2.7 Conditional Probability on the Independence of Events

Often we know that a particular event, event B say, is "known to have occurred". We can use this information to upgrade our knowledge about the probability of some other event, event A, say. More specifically, we calculate the probability of event A, conditional on the occurrence of event B. This probability is denoted $P(A|B)$. For any two events A and B with $P(B) > 0$, the *conditional* probability of A given that B has occurred is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Furthermore, two events are *independent* if any one of the following is true:

$$\begin{aligned} P(A \cap B) &= P(A)P(B), \\ P(A|B) &= P(A), \\ P(B|A) &= P(B). \end{aligned}$$

Exercise 2.7 (Conditional Probability on the Independence of Events)

1. *Conditional probability versus unconditional probability: Interviews.* Seven candidates, three of which are females, are being interviewed for a job. The three female's names are Kathy, Susan and Jamie; the four male's names are Tom, Tim, Tyler and Toothy. Assume the candidates are chosen at random for a job interview.

   (a) The chance that Tom is chosen *given that* a male is chosen, $\frac{1}{4}$, is an example of (i) **conditional probability** (ii) **unconditional probability**.

   (b) The chance Tom is chosen *given that* a female is chosen, $\frac{0}{3}$, is an example of (i) **conditional probability** (ii) **unconditional probability**.

   (c) The chance Tom is chosen, $\frac{1}{7}$, is an example of
   (i) **conditional probability** (ii) **unconditional probability**.

   (d) The chance Tom is *not* chosen, $\frac{6}{7}$, is an example of
   (i) **conditional probability** (ii) **unconditional probability**.

   (e) Conditional probability essentially involves taking a subset, defined by the conditional event, of the original sample space and then calculating the probability within this subset.
   (i) **True** (ii) **False**

2. *Using conditional probability formula: coins.* Consider the box of coins in Figure 2.12. A coin is chosen at random.

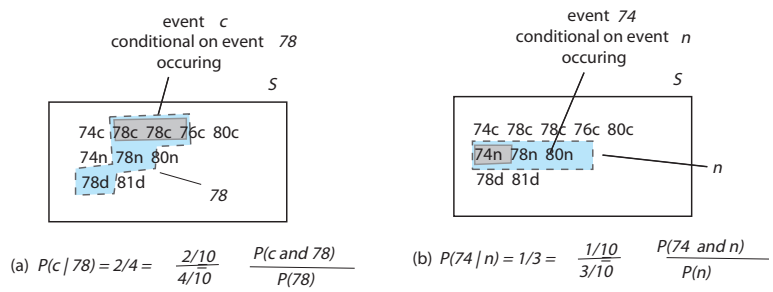

Figure 2.12: Conditional probability: coins

(a) *Calculating* $P(C|78)$.
The probability of picking a 1978 out of the box is
$P(78) =$ (circle one) (i) $\mathbf{\frac{3}{10}}$ (ii) $\mathbf{\frac{4}{10}}$ (iii) $\mathbf{\frac{6}{10}}$ (iv) $\mathbf{\frac{7}{10}}$.
The probability of picking out a coin which is both a cent and a 1978, is
$P(C \cap 78) =$ (circle one) (i) $\mathbf{\frac{2}{10}}$ (ii) $\mathbf{\frac{4}{10}}$ (iii) $\mathbf{\frac{6}{10}}$ (iv) $\mathbf{\frac{7}{10}}$.
The probability of picking out a cent *given that* it is a 1978 coin, is
$P(C|78) = \frac{P(C \cap 78)}{P(78)} = \frac{2/10}{4/10} =$ (circle one) (i) $\mathbf{\frac{2}{4}}$ (ii) $\mathbf{\frac{2}{5}}$ (iii) $\mathbf{\frac{6}{10}}$ (iv) $\mathbf{\frac{7}{10}}$.

(b) *Calculating* $P(74|N)$.
$P(N) =$ (circle one) (i) $\mathbf{\frac{3}{10}}$ (ii) $\mathbf{\frac{4}{10}}$ (iii) $\mathbf{\frac{6}{10}}$ (iv) $\mathbf{\frac{7}{10}}$.
$P(74 \cap N) =$ (circle one) (i) $\mathbf{\frac{1}{10}}$ (ii) $\mathbf{\frac{2}{10}}$ (iii) $\mathbf{\frac{5}{10}}$ (iv) $\mathbf{\frac{7}{10}}$.
$P(74|N) = \frac{P(74 \cap N)}{P(N)} = \frac{1/10}{3/10} =$ (circle one) (i) $\mathbf{\frac{1}{2}}$ (ii) $\mathbf{\frac{1}{3}}$ (iii) $\mathbf{\frac{2}{4}}$ (iv) $\mathbf{\frac{3}{4}}$.

(c) $P(78|N) = \frac{P(78 \cap N)}{P(N)} = \frac{1/10}{3/10} =$ (circle one) (i) $\mathbf{\frac{0}{3}}$ (ii) $\mathbf{\frac{1}{3}}$ (iii) $\mathbf{\frac{2}{3}}$ (iv) $\mathbf{\frac{3}{3}}$.

(d) $P(C|\bar{80}) = \frac{P(C \cap \bar{80})}{P(\bar{80})} = \frac{4/10}{8/10} =$ (circle one) (i) $\mathbf{\frac{1}{5}}$ (ii) $\mathbf{\frac{2}{5}}$ (iii) $\mathbf{\frac{4}{8}}$ (iv) $\mathbf{\frac{7}{10}}$.

3. *Independence (sampling with replacement)*
versus dependence (sampling without replacement): Box of Tickets.
Two things are *independent* if the chances for the second given the first are the same, no matter how the first turns out; otherwise, the two things are *dependent*. Consider the following box with six tickets.

$$\boxed{1_a}\ \boxed{2_a}\ \boxed{1_b}\ \boxed{3_b}\ \boxed{2_c}\ \boxed{3_c}$$

(a) Two tickets are drawn at random from the box. The first ticket drawn is *not* replaced in the box; that is, there are only five tickets remaining in the box when the second ticket is drawn. The chance the second ticket is a "2" *given that* the first ticket is a "2" is (circle one) (i) $\mathbf{\frac{1}{5}}$ (ii) $\mathbf{\frac{2}{5}}$ (iii) $\mathbf{\frac{3}{5}}$.

(b) Two tickets are sampled with*out* replacement at random from the box. That is, there are only five tickets remaining in the box when the second

ticket is drawn. The chance the second ticket is a "2" *given that* the first ticket is a "1" is (circle one) (i) $\frac{1}{5}$ (ii) $\frac{2}{5}$ (iii) $\frac{3}{5}$.

(c) When sampling at random with*out* replacement, the chance the second ticket of two drawn from the box is any given number will *depend* on what number was drawn on the first ticket.
(i) **True** (ii) **False**

(d) Two tickets are sampled *with* replacement at random from the box. That is, all *six* tickets remain in the box when the second ticket is drawn. The chance the second ticket is a "2" *given that* the first ticket is a "1" is (circle one) (i) $\frac{1}{6}$ (ii) $\frac{2}{6}$ (iii) $\frac{3}{6}$.

(e) When sampling at random with replacement, the chance the second ticket of two drawn from the box is "2", no matter what the first ticket is, will *always* be (circle one) (i) $\frac{1}{6}$ (ii) $\frac{2}{6}$ (iii) $\frac{3}{6}$.

(f) When sampling at random with replacement, the draws are independent of one another; without replacement, the draws are dependent.
(i) **True** (ii) **False**

4. *Independence versus dependence: fathers, sons and college.* The data from a sample of 80 families in a midwestern city gives the record of college attendance by fathers (F) and their oldest sons (S). A family is chosen at random.

	son attended college	son did not attend college	
father attended college	18	7	25
father did not attend college	22	33	55
	40	40	80

(a) *One way to demonstrate independence or dependence.*
The probability a son attended college, is $P(S) =$ (circle one)
(i) $\frac{18}{25}$ (ii) $\frac{18}{40}$ (iii) $\frac{25}{80}$ (iv) $\frac{40}{80}$.
The probability a father attended college, is $P(F) =$ (circle one)
(i) $\frac{18}{25}$ (ii) $\frac{18}{40}$ (iii) $\frac{25}{80}$ (iv) $\frac{40}{80}$.
The probability a son *and* a father both attended college is $P(S \cap F) =$
(i) $\frac{18}{25}$ (ii) $\frac{18}{80}$ (iii) $\frac{25}{80}$ (iv) $\frac{40}{80}$.
Since $P(S \cap F) = \frac{18}{80} \neq P(S) \times P(F) = \left(\frac{40}{80}\right) \times \left(\frac{25}{80}\right)$, the events "son attends college" and "father attends college" are (choose one)
(i) **independent** (ii) **dependent**.

(b) Events A and B are *in*dependent if and only if
$P(A \text{ and } B) = P(A) \cdot P(B)$.
Events A and B are dependent if and only if
$P(A \text{ and } B) \neq P(A) \cdot P(B)$.
(i) **True** (ii) **False**

(c) *Another way to demonstrate independence or dependence.*
$P(S) =$ (circle one) (i) $\mathbf{\frac{18}{25}}$ (ii) $\mathbf{\frac{18}{40}}$ (iii) $\mathbf{\frac{25}{80}}$ (iv) $\mathbf{\frac{40}{80}}$,
$P(S|F) =$ (circle one) (i) $\mathbf{\frac{18}{25}}$ (ii) $\mathbf{\frac{18}{40}}$ (iii) $\mathbf{\frac{25}{80}}$ (iv) $\mathbf{\frac{40}{80}}$.
Since $P(S|F) = \frac{18}{25} \neq P(S) = \frac{40}{80}$, the events "son attends college" and "father attends college" are (choose one)
(i) **independent** (ii) **dependent**.

(d) *A third way to demonstrate independence or dependence.*
$P(F) =$ (circle one) (i) $\mathbf{\frac{18}{25}}$ (ii) $\mathbf{\frac{18}{40}}$ (iii) $\mathbf{\frac{25}{80}}$ (iv) $\mathbf{\frac{40}{80}}$,
$P(F|S) =$ (circle one) (i) $\mathbf{\frac{18}{25}}$ (ii) $\mathbf{\frac{18}{40}}$ (iii) $\mathbf{\frac{25}{80}}$ (iv) $\mathbf{\frac{40}{80}}$.
Since $P(F|S) = \frac{18}{40} \neq P(F) = \frac{25}{80}$, the events "son attends college" and "father attends college" are (choose one)
(i) **independent** (ii) **dependent**.

2.8 Two Laws of Probability

The *multiplicative law of probability* for two events is given by

$$P(A \cap B) = P(B)P(A|B),$$

and, more generally, for n events,

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_n|A_1 \cap \cdots \cap A_{n-1}),$$

and if events $A_1, \ldots, A_n$ are independent, then, for every subset $A_{i_1}, \ldots, A_{i_r}$ of them,

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_r}).$$

The *additive law of probability* for two events is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B),$$

and, for three events, it is

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

Finally,

$$P(A) = 1 - P(\bar{A}).$$

Exercise 2.8 (Two Laws of Probability)

1. *Multiplicative law: cards.* Three cards are taken out of the deck at random. Let A_i represent the event the ith card is taken from the deck.

   (a) The chance the first card dealt is an ace is
   $P(A_1) =$ (circle one) (i) $\mathbf{\frac{1}{51}}$ (ii) $\mathbf{\frac{4}{51}}$ (iii) $\mathbf{\frac{3}{52}}$ (iv) $\mathbf{\frac{4}{52}}$.

(b) The chance the *second* card dealt is a jack, given the first card dealt is an ace, is $P(A_2|A_1) =$ (circle one)
(i) $\mathbf{\frac{1}{51}}$ (ii) $\mathbf{\frac{4}{51}}$ (iii) $\mathbf{\frac{3}{52}}$ (iv) $\mathbf{\frac{4}{52}}$.

(c) The probability the first card is an ace *and* the second card is a jack is[11]
$P(A_1 \cap A_2) = P(A_1)P(A_2|A_1) =$ (circle one)
(i) $\mathbf{\frac{1}{51}}$ (ii) $\mathbf{\frac{1}{52} \times \frac{3}{51}}$ (iii) $\mathbf{\frac{4}{52} \times \frac{4}{51}}$ (iv) $\mathbf{\frac{4}{52} \times \frac{4}{52}}$.

(d) The probability that the *third* card dealt is a jack, conditional on the first two cards dealt are a jack and an ace, is $P(A_3|A_1 \cap A_2) =$ (circle one)
(i) $\mathbf{\frac{3}{50}}$ (ii) $\mathbf{\frac{4}{50}}$ (iii) $\mathbf{\frac{4}{51}}$ (iv) $\mathbf{\frac{4}{52}}$.

(e) The probability that the first card is an ace *and* the second card is a jack *and* the third card is another jack is
$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) =$ (circle one)

i. $\frac{4}{50} \times \frac{3}{49} \times \frac{2}{48}$
ii. $\frac{4}{52} \times \frac{3}{51} \times \frac{2}{50}$
iii. $\frac{4}{52} \times \frac{4}{51} \times \frac{3}{50}$
iv. $\frac{4}{52} \times \frac{4}{51} \times \frac{4}{50}$

2. *Additive law for two events: dice.* As shown in Figure 2.13, in two rolls of fair die, let event A be the "sum of dice is five". Let event B be the event "no fours are rolled".

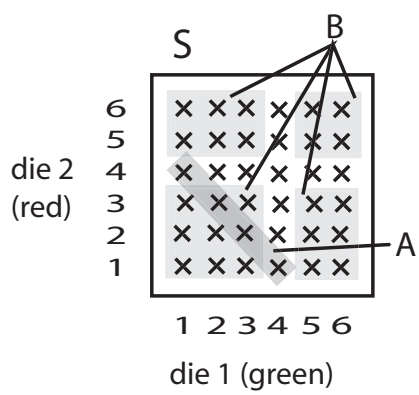

Figure 2.13: Venn diagram for tossing two dice

(a) $P(A) =$ (circle one) (i) $\mathbf{\frac{1}{36}}$ (ii) $\mathbf{\frac{2}{36}}$ (iii) $\mathbf{\frac{3}{36}}$ (iv) $\mathbf{\frac{4}{36}}$.

(b) $P(B) =$ (circle one) (i) $\mathbf{\frac{24}{36}}$ (ii) $\mathbf{\frac{25}{36}}$ (iii) $\mathbf{\frac{26}{36}}$ (iv) $\mathbf{\frac{27}{36}}$.

[11] Although both $P(A_1 \cap A_2) = P(A_2)P(A_1|A_2)$ and $P(A_1 \cap A_2) = P(A_1)P(A_2|A_1)$ are technically correct, it makes sense to use the second case here.

(c) $P(A \cap B) =$ (circle one) (i) $\frac{1}{36}$ (ii) $\frac{2}{36}$ (iii) $\frac{3}{36}$ (iv) $\frac{4}{36}$.

(d) So $P(A \cup B) = P(A) + P(B) - P(A \cap B) =$ (circle one)
(i) $\frac{26}{36}$ (ii) $\frac{27}{36}$ (iii) $\frac{28}{36}$ (iv) $\frac{29}{36}$.

(e) Event A and event B (circle one)
(i) **are mutually exclusive / are not mutually exclusive**.

3. *Additive law for three events: dice.* As shown in Figure 2.14, in two rolls of fair die, let event A be the "sum of dice is five" (4 outcomes). Let event B be the event "no fours are rolled" (25 outcomes). Let event C be "sum of dice is less than 3" (1 outcome).

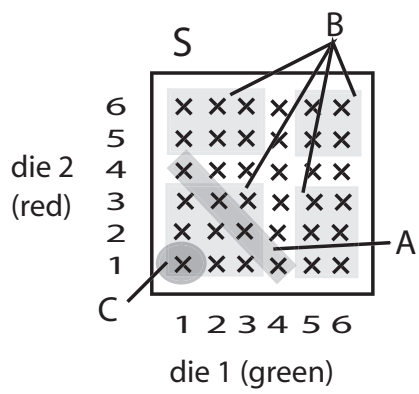

Figure 2.14: Venn diagram for tossing three dice

(a) $P(A) =$ (circle one) (i) $\frac{1}{36}$ (ii) $\frac{2}{36}$ (iii) $\frac{3}{36}$ (iv) $\frac{4}{36}$.

(b) $P(B) =$ (circle one) (i) $\frac{24}{36}$ (ii) $\frac{25}{36}$ (iii) $\frac{26}{36}$ (iv) $\frac{27}{36}$.

(c) $P(C) =$ (circle one) (i) $\frac{1}{36}$ (ii) $\frac{2}{36}$ (iii) $\frac{3}{36}$ (iv) $\frac{4}{36}$.

(d) $P(A \cap B) =$ (circle one) (i) $\frac{1}{36}$ (ii) $\frac{2}{36}$ (iii) $\frac{3}{36}$ (iv) $\frac{4}{36}$.

(e) $P(A \cap C) =$ (circle one) (i) **0** (ii) $\frac{2}{36}$ (iii) $\frac{3}{36}$ (iv) $\frac{4}{36}$.

(f) $P(B \cap C) =$ (circle one) (i) $\frac{1}{36}$ (ii) $\frac{2}{36}$ (iii) $\frac{3}{36}$ (iv) $\frac{4}{36}$.

(g) $P(A \cap B \cap C) =$ (circle one) (i) **0** (ii) $\frac{2}{36}$ (iii) $\frac{3}{36}$ (iv) $\frac{4}{36}$.

(h) Using the additive law for three events,

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) \\ &\quad -P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) = \\ &= \frac{4}{36} + \frac{25}{36} + \frac{1}{36} - \frac{2}{36} - \frac{0}{36} - \frac{1}{36} + \frac{0}{36} = \end{aligned}$$

(i) $\mathbf{\frac{24}{36}}$ (ii) $\mathbf{\frac{25}{36}}$ (iii) $\mathbf{\frac{26}{36}}$ (iv) $\mathbf{\frac{27}{36}}$.

(i) Since $C \subset B$, $P(C) \leq P(B)$
(i) **True** (ii) **False**

4. *Additive law, multiplicative law and independence: water pipes.* As shown in Figure 2.15, two systems of water pipes connecting city A to city B are given below. Water does not flow through a pipe if a valve is closed. Let V_i denote the event a value is *open*. Any value is open with probability $P(V_i) = 0.8$ and so a valve is closed with probability $P(\bar{V}_i) = 0.2$. Whether one valve is open is independent of whether any other value is open.

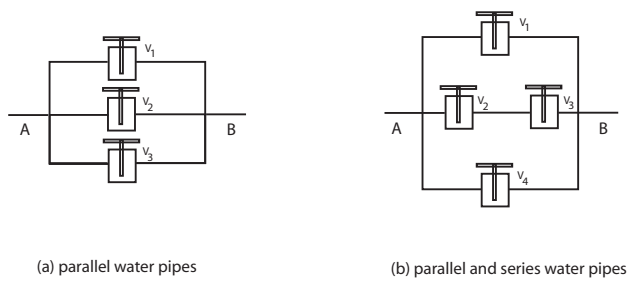

Figure 2.15: Multiplicative and additive rules: water pipes

(a) *Figure (a): Parallel Water Pipes.* Water flows from city A to city B in Figure 2.15(a) if either value V_1 *or* value V_2 *or* value V_3 is open.

$$\begin{aligned}
P(V_1 \cup V_2 \cup V_3) &= P(V_1) + P(V_2) + P(V_3) \\
&\quad -P(V_1 \cap V_2) - P(V_1 \cap V_3) - P(V_2 \cap V_3) + P(V_1 \cap V_2 \cap V_3) = \\
&= P(V_1) + P(V_2) + P(V_3) \\
&\quad -P(V_1)P(V_2) - P(V_1)P(V_3) - P(V_2)P(V_3) + P(V_1)P(V_2)P(V_3) = \\
&= 0.8 + 0.8 + 0.8 \\
&\quad -(0.8)(0.8) - (0.8)(0.8) - (0.8)(0.8) + (0.8)(0.8)(0.8) =
\end{aligned}$$

(i) **0.992** (ii) **0.993** (iii) **0.994** (iv) **0.995**.

(b) *Figure (b): Parallel and Series Water Pipes.* Water flows from city A to city B in Figure 2.15(b) if either value V_1 *or* values "V_2 *and* V_3" or value

V_4 is/are open.

$$\begin{aligned}
P(V_1 \cup (V_2 \cap V_3) \cup V_4) &= P(V_1) + P(V_2 \cap V_3) + P(V_4) \\
&\quad -P(V_1 \cap V_2 \cap V_3) - P(V_1 \cap V_4) \\
&\quad -P(V_2 \cap V_3 \cap V_4) + P(V_1 \cap V_2 \cap V_3 \cap V_4) = \\
&= P(V_1) + P(V_2)P(V_3) + P(V_4) \\
&\quad -P(V_1)P(V_2)P(V_3) - P(V_1)P(V_4) \\
&\quad -P(V_2)P(V_3)P(V_4) + P(V_1)P(V_2)P(V_3)P(V_4) = \\
&= 0.8 + (0.8)(0.8) + 0.8 \\
&\quad -(0.8)(0.8)(0.8) - (0.8)(0.8) \\
&\quad -(0.8)(0.8)(0.8) + (0.8)(0.8)(0.8)(0.8) =
\end{aligned}$$

(i) **0.9592** (ii) **0.9635** (iii) **0.9742** (iv) **0.9856**.

(c) *Figure (b) again: Parallel and Series Water Pipes.* The probability water does *not* flow from city A to city B is given by:

$$P(\overline{V_1 \cup (V_2 \cap V_3) \cup V_4}) = 1 - P(V_1 \cup (V_2 \cap V_3) \cup V_4) =$$

(i) **0.0144** (ii) **0.0185** (iii) **0.0294** (iv) **0.0300**.

5. *Properties.*

(a) If $P(A) = 0.7$ and $P(B) = 0.4$, then

$$P(A \cap B) \leq \min\{P(A), P(B)\} = \min\{0.7, 0.4\} = 0.4$$

(i) **True** (ii) **False**

(b) If events A and B are independent, $P(A) = 0.7$ and $P(B) = 0.4$, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A)P(B) = 0.7 + 0.4 - (0.7)(0.4) =$$

(i) **0.75** (ii) **0.78** (iii) **0.82** (iv) **0.86**.

(c) If events A and B are independent, $P(A) = 0.7$ and $P(B) = 0.4$, then

$$P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B) =$$

(i) **0.18** (ii) **0.20** (iii) **0.22** (iv) **0.24**.
(Hint: Recall DeMorgan's law: $\bar{A} \cap \bar{B} = \overline{A \cup B}$.)

2.9 Calculating the Probability of an Event: The Event-Composition Method

The *event-composition method* consists of equating an original event with the composition of "smaller" events, then using either multiplicative or additive rules to determine the probability of the original event. The point of this section, really, is just to give more practice calculating probabilities.

Exercise 2.9 (Calculating the Probability of an Event: The Event-Composition Method)

1. *Urns.* Urn A has 6 red and 10 blue marbles; urn B has 4 red and 14 blue marbles. A fair coin is tossed. If the coin comes up heads, a marble from urn A is chosen, otherwise a marble from urn B is chosen.

   (a) *Choosing a red marble.*

      i. The chance a head is tossed is $P(H) =$ (circle one)
      (i) $\mathbf{\frac{1}{2}}$ (ii) $\mathbf{\frac{1}{3}}$ (iii) $\mathbf{\frac{1}{4}}$ (iv) $\mathbf{\frac{1}{5}}$.

      ii. If a head is tossed, the chance a red marble is chosen is $P(R|H) =$
      (i) $\mathbf{\frac{6}{10}}$ (ii) $\mathbf{\frac{6}{12}}$ (iii) $\mathbf{\frac{6}{14}}$ (iv) $\mathbf{\frac{6}{16}}$.

      iii. If a tail is tossed, the chance a red marble is chosen is $P(R|T) =$
      (i) $\mathbf{\frac{4}{10}}$ (ii) $\mathbf{\frac{4}{12}}$ (iii) $\mathbf{\frac{4}{14}}$ (iv) $\mathbf{\frac{4}{18}}$.

      iv. The chance a red marble is chosen is
      $P(R) = P(R|H)P(H) + P(R|T)P(T) = \frac{6}{16} \times \frac{1}{2} + \frac{4}{18} \times \frac{1}{2} =$
      (i) $\mathbf{\frac{43}{144}}$ (ii) $\mathbf{\frac{45}{153}}$ (iii) $\mathbf{\frac{46}{165}}$ (iv) $\mathbf{\frac{53}{180}}$.

   (b) *Choosing a blue marble*[12].
   $P(B) = P(B|H)P(H) + P(B|T)P(T) = \frac{10}{16} \times \frac{1}{2} + \frac{14}{18} \times \frac{1}{2} =$
   (i) $\mathbf{\frac{43}{144}}$ (ii) $\mathbf{\frac{76}{144}}$ (iii) $\mathbf{\frac{97}{144}}$ (iv) $\mathbf{\frac{101}{144}}$.

   (c) *Choosing two red marbles, sample with replacement.*
   Let R_1 and R_2 represent first and second red marbles.
   $P(R_1 \cap R_2) = P(R_1)P(R_2) = \frac{43}{144} \times \frac{43}{144} =$
   (i) **0.012** (ii) **0.034** (iii) **0.089** (iv) **0.123**.

   (d) *Choosing two red marbles, sample without replacement.*
   $P(R_1 \cap R_2) = P(R_1)P(R_2|R_1) =$
   $\frac{43}{144} \times \left\{\left(\frac{5}{15} \times \frac{1}{2} + \frac{4}{18} \times \frac{1}{2}\right) + \left(\frac{6}{16} \times \frac{1}{2} + \frac{3}{17} \times \frac{1}{2}\right)\right\} =$
   (i) **0.042** (ii) **0.076** (iii) **0.099** (iv) **0.143**.

2. *Probability distributions.*

[12] Also, $P(B) = 1 - P(R) = 1 - \frac{43}{144} = \frac{101}{144}$

(a) *Geometric probability distribution: bull's eye.* There is a 15% ($P(H) = p = 0.15$) chance of hitting a bulls eye on a dart board. Assume each dart throw is independent of every other dart throw.

i. The chance that the first bulls eye will occur on the *first* try is, of course, 15% ($P(H_1) = p = 0.15$). The chance that the first bulls eye will occur on the *second* try is equal to the chance a miss occurs on the first try and a bulls eye occurs on the second try,
$P(H_2) = (1-p)p = (0.85)0.15 =$ (circle one)
(i) **0.1155** (ii) **0.1275** (iii) **0.1385** (iv) **0.1435**.

ii. The chance that the first bulls eye will occur on the *third* try is equal to the chance of two misses and then a bulls eye occurs on the third try, $P(H_3) = (1-p)^2p = (0.85)^2 0.15 =$ (circle one)
(i) **0.078** (ii) **0.099** (iii) **0.108** (iv) **0.143**.
(Use your calculator: 2nd DISTR geometpdf(0.15,3) ENTER.)

iii. In general, the probability of *first success*, where i is the number of the trial of *first success* and p is the probability of success on each trial, is given by

$$P(H_i) = p(1-p)^{i-1}, \quad i = 1, 2, \ldots.$$

(i) **True** (ii) **False**

(b) *More geometric: shooting hoops.* There is a 35% ($p = 0.35$) chance of making a basket on a free throw. All throws are independent of one another.

i. The chance that the first basket will occur *on* the *third* try is
$P(B_3) = (1-p)^2p = (0.65)^2(0.35) \approx$ (circle one)
(i) **0.078** (ii) **0.148** (iii) **0.198** (iv) **0.203**.
(Use your calculator: 2nd DISTR geometpdf(0.15,3) ENTER.)

ii. The chance that the first[13] basket occurs *no later* than the *third* try is

$$\begin{aligned} P(B_1 \cup B_2 \cup B_3) &= P(B_1) + P(B_2) + P(B_3) \\ &= (1-p)^0 p + (1-p)^1 p + (1-p)^2 p \\ &= (0.65)^0(0.35) + (0.65)^1(0.35) + (0.65)^2(0.35) = \end{aligned}$$

(i) **0.478** (ii) **0.648** (iii) **0.725** (iv) **0.886**.
(Use your calculator: 2nd DISTR geomet*cdf*(0.15,3) ENTER.)

(c) *Negative binomial: bull's eye.* There is a 15% ($p = 0.15$) chance of hitting a bulls eye on a dart board. Assume each dart throw is independent of every other dart throw.

[13] $P(B_1 \cup B_2 \cup B_3) = P(B_1) + P(B_2) + P(B_3) - P(B_1 \cap B_2) - P(B_1 \cap B_3) - P(B_2 \cap B_3) + P(B_1 \cap B_2 \cap B_3) = P(B_1) + P(B_2) + P(B_3)$ because the first basket can only occur once and so $P(B_1 \cap B_2) = P(B_1 \cap B_3) = P(B_2 \cap B_3) = P(B_1 \cap B_2 \cap B_3) = 0$.

i. The chance that the *second* bulls eye will occur on the *first* try is, of course, 0%. The chance that the *second* bulls eye will occur on the *second* try is equal to the chance a hit occurs on the first and second try, $P(H_1 \cap H_2) = p^2 = 0.15^2 =$ (circle one)
(i) **0.0225** (ii) **0.1275** (iii) **0.1385** (iv) **0.1635**.

ii. The chance that the *second* bulls eye will occur on the *third* try is equal to the chance of either a miss and a hit, in that order, or a hit and a miss, in that order, and then a bulls eye occurs on the third try, and so

$$\begin{aligned} P(\bar{H}_1 \cap H_2 \cap H_3) + P(H_1 \cap \bar{H}_2 \cap H_3) &= (1-p)(p)(p) + (p)(1-p)(p) \\ &= 2(1-p)p^2 \\ &= 2(0.85)0.15^2 \ = \end{aligned}$$

(i) **0.03825** (ii) **0.12755** (iii) **0.13860** (iv) **0.16375**.

iii. The chance that the second bulls eye will occur on the *fourth* try is equal to the chance of either a (hit,miss,miss), (miss,hit,miss) or (miss,miss,hit), and then a bulls eye occurs on the fourth try, and so

$$\begin{aligned} & P(H_1 \cap \bar{H}_2 \cap \bar{H}_3 \cap H_4) + P(\bar{H}_1 \cap H_2 \cap \bar{H}_3 \cap H_4) + P(\bar{H}_1 \cap \bar{H}_2 \cap H_3 \cap H_4) \\ = \ & 3(1-p)^2 p^2 \\ = \ & 3(0.85)^2 0.15^2 \ = \end{aligned}$$

(i) **0.0388** (ii) **0.0488** (iii) **0.0588** (iv) **0.0688**.

2.10 The Law of Total Probability and Bayes Rule

The law of total probability for two events is

$$P(A) = P(A|B)P(B) + P(A|\bar{B})P(\bar{B}),$$

and corresponding *Bayes Rule* is

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|\bar{B})P(\bar{B})}.$$

In general, let $B_1, \ldots, B_k$ be mutually exclusive and exhaustive events (a partition of sample space S) with $P(B_j) > 0$ for all j. Then for new evidence A, where $P(A) > 0$, *Bayes rule* for k events is

$$P(B_j|A) = \frac{P(A|B_j)P(B_j)}{P(A|B_1)P(B_1) + \cdots + P(A|B_k)P(B_k)}.$$

Exercise 2.10 (Total Probability and Bayes Rule)

1. *Total Probability and Bayes Theorem: Car Rental Agencies.* As shown in Figure 2.16, a firm rents cars from two rental agencies; 60% are from A, $P(A) = 0.6$, and so 40% are not, $P(\bar{A}) = 0.4$. Of the cars from agency A, 9% needed a tune–up, $P(T|A) = 0.09$; of the cars that are not from A, 20% needed a tune–up, $P(T|\bar{A}) = 0.2$. A car is chosen at random.

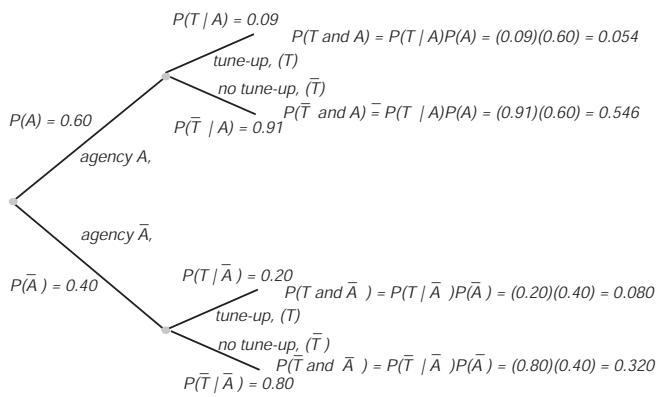

Figure 2.16: Total probability and Bayes rule: car rental

(a) *Total probability.* The probability a car needs a tune–up is

$$\begin{aligned} P(T) &= P(T \cap A) + P(T \cap \bar{A}) = \\ &= P(T|A)P(A) + P(T|\bar{A})P(\bar{A}) = \\ &= (0.09)(0.6) + (0.2)(0.4) = \end{aligned}$$

(i) **0.075** (ii) **0.092** (iii) **0.125** (iv) **0.134**.

(b) *More total probability.* The probability a car does *not* need a tune–up[14] is

$$\begin{aligned} P(\bar{T}) &= P(\bar{T} \cap A) + P(\bar{T} \cap \bar{A}) = \\ &= P(\bar{T}|A)P(A) + P(\bar{T}|\bar{A})P(\bar{A}) = \\ &= (0.91)(0.6) + (0.8)(0.4) = \end{aligned}$$

(i) **0.866** (ii) **0.892** (iii) **0.925** (iv) **0.934**.

(c) *Bayes rule.* Given a car needs a tune–up, what is the chance this car came from agency A?

$$P(A|T) = \frac{P(T|A)P(A)}{P(T|A)P(A) + P(T|\bar{A})P(\bar{A})} = \frac{(0.09)(0.60)}{(0.09)(0.6) + (0.2)(0.4)} =$$

(i) **0.304** (ii) **0.403** (iii) **0.525** (iv) **0.634**.

[14] Notice also $P(\bar{T}) = 1 - P(T) = 1 - 0.134 = 0.866$.

(d) *More Bayes rule.* Given a car needs a tune–up, what is the chance this car does *not* come from agency A?

$$P(\bar{A}|T) = \frac{P(T|\bar{A})P(\bar{A})}{P(T)} = \frac{(0.20)(0.40)}{0.134} =$$

(i) **0.454** (ii) **0.503** (iii) **0.525** (iv) **0.597**.

2. *Total probability and Bayes rule: three car rental agencies.* A firm rents cars from three rental agencies where $P(A_1) = \alpha$, $P(A_2) = \alpha$, $P(A_3) = \frac{1}{2}\alpha$. Also, $P(T|A_1) = 0.09$, $P(T|A_2) = 0.2$ and $P(T|A_3) = 0.13$. A car is chosen at random.

(a) *Bayes rule.* The probability a car comes from agency A_1, conditional on this car needs a tune–up, is

$$\begin{aligned} P(A_1|T) &= \frac{P(T|A_1)P(A_1)}{P(T|A_1)P(A_1) + P(T|A_2)P(A_2) + P(T|A_3)P(A_3)} \\ &= \frac{(0.09)(\alpha)}{(0.09)(\alpha) + (0.2)(\alpha) + (0.13)\left(\frac{1}{2}\alpha\right)} = \end{aligned}$$

(i) **0.254** (ii) **0.323** (iii) **0.426** (iv) **0.547**.

(b) *More Bayes rule.*

$$\begin{aligned} P(A_3|T) &= \frac{P(T|A_3)P(A_3)}{P(T|A_1)P(A_1) + P(T|A_2)P(A_2) + P(T|A_3)P(A_3)} \\ &= \frac{(0.13)\left(\frac{1}{2}\alpha\right)}{(0.09)(\alpha) + (0.2)(\alpha) + (0.13)\left(\frac{1}{2}\alpha\right)} = \end{aligned}$$

(i) **0.154** (ii) **0.183** (iii) **0.226** (iv) **0.446**.

(c) *Still more Bayes rule.* The probability a car comes from agency A_1, conditional on this car does *not* need a tune–up, is

$$\begin{aligned} P(A_1|\bar{T}) &= \frac{P(\bar{T}|A_1)P(A_1)}{P(\bar{T}|A_1)P(A_1) + P(\bar{T}|A_2)P(A_2) + P(\bar{T}|A_3)P(A_3)} \\ &= \frac{(0.91)(\alpha)}{(0.91)(\alpha) + (0.8)(\alpha) + (0.87)\left(\frac{1}{2}\alpha\right)} = \end{aligned}$$

(i) **0.244** (ii) **0.424** (iii) **0.476** (iv) **0.847**.

2.11 Numerical Events and Random Variables

A random variable is a "rule" (or, more technically, a real-valued function) which assigns a number to each outcome in the sample space of an experiment. Probabilities are then assigned to the values of the random variable.

Exercise 2.11 (Numerical Events and Random Variables)

1. *Random variable: flipping a coin twice.* Let random variable Y be the *number* of heads that come up. Assume the coin is fair.

   (a) If *two* heads appear in two flips of a coin, HH, then $Y =$ (circle one)
   (i) **0** (ii) **1** (iii) **2** (iv) **3**.
   Associated with this, $P\{HH\} = P(Y = 2) =$ (circle one)
   (i) $\mathbf{\frac{1}{4}}$ (ii) $\mathbf{\frac{2}{4}}$ (iii) $\mathbf{\frac{3}{4}}$ (iv) $\mathbf{\frac{4}{4}}$.

   (b) If *one* head appear in two flips of a coin, HT or TH, then $Y =$
   (i) **0** (ii) **1** (iii) **2** (iv) **3**.
   Associated with this, $P\{HT, TH\} = P(Y = 1) =$ (circle one)
   (i) $\mathbf{\frac{1}{4}}$ (ii) $\mathbf{\frac{2}{4}}$ (iii) $\mathbf{\frac{3}{4}}$ (iv) $\mathbf{\frac{4}{4}}$.

   (c) If *no* heads appear in two flips of a coin, TT, then $Y =$ (circle one)
   (i) **0** (ii) **1** (iii) **2** (iv) **3**.
   Associated with this, $P\{TT\} = P(Y = 0) =$ (circle one)
   (i) $\mathbf{\frac{1}{4}}$ (ii) $\mathbf{\frac{2}{4}}$ (iii) $\mathbf{\frac{3}{4}}$ (iv) $\mathbf{\frac{4}{4}}$.

   (d) A summary of this information is given in the following table.

y	0	1	2
$P(Y = y)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

   (i) **True** (ii) **False**

2. *Random variable: televisions and hypergeometric.* Seven television tubes are chosen at random from a shipment of 240 television tubes of which 15 are defective. Let random variable Y be the *number* of defective tubes chosen.

   (a) If *four* defective tubes are chosen, then $Y =$ (circle one)
   (i) **1** (ii) **2** (iii) **3** (iv) **4**.
   Associated with this, $P(Y = 4) =$ (circle one)
   (i) $\dfrac{\binom{\mathbf{15}}{\mathbf{4}} \times \binom{\mathbf{225}}{\mathbf{3}}}{\binom{\mathbf{240}}{\mathbf{7}}}$ (ii) $\dfrac{\binom{\mathbf{15}}{\mathbf{3}} \times \binom{\mathbf{225}}{\mathbf{4}}}{\binom{\mathbf{240}}{\mathbf{7}}}$ (iii) $\dfrac{\binom{\mathbf{15}}{\mathbf{2}} \times \binom{\mathbf{225}}{\mathbf{5}}}{\binom{\mathbf{240}}{\mathbf{7}}}$.
   In other words, $P(Y = 4) =$ (circle one)
   (i) $\mathbf{3.07 \times 10^{-4}}$ (ii) $\mathbf{3.07 \times 10^{-5}}$ (iii) $\mathbf{3.07 \times 10^{-6}}$ (iv) $\mathbf{3.07 \times 10^{-7}}$.
   (Hint: Use your calculator: (15 nCr 4) × (225 nCr 3) / (240 nCr 7))

(b) If *seven* defective tubes are chosen, then $Y =$ (circle one)
(i) **4** (ii) **5** (iii) **6** (iv) **7**.
Associated with this, $P(Y = 7) =$ (circle one)

(i) $\dfrac{\binom{\mathbf{15}}{\mathbf{7}} \times \binom{\mathbf{225}}{\mathbf{0}}}{\binom{\mathbf{240}}{\mathbf{7}}}$ (ii) $\dfrac{\binom{\mathbf{15}}{\mathbf{6}} \times \binom{\mathbf{225}}{\mathbf{1}}}{\binom{\mathbf{240}}{\mathbf{7}}}$ (iii) $\dfrac{\binom{\mathbf{15}}{\mathbf{5}} \times \binom{\mathbf{225}}{\mathbf{2}}}{\binom{\mathbf{240}}{\mathbf{7}}}$.

In other words, $P(Y = 7) =$ (circle one)
(i) $\mathbf{7.72 \times 10^{-7}}$ (ii) $\mathbf{7.72 \times 10^{-8}}$ (iii) $\mathbf{7.72 \times 10^{-9}}$ (iv) $\mathbf{7.72 \times 10^{-10}}$.

2.12 Random Sampling

The *simple random sampling* (SRS) procedure is a method of selecting n units out of N population units such that every one of the distinct samples has an *equal* chance of being drawn. The sampling occurs with*out* replacement. It is in this way the sample is assured to be *representative* of the population and for there to be *no* bias in the sample. Other techniques include stratified, cluster and systematic sampling.

2.13 Summary