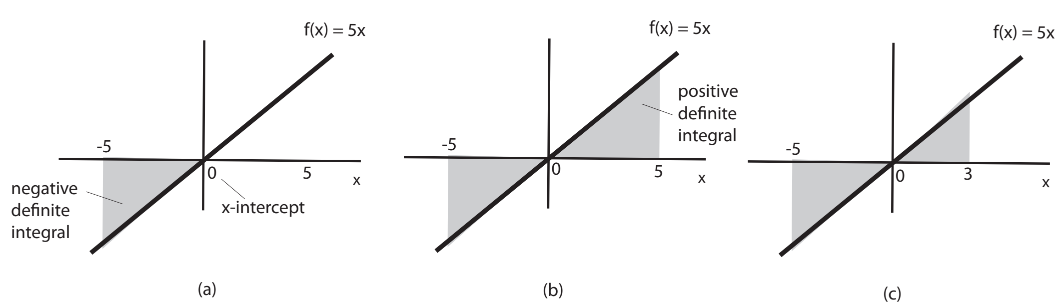


Figure 7.4 ($f(x) = 5x$, area and definite integrals)

(a) $f(x) = 5x,\ [-5, 0]$, *graph (b).*

$$\int_{-5}^{0} 5x\,dx = \left[\frac{5}{2}x\right]_{x=-5}^{x=0} = \frac{5}{2}(0)^2 - \frac{5}{2}(-5)^2 =$$

(i) **62.5** (ii) **−62.5** (iii) **−40**

GRAPH 2nd CALC 7:$\int f(x)\,dx$ ENTER -5 ENTER 0 ENTER

OR MATH fnInt(Y_2,X,-5,0) instead.

If considered *area*,

$$\int_{-5}^{0} 5x\,dx = \left|\left[\frac{5}{2}x\right]_{x=-5}^{x=0}\right| = \left|\frac{5}{2}(0)^2 - \frac{5}{2}(-5)^2\right| =$$

(i) **62.5** (ii) **−62.5** (iii) **−40**

(b) $f(x) = 5x,\ [-5, 5]$, *graph (b).*

$$\int_{-5}^{5} 5x\,dx = \left[\frac{5}{2}x\right]_{x=-5}^{x=5} = \frac{5}{2}(5)^2 - \frac{5}{2}(-5)^2 =$$

(i) **0** (ii) **−62.5** (iii) **−40**

GRAPH 2nd CALC 7:$\int f(x)\,dx$ ENTER -5 ENTER 5ENTER

OR MATH fnInt(Y_2,X,-5,5) instead.

If considered area, split into *two* integrals

$$\begin{aligned}\int_{-5}^{5} 5x\,dx &= \int_{-5}^{0} 5x\,dx + \int_{0}^{5} 5x\,dx \\ &= \left|\left[\frac{5}{2}x\right]_{x=-5}^{x=0}\right| + \left|\left[\frac{5}{2}x\right]_{x=0}^{x=5}\right| \\ &= \left|\frac{5}{2}(0)^2 - \frac{5}{2}(-5)^2\right| + \left|\frac{5}{2}(5)^2 - \frac{5}{2}(0)^2\right| =\end{aligned}$$

(i) **62.5** (ii) **−62.5** (iii) **125**

(c) $f(x) = 5x,\ [-5, 3]$, *graph (c).*

$$\int_{-5}^{3} 5x\,dx = \left[\frac{5}{2}x\right]_{x=-5}^{x=3} = \frac{5}{2}(3)^2 - \frac{5}{2}(-5)^2 =$$

(i) **62.5** (ii) **−62.5** (iii) **−40**

GRAPH 2nd CALC 7:$\int f(x)\,dx$ ENTER -5 ENTER 3 ENTER

OR MATH fnInt(Y_2,X,-5,3) instead.

If considered area, split into two integrals

$$\begin{aligned}\int_{-5}^{3} 5x\,dx &= \int_{-5}^{0} 5x\,dx + \int_{0}^{3} 5x\,dx \\ &= \left|\left[\frac{5}{2}x\right]_{x=-5}^{x=0}\right| + \left|\left[\frac{5}{2}x\right]_{x=0}^{x=3}\right| \\ &= \left|\frac{5}{2}(0)^2 - \frac{5}{2}(-5)^2\right| + \left|\frac{5}{2}(3)^2 - \frac{5}{2}(0)^2\right| =\end{aligned}$$

(i) **25** (ii) **85** (iii) $\mathbf{-40}$

(d) (i) **True** (ii) **False**. If considered *area*, divide the definite integral into positive (regions *above* the x-axis) and negative regions (regions *below* the x-axis), convert the negative region values to positive values, then sum all of the positive values to determine the total area.

5. *More area and definite integrals.*

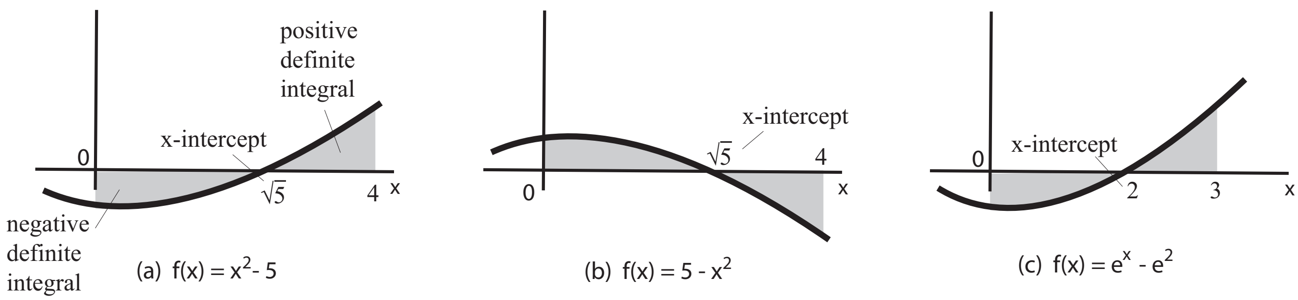


Figure 7.5 (More definite integrals and area)

(a) $f(x) = x^2 - 5$, $[0, 4]$, *graph (a).*

$$\int_0^4 \left(x^2 - 5\right)\, dx = \left[\frac{1}{2+1}x^{2+1} - 5 \cdot \frac{1}{0+1}x^{0+1}\right]_{x=0}^{x=4}$$
$$= \left[\frac{1}{3}(4)^3 - 5(4)\right] - \left[\frac{1}{3}(0)^3 - 5(0)\right] =$$

(i) $-\frac{\mathbf{4}}{\mathbf{3}}$ (ii) $\frac{\mathbf{4}}{\mathbf{3}}$ (iii) $\frac{\mathbf{1}}{\mathbf{3}}$

$Y_6 = x^2 - 5$ WINDOW -2 5 1 -20 20 2

GRAPH 2nd CALC 7:$\int f(x)\, dx$ ENTER 0 ENTER 4 ENTER

OR MATH fnInt(Y_6,X,0,4) instead.

If considered area, split into positive and negative regions;
find x-intercept(s) in [0,4],

$$f(x) = x^2 - 5 = 0$$

so $x =$ (i) $-\sqrt{\mathbf{5}}$ (ii) **0** (iii) $\sqrt{\mathbf{5}}$

since only $\sqrt{5}$ is in [0,4], use only $\sqrt{5}$

and so

$$\int_0^4 \left(x^2 - 5\right)\, dx = \int_0^{\sqrt{5}} \left(x^2 - 5\right)\, dx + \int_{\sqrt{5}}^4 \left(x^2 - 5\right)\, dx$$

$$\begin{aligned}
&= \left[\frac{1}{3}x^3 - 5x\right]_{x=0}^{x=\sqrt{5}} + \left[\frac{1}{3}x^3 - 5x\right]_{x=\sqrt{5}}^{x=4} \\
&= \left|\left[\frac{1}{3}(\sqrt{5})^3 - 5(\sqrt{5})\right] - \left[\frac{1}{3}(0)^3 - 5(0)\right]\right| \\
&\quad + \left|\left[\frac{1}{3}(4)^3 - 5(4)\right] - \left[\frac{1}{3}(\sqrt{5})^3 - 5(\sqrt{5})\right]\right| =
\end{aligned}$$

(i) **16.24** (ii) $\mathbf{\frac{4}{3}}$ (iii) **15.24**

MATH NUM abs(ENTER MATH fnInt(Y_6,X,0,$\sqrt{5}$)
+ abs(ENTER MATH fnInt(Y_6,X,$\sqrt{5}$,4) ENTER.

(b) $f(x) = 5 - x^2$, $[0, 4]$, *graph (b).*

$$\begin{aligned}
\int_0^4 \left(5 - x^2\right)\, dx &= \left[5 \cdot \frac{1}{0+1}x^{0+1} - \frac{1}{2+1}x^{2+1}\right]_{x=0}^{x=4} \\
&= \left[5(4) - \frac{1}{3}(4)^3\right] - \left[5(0) - \frac{1}{3}(0)^3\right] =
\end{aligned}$$

(i) $\mathbf{-\frac{4}{3}}$ (ii) $\mathbf{\frac{4}{3}}$ (iii) $\mathbf{\frac{1}{3}}$

$Y_7 = 5 - x^2$ WINDOW −2 5 1 −20 20 2
GRAPH 2nd CALC 7:$\int f(x)\, dx$ ENTER 0 ENTER 4 ENTER
OR MATH fnInt(Y_7,X,0,4) instead.

If considered area, split into positive and negative regions; find x-intercept(s) in [0,4],

$$f(x) = 5 - x^2 = 0$$

so $x =$ (i) $\mathbf{-\sqrt{5}}$ (ii) **0** (iii) $\mathbf{\sqrt{5}}$

since only $\sqrt{5}$ is in [0,4], use only $\sqrt{5}$

and so

$$\begin{aligned}
\int_0^4 \left(5 - x^2\right)\, dx &= \int_0^{\sqrt{5}} \left(5 - x^2\right)\, dx + \int_{\sqrt{5}}^4 \left(5 - x^2\right)\, dx \\
&= \left[5x - \frac{1}{3}x^3\right]_{x=0}^{x=\sqrt{5}} + \left[5x - \frac{1}{3}x^3\right]_{x=\sqrt{5}}^{x=4} \\
&= \left|\left[5(\sqrt{5}) - \frac{1}{3}(\sqrt{5})^3\right] - \left[5(0) - \frac{1}{3}(0)^3\right]\right| \\
&\quad + \left|\left[5(4) - \frac{1}{3}(4)^3\right] - \left[\frac{1}{3}(5(\sqrt{5}) - \sqrt{5})^3\right]\right| \approx
\end{aligned}$$

(i) **16.24** (ii) $\mathbf{\frac{4}{3}}$ (iii) **15.24**

MATH NUM abs(ENTER MATH fnInt(Y_7,X,0,$\sqrt{5}$)
+ abs(ENTER MATH fnInt(Y_7,X,$\sqrt{5}$,4) ENTER.

(c) $f(x) = e^x - e^2$, $[0, 3]$, *graph (c).*

$$\begin{aligned}\int_0^3 \left(e^x - e^2\right)\, dx &= \left[e^x - e^2 \cdot \frac{1}{0+1} x^{0+1}\right]_{x=0}^{x=3} \\ &= \left[e^x - xe^2\right]_{x=0}^{x=3} \\ &= \left[e^3 - 3e^2\right] - \left[e^0 - 0e^2\right] \approx\end{aligned}$$

(i) $\mathbf{-3.08}$ (ii) $\mathbf{0}$ (iii) $\mathbf{3.08}$

$Y_8 = e^x - e^2$ WINDOW −2 5 1 −20 20 2

GRAPH 2nd CALC 7:$\int f(x)\, dx$ ENTER 0 ENTER 3 ENTER

OR MATH fnInt(Y_8,X,0,3) instead.

If considered area, split into positive and negative regions;
find x-intercept(s) in [0,3],

$$f(x) = e^x - e^2 = 0$$

so $x =$ (i) $\boldsymbol{e}$ (ii) $\mathbf{1}$ (iii) $\mathbf{2}$

and so

$$\begin{aligned}\int_0^3 \left(e^x - e^2\right)\, dx &= \int_0^2 \left(e^x - e^2\right)\, dx + \int_2^3 \left(e^x - e^2\right)\, dx \\ &= \left[e^x - xe^2\right]_{x=0}^{x=2} + \left[e^x - xe^2\right]_{x=2}^{x=3} \\ &= \left|\left[e^2 - 2e^2\right] - \left[e^0 - (0)e^2\right]\right| \\ &\quad + \left|\left[e^3 - 3e^2\right] - \left[e^2 - 2e^2\right]\right| \approx\end{aligned}$$

(i) **12.70** (ii) **13.70** (iii) **14.70**

MATH NUM abs(ENTER MATH fnInt(Y_8,X,0,2)

+ MATH fnInt(Y_8,X,2,3) ENTER.

6. *Substitution and definite integrals, method 1.*

(a) Find $\int_1^4 \frac{1}{2} \frac{3+2x}{\sqrt{3x+x^2}}\, dx = \frac{1}{2} \int_1^4 (3x + x^2)^{-\frac{1}{2}} (3 + 2x)\, dx$.

guess $u =$ (i) $\mathbf{3x + x^2}$ (ii) $\mathbf{3 + 2x}$ (iii) $\mathbf{\sqrt{3x + x^2}}$

then $\frac{du}{dx} = 3(1)x^{1-1} + 2x^{2-1} = 3 + 2x$ or $du = (3 + 2x)\, dx$

but also *limits* $x = [1, 4]$ should also be changed;
when $x = 1$, $u = 3(1) + 1^2 = 4$

and when $x = 4$, $u = 3(4) + 4^2 = 28$

substituting u, du and new limits into $\int f(x)\,dx$,

$$\begin{aligned}\frac{1}{2}\int_1^4 \left(3x + x^2\right)^{-\frac{1}{2}}(3+2x)\,dx &= \frac{1}{2}\int_4^{28} u^{-\frac{1}{2}}\,du \\ &= \frac{1}{2}\left(\frac{1}{-\frac{1}{2}+1}u^{-\frac{1}{2}+1}\right)_{u=4}^{u=28} \\ &= \left(u^{\frac{1}{2}}\right)_{u=4}^{u=28} \\ &= \sqrt{28} - \sqrt{4} \approx\end{aligned}$$

(i) **2.29** (ii) **2.79** (iii) **3.29**

$Y_9 = \frac{1}{2}\frac{3+2x}{\sqrt{3x+x^2}}$ MATH ENTER MATH fnInt(Y9,X,1,4).

(b) Find $\int_1^4 \frac{\ln x}{x}\,dx = \int_1^4 \ln x\,(x^{-1})\,dx$.

guess $u =$ (i) $\boldsymbol{\ln x}$ (ii) $\boldsymbol{x^{-1}}$

then $du =$ (i) $\boldsymbol{(x^{-1})\,dx}$ (ii) $\boldsymbol{(-x^{-2})\,dx}$

but also *limits* $x = [1, 4]$ should also be changed;
when $x = 1$, $u = \ln 1 = 0$
and when $x = 4$, $u = \ln 4$

substituting u, du and new limits into $\int f(x)\,dx$,

$$\begin{aligned}\int_1^4 \ln x\left(x^{-1}\right)\,dx &= \int_0^{\ln 4} u\,du \\ &= \left(\frac{1}{1+1}u^{1+1}\right)_{u=0}^{u=\ln 4} \\ &= \left(\frac{1}{2}u^2\right)_{u=0}^{u=\ln 4} \\ &= \frac{1}{2}(\ln 4)^2 - \frac{1}{2}(0)^2 \approx\end{aligned}$$

(i) **0.36** (ii) **0.66** (iii) **0.96**

$Y_{10} = \frac{\ln x}{x}$ MATH ENTER MATH fnInt(Y10,X,1,4).

7. *Substitution and definite integrals, method 2.*

(a) Find $\int_1^4 \frac{1}{2}\frac{3+2x}{\sqrt{3x+x^2}}\,dx = \frac{1}{2}\int_1^4 (3x+x^2)^{-\frac{1}{2}}(3+2x)\,dx$.

guess $u =$ (i) $\boldsymbol{3x + x^2}$ (ii) $\boldsymbol{3 + 2x}$ (iii) $\boldsymbol{\sqrt{3x + x^2}}$

then $\frac{du}{dx} = 3(1)x^{1-1} + 2x^{2-1} = 3 + 2x$ or $du = (3 + 2x)\,dx$
substituting u and du into $\int f(x)\,dx$,

$$\frac{1}{2}\int \left(3x + x^2\right)^{-\frac{1}{2}}(3 + 2x)\,dx = \frac{1}{2}\int u^{-\frac{1}{2}}\,du = \frac{1}{2}\left(\frac{1}{-\frac{1}{2}+1}u^{-\frac{1}{2}+1} + C\right) =$$

(i) $\boldsymbol{\frac{3}{2} + C}$ (ii) $\boldsymbol{\frac{1}{2}u^{\frac{3}{2}} + C}$ (iii) $\boldsymbol{u^{\frac{1}{2}} + C}$

but $u = 3x + x^2$, so

$$\int f(x)\,dx = u^{\frac{1}{2}} + C =$$

(i) $\boldsymbol{(3 + 2x)^{\frac{3}{2}} + C}$ (ii) $\boldsymbol{(3x + x^2)^{\frac{1}{2}} + C}$ (iii) $\boldsymbol{(3x + x^3)^{\frac{5}{2}} + C}$

and so

$$\begin{aligned} \frac{1}{2}\int_1^4 \left(3x + x^2\right)^{-\frac{1}{2}}(3 + 2x)\,dx &= \left(\left(3x + x^2\right)^{\frac{1}{2}}\right)_{x=1}^{x=4} \\ &= \sqrt{3(4) + (4)^2} - \sqrt{3(1) + (1)^2} \approx \end{aligned}$$

(i) **2.29** (ii) **2.79** (iii) **3.29**

(b) Find $\int_1^4 \frac{\ln x}{x}\,dx = \int_1^4 \ln x\,(x^{-1})\,dx$.

guess $u =$ (i) $\boldsymbol{\ln x}$ (ii) $\boldsymbol{x^{-1}}$

then $du = (x^{-1})\,dx =$ (i) $\boldsymbol{(x^{-1})\,dx}$ (ii) $\boldsymbol{(-x^{-2})\,dx}$

substituting u and du into $\int f(x)\,dx$,

$$\int \ln x\left(x^{-1}\right)\,dx =$$

(i) $\boldsymbol{\int u\,du = \frac{1}{1+1}u^{1+1} + C = \frac{u^2}{2} + C}$
(ii) $\boldsymbol{\int u^{-1}\,du = \ln|u| + C}$
(iii) $\boldsymbol{2\int u\,du = 2\left(\frac{1}{1+1}u^{1+1} + C\right) = u^2 + C}$

but $u = \ln x$, so

$$\int f(x)\,dx = \frac{u^2}{2} + C =$$

(i) $\frac{1}{2}(\ln x)^2 + C$ (ii) $\ln|\ln x| + C$ (iii) $(\ln x)^2 + C$

and so

$$\begin{aligned}\int_1^4 \ln x \left(x^{-1}\right)\, dx &= \left(\frac{1}{2}(\ln x)^2\right)_{x=1}^{x=4} \\ &= \frac{1}{2}(\ln 4)^2 - \frac{1}{2}(\ln 1)^2 \approx\end{aligned}$$

(i) **0.36** (ii) **0.66** (iii) **0.96**

8. *Application: bacterial growth.* A type of bacteria grows at a rate given by

$$w'(t) = e^{\left(7+t^3\right)}\left(3t^2\right)$$

where $w(t)$ is weight after t hours. Determine weight from $t = 0$ to $t = 1$.

Find $\int_0^1 e^{\left(7+t^3\right)}\left(3t^2\right)\, dt$.

guess $u =$ (i) $7 + t^3$ (ii) $3t^2$

then $du = (0 + 3t^{3-1})\, dt =$ (i) $(1 + 3t^2)\, dt$ (ii) $(3t^2)\, dt$

substituting u and du into $\int w'(t)\, dt$,

$$\int e^{\left(7+t^3\right)}\left(3t^2\right)\, dt = \int e^u\, du = e^u + C$$

but $u = 7 + t^3$, so

$$\int w'(t)\, dt = e^u + C =$$

(i) $t^3 e^{7+t^3} + C$ (ii) $7e^{7+t^3} + C$ (iii) $e^{7+t^3} + C$

and so

$$\begin{aligned}\int_0^1 e^{\left(7+t^3\right)}\left(3t^2\right)\, dt &= \left(e^{7+t^3}\right)_{t=0}^{t=1} \\ &= e^{7+1^3} - e^{7+0^3} \approx\end{aligned}$$

(i) **1684.32** (ii) **1784.32** (iii) **1884.32**

7.5 The Area Between Two Curves

The area *between* two functions, $f(x)$ and $g(x)$, where $f(x) \geq g(x)$ on $[a, b]$, is

$$\int_a^b [f(x) - g(x)]\, dx.$$

Two related applications of this, given demand function $D(q)$, supply function $S(q)$, equilibrium price p_0 and equilibrium demand q_0, are

- *consumer's surplus, difference between what consumers would and actually pay,*

$$\int_0^{q_0} [D(q) - p_0]\, dq$$

- *producer's surplus, difference between what producers should and actually receive,*

$$\int_0^{q_0} [p_0 - S(q)]\, dq$$

Exercise 7.5 (The Area Between Two Curves)

1. *Area Between* $f(x) = 3x + 5$ *and* $g(x) = 1 + x^2$

   (a) $f(x) = 3x + 5$, $g(x) = 1 + x^2$ *on [1,3].*

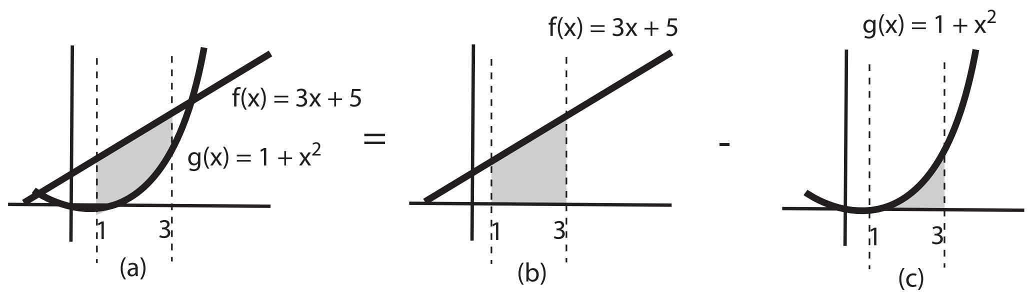


Figure 7.6 ($f(x) = 3x + 5$, $g(x) = 1 + x^2$ on [1,3])

verify $f(x) \geq g(x)$ on [1,3] by graphing both functions,
and noticing $f(x) \geq g(x)$ on [1,3].
$Y_1 = 3x + 5$, $Y_2 = 1 + x^2$, WINDOW -2 5 1 1 30 1 GRAPH.

so area bounded by $f(x) = 3x + 5$ and $g(x) = 1 + x^2$ on [1,3]

$$\begin{aligned}
\int_a^b [f(x) - g(x)]\, dx &= \int_1^3 [(3x + 5) - (1 + x^2)]\, dx \\
&= \int_1^3 (-x^2 + 3x + 4)\, dx \\
&= \left(-\frac{1}{2+1}x^{2+1} - 3 \cdot \frac{1}{1+1}x^{1+1} + 4 \cdot \frac{1}{0+1}x^{0+1}\right)_{x=1}^{x=3} \\
&= \left(-\frac{1}{3}x^3 + \frac{3}{2}x^2 + 4x\right)_{x=1}^{x=3} =
\end{aligned}$$

(i) $\mathbf{\frac{34}{3}}$ (ii) $\mathbf{\frac{35}{3}}$ (iii) $\mathbf{\frac{36}{3}}$
MATH ENTER MATH fnInt(Y_1 - Y_2,X,1,3), then MATH ENTER for fraction.

(b) *$f(x) = 3x + 5$, $g(x) = 1 + x^2$ on [-1,4].*

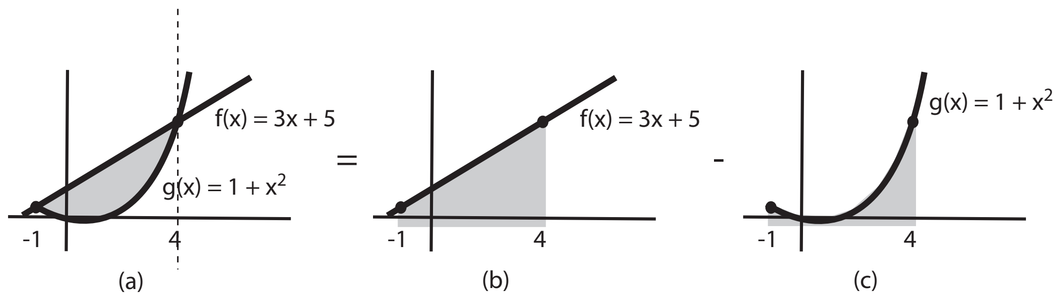


Figure 7.7 ($f(x) = 3x + 5$, $g(x) = 1 + x^2$ on [-1,4])

verify $f(x) \geq g(x)$ on [-1,4] by graphing both functions,
and noticing $f(x) \geq g(x)$ on [-1,4].
$Y_1 = 3x + 5$, $Y_2 = 1 + x^2$, WINDOW -2 5 1 1 30 1 GRAPH.

and since

$$\begin{aligned} f(x) &= g(x) \\ 3x + 5 &= 1 + x^2 \\ x^2 - 3x - 4 &= 0 \\ (x - 4)(x + 1) &= 0 \end{aligned}$$

intersections occurs at $x =$ (choose two!) (i) $\mathbf{-1}$ (ii) $\mathbf{0}$ (iii) $\mathbf{4}$

so area bounded by $f(x) = 3x + 5$ and $g(x) = 1 + x^2$ on [-1,4]

$$\begin{aligned} \int_a^b [f(x) - g(x)]\, dx &= \int_{-1}^4 [(3x + 5) - (1 + x^2)]\, dx \\ &= \int_{-1}^4 (-x^2 + 3x + 4)\, dx \\ &= \left(-\frac{1}{3}x^3 + \frac{3}{2}x^2 + 4x \right)_{x=-1}^{x=4} = \end{aligned}$$

(i) $\mathbf{\frac{123}{6}}$ (ii) $\mathbf{\frac{125}{6}}$ (iii) $\mathbf{\frac{127}{6}}$
MATH ENTER MATH fnInt(Y_1 - Y_2,X,-1,4), then MATH ENTER for fraction.

(c) *$f(x) = 3x + 5$, $g(x) = 1 + x^2$ on [1,5].*

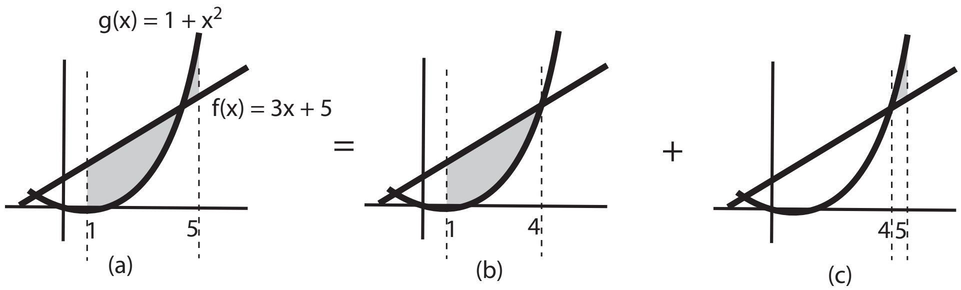


Figure 7.8 ($f(x) = 3x + 5$, $g(x) = 1 + x^2$ on [1,5])

by graphing both functions,
notice $f(x) \geq g(x)$ on [1,4] but $g(x) \geq f(x)$ on [4,5]
$Y_1 = 3x + 5$, $Y_2 = 1 + x^2$, WINDOW -2 5 1

so area bounded by $f(x) = 3x + 5$ and $g(x) = 1 + x^2$ on [1,4]

$$\begin{aligned}\int_a^b [f(x) - g(x)]\,dx &= \int_1^4 [(3x+5) - (1+x^2)]\,dx \\ &= \int_1^4 (-x^2 + 3x + 4)\,dx \\ &= \left(-\frac{1}{3}x^3 + \frac{3}{2}x^2 + 4x\right)_{x=1}^{x=4} =\end{aligned}$$

(i) $\mathbf{\frac{25}{2}}$ (ii) $\mathbf{\frac{26}{2}}$ (iii) $\mathbf{\frac{27}{2}}$
MATH ENTER MATH fnInt(Y_1 - Y_2,X,1,4), then MATH ENTER for fraction.

and area bounded by $f(x) = 3x + 5$ and $g(x) = 1 + x^2$ on [4,5]

$$\begin{aligned}\int_a^b [f(x) - g(x)]\,dx &= \left|\int_4^5 [(3x+5) - (1+x^2)]\,dx\right| \\ &= \left|\int_4^5 (-x^2 + 3x + 4)\,dx\right| \\ &= \left|\left(-\frac{1}{3}x^3 + \frac{3}{2}x^2 + 4x\right)_{x=4}^{x=5}\right| =\end{aligned}$$

(i) $\mathbf{\frac{16}{6}}$ (ii) $\mathbf{\frac{17}{6}}$ (iii) $\mathbf{\frac{18}{6}}$
MATH abs ENTER MATH fnInt(Y_1 - Y_2,X,4,5), then MATH ENTER for fraction.

so total area is $\frac{27}{2} + \frac{17}{6} =$ (i) $\mathbf{\frac{49}{3}}$ (ii) $\mathbf{\frac{50}{3}}$ (iii) $\mathbf{\frac{51}{3}}$

2. *Area Between* $f(x) = -x^2 + 6x + 5$ *and* $g(x) = e^x$.

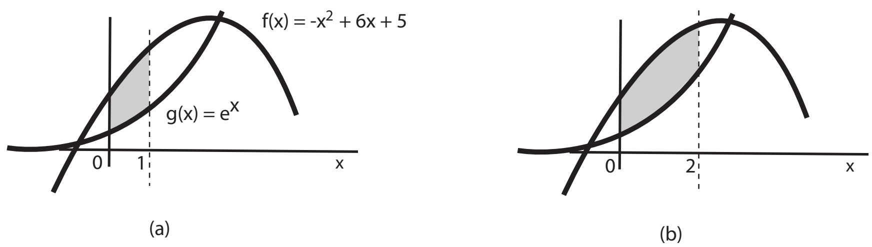


Figure 7.9 (Area Between $f(x) = -x^2 + 6x + 5$ and $g(x) = e^x$)

$\text{Y}_1 = -x^2 + 6x + 5$, $\text{Y}_2 = e^x$, WINDOW −2 4 1 -2 15 1 GRAPH.

(a) *$f(x) = -x^2 + 6x + 5$, $g(x) = e^x$ on [0,1], figure (a).*

verify $f(x) \geq g(x)$ on [0,1] by graphing both functions,
and noticing, on [0,1] (i) $\boldsymbol{f(x) \geq g(x)}$ (ii) $\boldsymbol{g(x) \geq f(x)}$

so area bounded by [0,1]

$$\begin{aligned}
\int_a^b [f(x) - g(x)]\, dx &= \int_0^1 [(-x^2 + 6x + 5) - (e^x)]\, dx \\
&= \int_0^1 (-x^2 + 6x + 5 - e^x)\, dx \\
&= \left(-\frac{1}{2+1}x^{2+1} + 6 \cdot \frac{1}{1+1}x^{1+1} + 5 \cdot \frac{1}{0+1}x^{0+1} - e^x\right)_{x=0}^{x=1} \\
&= \left(-\frac{1}{3}x^3 + 3x^2 + 5x - e^x\right)_{x=0}^{x=1} =
\end{aligned}$$

(i) **5.95** (ii) **6.05** (iii) **6.15**
MATH ENTER MATH fnInt(Y_1 - Y_2,X,0,1).

(b) *$f(x) = -x^2 + 6x + 5$, $g(x) = e^x$ on [0,2], figure (b).*

verify $f(x) \geq g(x)$ on [0,2] by graphing both functions,
and noticing, on [0,2] (i) $\boldsymbol{f(x) \geq g(x)}$ (ii) $\boldsymbol{g(x) \geq f(x)}$

so area bounded by [0,2]

$$\begin{aligned}
\int_a^b [f(x) - g(x)]\, dx &= \int_0^2 [(-x^2 + 6x + 5) - (e^x)]\, dx \\
&= \int_0^2 (-x^2 + 6x + 5 - e^x)\, dx \\
&= \left(-\frac{1}{3}x^3 + 3x^2 + 5x - e^x\right)_{x=0}^{x=2} =
\end{aligned}$$

(i) **10.94** (ii) **11.94** (iii) **12.94**
MATH ENTER MATH fnInt(Y_1 - Y_2,X,0,2).

3. *Consumers' surplus and producers' surplus: vacuum cleaners.*

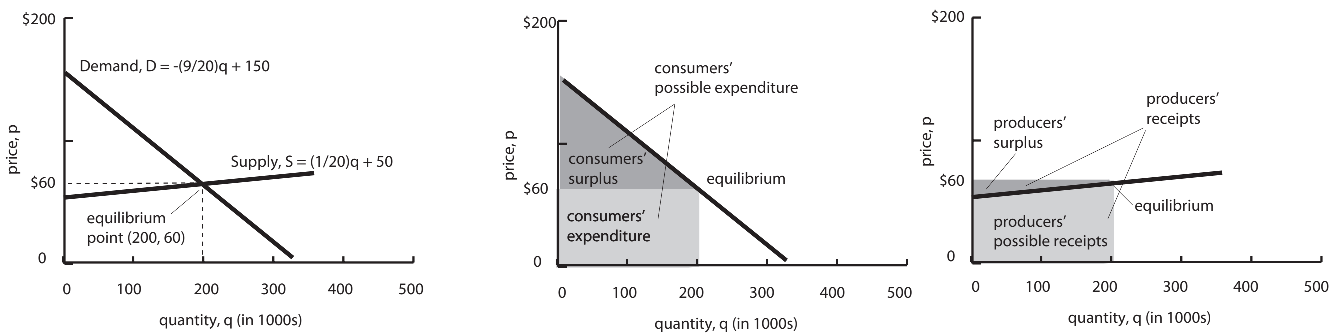


Figure 7.10 (Consumers' surplus and producers' surplus for vacuum cleaners)

(a) *Equilibrium point.*

Supply function for vacuum cleaners is
(i) $\boldsymbol{p = S(q) = \frac{1}{20}q + 50}$
(ii) $\boldsymbol{p = D(q) = -\frac{9}{20}q + 150}$
sellers increase *supply*, produce more quantity q, if price p increases

Demand function for vacuum cleaners is
(i) $\boldsymbol{p = S(q) = \frac{1}{20}q + 50}$
(ii) $\boldsymbol{p = D(q) = -\frac{9}{20}q + 150}$
buyers *de*crease *demand*, buy less quantity q, if price p increases

Equilibrium occurs at intersection of supply and demand

$$\frac{1}{20}q + 50 = -\frac{9}{20}q + 150,$$

so $\frac{10}{20}q = 100$ and $q = \frac{100}{0.5} =$ (i) **100** (ii) **200** (iii) **300** (1000s) units
where $p = \left(\frac{1}{20}\right)(200) + 50 =$ (i) **$50** (ii) **$60**
so equilibrium is (i) $\mathbf{(200, \$50)}$ (ii) $\mathbf{(200, \$60)}$
Enter $Y_1 = \left(\frac{1}{20}\right)x + 50$ and $Y_2 = -\left(\frac{9}{20}\right)x + 150$, WINDOW, set 0, 500, 50, 0, 200, 50, 1, GRAPH
Determine intersection: 2nd CALC, intersect, ENTER to First curve? and ENTER to Second curve?, arrow close to intersection, ENTER, and intersection is X = 200, Y = 60.

(b) *Consumers' surplus.*

Consumers' possible expenditure at equilibrium is

$$\begin{aligned}
\int_0^{q_0} D(q)\,dq &= \int_0^{200} \left(-\frac{9}{20}q + 150\right)\,dq \\
&= \left(-\frac{9}{20}\cdot\frac{1}{1+1}q^{1+1} + 150\cdot\frac{1}{0+1}q^{0+1}\right)_{q=0}^{q=200} \\
&= \left(-\frac{9}{40}q^2 + 150q\right)_{q=0}^{q=200} \\
&= \left(-\frac{9}{40}(200)^2 + 150(200)\right) - \left(-\frac{9}{40}(0)^2 + 150(0)\right) =
\end{aligned}$$

(i) **9000** (ii) **12000** (iii) **21000**
what consumers *would* pay at equilibrium price $p_0 = \$60$

Consumers' expenditure at equilibrium is

$$\begin{aligned}
\int_0^{q_0} p_0\,dq &= \int_0^{200} 60\,dq \\
&= \left(60\cdot\frac{1}{0+1}q^{0+1}\right)_{q=0}^{q=200} \\
&= (60q)_{q=0}^{q=200} \\
&= (60(200)) - (60(0)) =
\end{aligned}$$

(i) **9000** (ii) **12000** (iii) **21000**
what consumers *actually* pay at equilibrium price $p_0 = \$60$

Consumers' surplus at equilibrium is

$$\begin{aligned}
\int_0^{q_0} [D(q) - p_0]\,dq &= \int_0^{200} \left[\left(-\frac{9}{20}q + 150\right) - 60\right]\,dq \\
&= \int_0^{200} \left[-\frac{9}{20}q + 90\right]\,dq \\
&= \left[-\frac{9}{20}\cdot\frac{1}{1+1}q^{1+1} + 90\cdot\frac{1}{0+1}q^{0+1}\right]_{q=0}^{q=200} \\
&= \left[-\frac{9}{40}q^2 + 90q\right]_{q=0}^{q=200} \\
&= \left[-\frac{9}{40}(200)^2 + 90(200)\right] - \left[-\frac{9}{40}(0)^2 + 90(0)\right] =
\end{aligned}$$

(i) **9000** (ii) **12000** (iii) **21000**
difference between what consumers would pay and actually pay

(c) *Producers' surplus.*

Producers' possible receipts at equilibrium is

$$\begin{aligned}\int_0^{q_0} S(q)\,dq &= \int_0^{200}\left(\frac{1}{20}x+50\right)\,dq \\ &= \left(\frac{1}{20}\cdot\frac{1}{1+1}q^{1+1}+50\cdot\frac{1}{0+1}q^{0+1}\right)_{q=0}^{q=200} \\ &= \left(\frac{1}{40}q^2+50q\right)_{q=0}^{q=200} \\ &= \left(\frac{1}{40}(200)^2+50(200)\right)-\left(\frac{1}{40}(0)^2+50(0)\right)=\end{aligned}$$

(i) **1000** (ii) **11000** (iii) **12000**
what producers *should* receive at equilibrium price $p_0 = \$60$

Producers' expenditure at equilibrium is

$$\begin{aligned}\int_0^{q_0} p_0\,dq &= \int_0^{200} 60\,dq \\ &= \left(60\cdot\frac{1}{0+1}q^{0+1}\right)_{q=0}^{q=200} \\ &= (60q)_{q=0}^{q=200} \\ &= (60(200))-(60(0))=\end{aligned}$$

(i) **1000** (ii) **11000** (iii) **12000**
what producers *actually* receive at equilibrium price $p_0 = \$60$
which, notice, is *more* than what they "should" receive

Producers' surplus at equilibrium is

$$\begin{aligned}\int_0^{q_0}[p_0-S(q)]\,dq &= \int_0^{200}\left[60-\left(\frac{1}{20}x+50\right)\right]\,dq \\ &= \int_0^{200}\left[-\frac{1}{20}x+10\right]\,dq \\ &= \left[-\frac{1}{20}\cdot\frac{1}{1+1}q^{1+1}+10\cdot\frac{1}{0+1}q^{0+1}\right]_{q=0}^{q=200} \\ &= \left[-\frac{1}{40}q^2+10q\right]_{q=0}^{q=200} \\ &= \left[-\frac{1}{40}(200)^2+10(200)\right]-\left[-\frac{1}{40}(0)^2+10(0)\right]=\end{aligned}$$

(i) **1000** (ii) **11000** (iii) **12000**
difference between what producers should receive and actually receive

4. *Another Example:* $D(q) = (q-6)^2 = q^2 - 12q + 36$ *and* $S(q) = q^2 + 6q$.

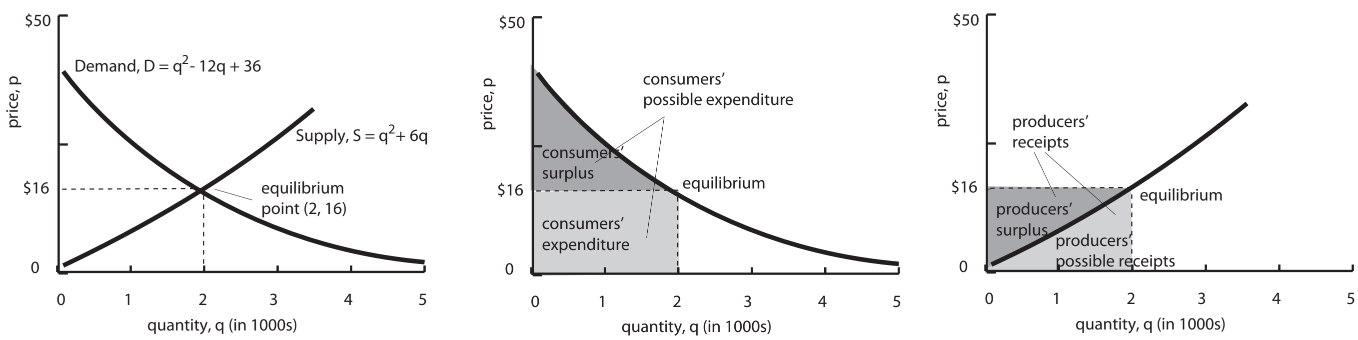


Figure 7.11 (Consumers' surplus and producers' surplus)

(a) *Equilibrium point.*

Equilibrium occurs at intersection of supply and demand

$$q^2 + 6q = q^2 - 12q + 36,$$

so $18q = 36$ and $q = \frac{36}{18} =$ (i) **1** (ii) **2** (iii) **3** (1000s) units
where $p = q^2 + 6q = 2^2 + 6(2) =$ (i) **\$10** (ii) **\$16**
so equilibrium is (i) $\mathbf{(1, \$7)}$ (ii) $\mathbf{(2, \$16)}$

Enter $Y_1 = x^2 - 12x + 36$ and $Y_2 = x^2 + 6x$, WINDOW, set 0, 5, 1, 0, 50, 10, 1, GRAPH
Determine intersection: 2nd CALC, intersect, ENTER to First curve? and ENTER to Second curve?, arrow close to intersection, ENTER, and intersection is X = 2, Y = 16.

(b) *Consumers' surplus.*

Consumers' possible expenditure at equilibrium is

$$\begin{aligned}
\int_0^{q_0} D(q)\, dq &= \int_0^2 \left(q^2 - 12q + 36\right) dq \\
&= \left(\frac{1}{2+1}q^{2+1} - 12 \cdot \frac{1}{1+1}q^{1+1} + 36 \cdot \frac{1}{0+1}q^{0+1}\right)_{q=0}^{q=2} \\
&= \left(\frac{1}{3}q^3 - 6q^2 + 36q\right)_{q=0}^{q=2} \\
&= \left(\frac{1}{3}(2)^3 - 6(2)^2 + 36(2)\right) - \left(\frac{1}{3}(0)^3 - 6(0)^2 + 36(0)\right) =
\end{aligned}$$

(i) $\mathbf{\frac{56}{3}}$ (ii) **32** (iii) $\mathbf{\frac{152}{3}}$
what consumers *would* pay at equilibrium price $p_0 = \$16$

Consumers' expenditure at equilibrium is

$$\begin{aligned}\int_0^{q_0} p_0 \, dq &= \int_0^2 16 \, dq \\ &= \left(16 \cdot \frac{1}{0+1} q^{0+1}\right)_{q=0}^{q=2} \\ &= (16q)_{q=0}^{q=2} \\ &= (16(2)) - (16(0)) =\end{aligned}$$

(i) $\mathbf{\frac{56}{3}}$ (ii) **32** (iii) $\mathbf{\frac{152}{3}}$
what consumers *actually* pay at equilibrium price $p_0 = \$60$

Consumers' surplus at equilibrium is

$$\begin{aligned}\int_0^{q_0} [D(q) - p_0] \, dq &= \int_0^2 \left[\left(q^2 - 12q + 36\right) - 6\right] \, dq \\ &= \left[\frac{1}{2+1} q^{2+1} - 12 \cdot \frac{1}{1+1} q^{1+1} + 30 \cdot \frac{1}{0+1} q^{0+1}\right]_{q=0}^{q=2} \\ &= \left(\frac{1}{3} q^3 - 6q^2 + 30q\right)_{q=0}^{q=2} \\ &= \left(\frac{1}{3}(2)^3 - 6(2)^2 + 30(2)\right) - \left(\frac{1}{3}(0)^3 - 6(0)^2 + 30(0)\right) =\end{aligned}$$

(i) $\mathbf{\frac{56}{3}}$ (ii) **32** (iii) $\mathbf{\frac{152}{3}}$
difference between what consumers would pay and actually pay

(c) *Producers' surplus.*

Producers' possible receipts at equilibrium is

$$\begin{aligned}\int_0^{q_0} S(q) \, dq &= \int_0^2 \left(q^2 + 6q\right) \, dq \\ &= \left(\frac{1}{2+1} q^{2+1} + 6 \cdot \frac{1}{1+1} q^{1+1}\right)_{q=0}^{q=2} \\ &= \left(\frac{1}{3} q^3 + 3q^2\right)_{q=0}^{q=2} \\ &= \left(\frac{1}{3}(2)^3 + 3(2)^2\right) - \left(\frac{1}{3}(0)^3 + 3(0)^2\right) =\end{aligned}$$

(i) $\mathbf{\frac{44}{3}}$ (ii) $\mathbf{\frac{52}{3}}$ (iii) **32**
what producers *should* receive at equilibrium price $p_0 = \$16$

Producers' expenditure at equilibrium is

$$\begin{aligned}\int_0^{q_0} p_0\,dq &= \int_0^2 16\,dq \\ &= \left(16 \cdot \frac{1}{0+1} q^{0+1}\right)_{q=0}^{q=2} \\ &= (16q)_{q=0}^{q=2} \\ &= (16(2)) - (16(0)) = \end{aligned}$$

(i) $\mathbf{\frac{44}{3}}$ (ii) $\mathbf{\frac{52}{3}}$ (iii) **32**
what producers *actually* receive at equilibrium price $p_0 = \$16$
which, notice, is *more* than what they "should" receive

Producers' surplus at equilibrium is

$$\begin{aligned}\int_0^{q_0} [p_0 - S(q)]\,dq &= \int_0^2 \left[16 - \left(q^2 + 6q\right)\right]\,dq \\ &= \int_0^2 \left[-q^2 - 6q + 16\right]\,dq \\ &= \left[-\frac{1}{2+1}q^{2+1} - 6 \cdot \frac{1}{1+1}q^{1+1} + 16 \cdot \frac{1}{0+1}q^{0+1}\right]_{q=0}^{q=2} \\ &= \left[-\frac{1}{3}q^3 - 3q^2 + 16q\right]_{q=0}^{q=2} \\ &= \left[-\frac{1}{3}(2)^3 - 3(2)^2 + 16(2)\right] - \left[-\frac{1}{3}(0)^3 - 3(0)^2 + 16(0)\right] = \end{aligned}$$

(i) $\mathbf{\frac{44}{3}}$ (ii) $\mathbf{\frac{52}{3}}$ (iii) **32**
difference between what producers should receive and actually receive

7.6 Numerical Integration

Not covered.