2.4 Exponential Functions

An exponential function is given by

$$f(x) = a^x$$

where x is any real number, $a > 0$ and $a \neq 1$. If base $a = e \approx 2.718$, the exponential function becomes the (natural) exponential function, $f(x) = e^x$. Related to this, as m gets larger, $(1 + \frac{1}{m})^m$ approaches e.

Exponential functions are used in financial formulas. If principal (present value) amount P is invested at interest rate r per year over time t, *simple interest*, I, is $I = Prt$. If P is invested at interest rate r per year, compounded m times per year for t years, *compound amount* is

$$A = P\left(1 + \frac{r}{m}\right)^{mt}.$$

If interest rate r is compounded *continuously*, compound amount after t years is

$$A = Pe^{rt}.$$

Exercise 2.4 (Exponential functions)

1. *Properties of exponential functions.*

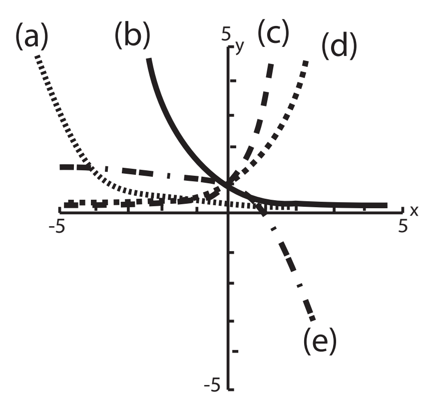

Figure 2.11 (Various Exponential Functions)

(Use WINDOW -5 5 1 -5 5 1 1)

(a) Exponential function $f(x) = 2^x$ corresponds to
graph (i) **(a)** (ii) **(b)** (iii) **(c)** (iv) **(d)** (v) **(e)**

(b) Exponential function $f(x) = 3^x$ corresponds to
graph (i) **(a)** (ii) **(b)** (iii) **(c)** (iv) **(d)** (v) **(e)**
$f(x) = 3^x$ increases more rapidly than $f(x) = 2^x$.

(c) Exponential function $f(x) = 0.5^x$ corresponds to
graph (i) **(a)** (ii) **(b)** (iii) **(c)** (iv) **(d)** (v) **(e)**
$f(x) = 0.5^x = \left(\frac{1}{2}\right)^x = \frac{1^x}{2^x} = \frac{1}{2^x} = 2^{-x} = f(-x)$, which is reflection in y-axis to $f(x) = 2^x$.

(d) Exponential function $f(x) = 2^{-x-4}$ corresponds to
graph (i) **(a)** (ii) **(b)** (iii) **(c)** (iv) **(d)** (v) **(e)**
$f(x) = 2^{-x-4} = 2^{-(x+4)} = f(-(x+4))$, which a left translation 4 units from $f(-x) = 2^{-x}$.

(e) Exponential function $f(x) = -2^x + 2$ corresponds to
graph (i) **(a)** (ii) **(b)** (iii) **(c)** (iv) **(d)** (v) **(e)**
$-f(x) = -2^x$ is a reflection of $f(x) = 2^x$ in x-axis and $-f(x) + 2$ is a translation up 2 units.

2. *Exponential function applications: radioactive decay.* Quantity (in ounces) present at time t (in years) is

$$Q(t) = 500(5^{-0.2t})$$

(a) Quantity left in $t = 8$ years is approximately
(i) **28** (ii) **38** (iii) **48** ounces.
Calculator: $500 * (5 \wedge (-0.2 * 8))$

(b) Length of time until quantity reduces to 10 ounces:
(i) **12.12** (ii) **12.34** (iii) **12.67** years.
Calculator: Type $500 * (5 \wedge (-0.2 * X))$ into Y=, set WINDOW 0 30 1 0 500 1 1,
GRAPH, TRACE, arrow close to Y= 10. Later, we will use logarithms to solve this sort of question.

3. *Compound Interest:* $A = P\left(1 + \frac{r}{m}\right)^{mt}$.

(a) If \$700 is invested at 11% interest compounded *yearly* (or annually), calculate its value after 8 years.
$A = P\left(1 + \frac{r}{m}\right)^{mt} = 700\left(1 + \frac{0.11}{1}\right)^{1(8)} =$ **1513.18 / 1613.18 / 1713.18**

(b) If \$700 is invested at 11% interest compounded *monthly*, calculate its value after 8 years.
$A = P\left(1 + \frac{r}{m}\right)^{mt} = 700\left(1 + \frac{0.11}{12}\right)^{(12)8} =$ **1580.88 / 1680.88 / 1780.88**

4. *Compound interest: related questions.*

(a) *Interest rate, r?*

i. If $A = 700$, $P = 15$, $t = 10$ years, interest compounded yearly
Since $A = P\left(1 + \frac{r}{m}\right)^{mt}$, then $700 = 15\left(1 + \frac{r}{1}\right)^{1(10)}$ or $(1 + r)^{10} = \frac{700}{15}$
or taking tenth root of both sides,
$1 + r = \left(\frac{700}{15}\right)^{1/10}$ or $r = \left(\frac{700}{15}\right)^{1/10} - 1 \approx$ **0.15 / 0.39 / 0.47**.
Calculator: $(700/15) \wedge (0.1) - 1$

ii. If $A = 700$, $P = 15$, $t = 10$ years, interest compounded monthly
Since $A = P\left(1+\frac{r}{m}\right)^{mt}$, $700 = 15\left(1+\frac{r}{12}\right)^{12(10)}$ or $\left(1+\frac{r}{12}\right)^{120} = \frac{700}{15}$
or taking 120th root of both sides,
$1+\frac{r}{12} = \left(\frac{700}{15}\right)^{1/120}$ or $r = 12\left(\left(\frac{700}{15}\right)^{1/120} - 1\right) \approx$ **0.15 / 0.39 / 0.47**.
Calculator: $12 * ((700/15) \wedge (1/120) - 1)$

(b) *Principal, P?*

i. If $A = 700$, $t = 5$ years, $r = 0.08$ interest compounded yearly
Since $A = P\left(1+\frac{r}{m}\right)^{mt}$, $700 = P\left(1+\frac{0.08}{1}\right)^{1(5)}$
or $P = 700(1+0.08)^{-5} \approx$ **476.41 / 500.00 / 528.89**.
Calculator: $700 * 1.08 \wedge (-5)$

ii. If $A = 700$, $t = 5$ years, $r = 0.08$ interest compounded monthly
Since $A = P\left(1+\frac{r}{m}\right)^{mt}$, $700 = P\left(1+\frac{0.08}{12}\right)^{12(5)}$
or $P = 700\left(1+\frac{0.08}{12}\right)^{-60} \approx$ **469.85 / 499.00 / 518.89**.
Calculator: $700 * (1 + 0.08/12) \wedge (-60)$

5. *Compound interest (continuously):* $A = Pe^{rt}$

(a) If \$700 is invested at 11% interest compounded continuously, calculate its value after 8 years.
$A = Pe^{rt} = 700e^{0.11(8)} =$ **1687.63 / 1967.36 / 2267.36**.

(b) An amount \$700 invested at 11% interest *compounded annually* (\$1613.18)
is **less** / **greater**
than \$700 invested at 11% interest *compounded monthly* (\$1680.88)
is **less** / **greater**
than \$700 invested at 11% interest *compounded continuously* (\$1687.63)
after 8 years.

2.5 Logarithmic Functions

Logarithmic functions are related to exponential functions. Assume $a > 0$, $a \neq 1$ and $x > 0$

$$y = \log_a x \quad \text{if and only if} \quad a^y = x \ \ (\text{or } a^y - x = 0)$$

where "$\log_a x$" is read "logarithm of x to the base a". If base $a = e$, the logarithmic function becomes the (natural) logarithmic function, $f(x) = \log_e x = \ln x$; if base $a = 10$, the logarithmic function becomes the (common) logarithmic function, $f(x) = \log_{10} x = \log_{10} x$. For any positive x, y and a, $a \neq 1$, and any real number r,

- $\log_a xy = \log_a x + \log_a y$

- $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$
- $\log_a x^r = r \log_a x$

Also, $\log_a a = 1$, $\log_a 1 = 0$ and $\log_a a^r = r$. The change-of-base theorem for logarithms

$$\log_a x = \frac{\log_b x}{\log_b a} = \frac{\ln x}{\ln a}$$

and change-of-base theorem for exponentials is

$$a^x = e^{(\ln a)x}$$

Exercise 2.5 (Logarithmic Functions)

1. *Graphs of Logarithmic Functions.* Consider the following graphs of various logarithm functions.

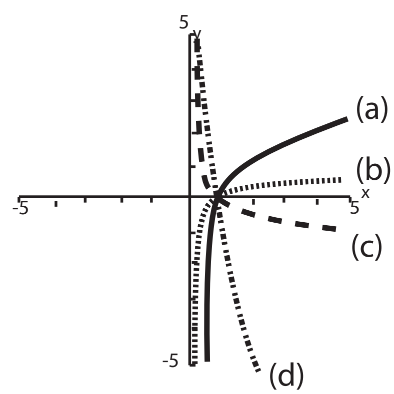

Figure 2.12 (Various Logarithmic Functions)

(Calculator: Use WINDOW -5 5 1 -5 5 1 1)

(a) Logarithm function $f(x) = \log_2 x$ corresponds to
graph (i) **(a)** (ii) **(b)** (iii) **(c)** (iv) **(d)**

Use change-of-base theorem: for $f(x) = \log_2 x$, type $\ln(X)/\ln(2)$ into Y=, then GRAPH
or your calculator may have MATH logBASE 2 X ENTER

(b) Logarithm function $f(x) = \log_3 x$ corresponds to
graph (i) **(a)** (ii) **(b)** (iii) **(c)** (iv) **(d)**

Type $\ln(X)/\ln(3)$ into Y=, then GRAPH

(c) Logarithm function $f(x) = \log_{0.9} x$ corresponds to
graph (i) **(a)** (ii) **(b)** (iii) **(c)** (iv) **(d)**

Type $\ln(X)/\ln(0.9)$ into Y=, then GRAPH

(d) Logarithm function $f(x) = \log_{0.1} x$ corresponds to
graph (i) **(a)** (ii) **(b)** (iii) **(c)** (iv) **(d)**
Type ln(X)/ ln(0.1) into Y=, then GRAPH

(e) In general, the x–intercept of $f(x) = \log_a x$ is (i) **0** (ii) **1** (iii) **2**.

2. *Logarithmic function inverse of exponential function*

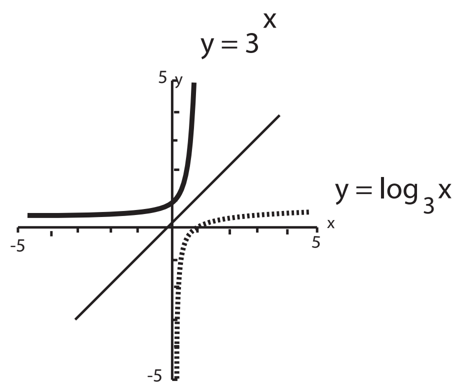

Figure 2.13 (Logarithmic Versus Exponential Function)

(a) Functions $y = \log_3 x$ and $y = 3^x$ reflections of one another through
(i) y–**axis** (ii) x–**axis** (iii) $\mathbf{45^o}$ **degree line**.
See figure above.

(b) Inverse function of natural logarithmic function $y = \ln x$ is
(i) $\mathbf{10^x}$ (ii) $\boldsymbol{e^x}$ (iii) $\mathbf{5^x}$

(c) Inverse function of common logarithmic function $y = \log_{10} x$ is
(i) $\mathbf{10^x}$ (ii) $\boldsymbol{e^x}$ (iii) $\mathbf{5^x}$

(d) Since $y = \log_a x$ means $x = a^y$, then $y = \log_{0.5} x$ means
$x =$ (circle one or *more*!) (i) $\mathbf{(0.5)^y}$ (ii) $\left(\frac{1}{2}\right)^{y}$ (iii) $\frac{1^y}{2^y}$ (iv) $\frac{1}{2^y}$ (v) $\mathbf{2^{-y}}$

(e) $10 = \log_{0.2} x$ means
$x =$ (one or *more*!) (i) $\mathbf{(0.2)^y}$ (ii) $\left(\frac{1}{5}\right)^{10}$ (iii) $\frac{1^{10}}{5^{10}}$ (iv) $\frac{1}{5^{10}}$ (v) $\mathbf{5^{-10}}$

(f) $2 = \log_{0.1} x$ means
$x =$ (circle one) (i) $\mathbf{2^{-2}}$ (ii) $\mathbf{7^{-0.1}}$ (iii) $\mathbf{10^{-2}}$ (iv) $\mathbf{12^{-10}}$

(g) Write $\log_4 16384 = 7$ in exponential form
(i) $\mathbf{7^4 = 16384}$ (ii) $\mathbf{4^7 = 16384}$ (iii) $\mathbf{16384^4 = 7}$

(h) Write $\log 100000 = 5$ in exponential form
(i) $\mathbf{100000^{10} = 5}$ (ii) $\mathbf{10^5 = 100000}$ (iii) $\mathbf{5^{10} = 100000}$

(i) Write $5^2 = 25$ in logarithmic form
(i) $\mathbf{\log_5 25 = 2}$ (ii) $\mathbf{\log_{25} 5 = 2}$ (iii) $\mathbf{\log_2 25 = 5}$

(j) Write $4^{-2} = \frac{1}{16}$ in logarithmic form
(i) $\mathbf{\log_4 \frac{1}{16} = -2}$ (ii) $\mathbf{\log_{-2} 4 = \frac{1}{16}}$ (iii) $\mathbf{\log_{-2} \frac{1}{16} = 4}$

3. *Properties of logarithms.*

(a) $\log_b x + \log_b y - \log_b z =$ (i) $\mathbf{\log_b \frac{xy}{z}}$ (ii) $\mathbf{\log_b \frac{xz}{y}}$ (ii) $\mathbf{\log_b \frac{z}{xy}}$.

(b) $5\log_b x - 4\log_b y =$ (i) $\mathbf{\log_b \frac{5x}{4y}}$ (ii) $\mathbf{\log_b \frac{x^5}{y^4}}$ (iii) $\mathbf{\log_b \frac{y^4}{x^5}}$.

(c) $2\log_b(x+3) - 4\log_b(x-3) =$
(i) $\mathbf{\log_b \frac{2(x+3)}{4(x-3)}}$ (ii) $\mathbf{\log_b \frac{(x+3)^2}{(x-3)^4}}$ (iii) $\mathbf{\log_b \frac{(x-3)^2}{(x+3)^5}}$.

(d) $\log_a a =$ (i) $\mathbf{0}$ (ii) $\mathbf{1}$ (iii) $\boldsymbol{e}$.
(Hint: $\log_a a = y$ means $a^y = a$ and so $y = ?$)

(e) $\log_a a^k =$ (i) $\mathbf{0}$ (ii) $\mathbf{1}$ (iii) $\boldsymbol{k}$.
(Hint: $\log_a a^k = y$ means $a^y = a^k$ and so $y = ?$)

(f) $\log_a 1 =$ (i) $\mathbf{0}$ (ii) $\mathbf{1}$ (iii) $\boldsymbol{k}$.
(Hint: $\log_a 1 = y$ means $a^y = 1$ and so $y = ?$)

(g) $\log_a -3 =$ (i) $\mathbf{0}$ (ii) $\mathbf{1}$ (iii) **undefined**.
(Hint: $\log_a -3 = y$ means $a^y = -3$, $a > 0$, and so $y = ?$)

(h) $\ln x + \ln y =$ (i) $\mathbf{\ln \boldsymbol{xy}}$ (ii) $\mathbf{\ln \frac{xz}{y}}$ (iii) $\mathbf{\ln \frac{z}{xy}}$.

(i) $5\ln x - 4\ln y =$ (i) $\mathbf{\ln \frac{5x}{4y}}$ (ii) $\mathbf{\ln \frac{x^5}{y^4}}$ (iii) $\mathbf{\ln \frac{y^4}{x^5}}$.

(j) $\ln e =$ (i) $\mathbf{0}$ (ii) $\mathbf{1}$ (iii) $\boldsymbol{e}$
(Hint: $\ln e = \log_e e = y$ means $e^y = e$ and so $y = ?$)

(k) $2\log(x+3) - 4\log(x-3) =$
(i) $\mathbf{\log_{10} \frac{2(x+3)}{4(x-3)}}$ (ii) $\mathbf{\log_{10} \frac{(x+3)^2}{(x-3)^4}}$ (iii) $\mathbf{\log_{10} \frac{(x-3)^2}{(x+3)^5}}$

(l) $\log 10 =$ (i) $\mathbf{0}$ (ii) $\mathbf{1}$ (iii) $\boldsymbol{e}$
(Hint: $\log_{10} 10 = y$ means $10^y = 10$ and so $y = ?$)

4. *Working with logarithmic functions.* Let

$$\log_a 2 = 0.245 \qquad \log_a 7 = 0.404$$

(a) $\log_a 14 =$ (circle one or more)
(i) $\mathbf{\log_a(2 \cdot 7)}$ (ii) $\mathbf{\log_a 2 + \log_a 7}$ (iii) $\mathbf{0.245 + 0.404 = 0.649}$

(b) $\log_a \frac{2}{7} =$ (circle one or more)
(i) $\mathbf{\log_a 2 + \log_a 7}$ (ii) $\mathbf{\log_a 2 - \log_a 7}$ (iii) $\mathbf{0.245 - 0.404 = -0.159}$

(c) $\log_a \frac{8}{7} =$ (circle one or more)
(i) $\mathbf{\log_a 8 - \log_a 7}$
(ii) $\mathbf{\log_a 2^3 - \log_a 7}$
(iii) $\mathbf{3\log_a 2 - \log_a 7}$
(iv) $\mathbf{3(0.245) - 0.404 = 0.331}$

(d) $\log_a 49 =$ (circle one or more)
(i) $\boldsymbol{\log_a 7^2}$ (ii) $\boldsymbol{2\log_a 7}$ (iii) $\mathbf{2(0.404) = 0.808}$

(e) $\log_a 49a =$ (circle one or more)
(i) $\boldsymbol{\log_a 7^2 a}$ (ii) $\boldsymbol{2\log_a 7 + \log_a a}$ (iii) $\mathbf{2(0.404) + 1 = 1.808}$

(f) $\log_a \sqrt[3]{a} =$ (circle one or more) (i) $\boldsymbol{\log_a a^{1/3}}$ (ii) $\boldsymbol{\frac{1}{3}\log_a a}$ (iii) $\mathbf{\frac{1}{3}(1) = \frac{1}{3}}$

(g) $\log_a \sqrt[5]{a} =$ (circle one) (i) $\mathbf{\frac{1}{3}}$ (ii) $\mathbf{\frac{1}{5}}$ (iii) $\mathbf{\frac{1}{7}}$

(h) $\log_a 7^7 =$ (circle one)
(i) $\mathbf{7(0.404) = 2.828}$
(ii) $\mathbf{7(7) = 49}$
(iii) $\mathbf{7^7 = 823543}$

(i) $\log_a 64 =$ (circle one) (i) $\mathbf{6(0.245) = 1.47}$ (ii) $\mathbf{2(6) = 12}$ (iii) $\mathbf{2^6 = 64}$

5. *Change-of-base for logarithms and exponentials.* Evaluate to 3 decimal points of accuracy if necessary.

(a) $\log_{11} 345 =$ (i) **2.437** (ii) **2.447** (iii) **2.457**
Since $\log_a x = \frac{\ln x}{\ln a}$, $\log_{11} 345 = \frac{\ln 345}{\ln 11}$

(b) $\log_{1.1} 345 =$ (i) **61.321** (ii) **61.301** (iii) **61.311**
$\log_{1.1} 345 = \frac{\ln 345}{\ln 1.1}$

(c) Write 9^{x-4} using base e: (i) $\boldsymbol{e^{(\ln 9)(x+4)}}$ (ii) $\boldsymbol{e^{(\ln 9)(x-4)}}$ (ii) $\boldsymbol{e^{(9)(x-4)}}$
Hint: $a^x = e^{(\ln a)(x)}$

6. *Logarithm and exponential equations.*

(a) Solve $\log_9 81 = y$ for y.
i. **True** / **False** $9^y = 81$
ii. **True** / **False** $(3^2)^y = 3^4$
iii. **True** / **False** $3^{2y} = 3^4$
iv. and so $y =$ (i) **4** (ii) **6**. (iii) **2**

(b) Solve $\log_q 6 = \frac{1}{2}$ for q.
i. **True** / **False** $q^{1/2} = 6$
ii. **True** / **False** $\sqrt{q} = \sqrt{36}$
iii. and so $q =$ (i) **6** (ii) $\boldsymbol{\sqrt{6}}$. (iii) **36**

(c) Solve $\log(x + 5)(x + 1) = 1$ for x.
i. **True** / **False** $10^1 = (x + 5)(x + 1) = x^2 + 6x + 5$
ii. **True** / **False** $0 = x^2 + 6x - 5 = (x - 3)(x - 2)$
iii. and so $x =$ (i) $\mathbf{1, 3}$ (ii) $\mathbf{2, 4}$. (iii) $\mathbf{2, 3}$

(d) Solve $\ln x + \ln 2x = -2$ for x.

i. **True** / **False** $\ln 2x^2 = -2$
ii. **True** / **False** $e^{-2} = 2x^2$
iii. **True** / **False** $\frac{1}{2}e^{-2} = x^2$
iv. and so $x =$ (i) $\mathbf{\frac{1}{\sqrt{2}e}}$ (ii) $\mathbf{\frac{1}{2e^2}}$ (iii) $\mathbf{\frac{2}{\sqrt{e}}}$.

(e) Solve $21^{2x} = 27$ for x.

i. **True** / **False** $\ln 21^{2x} = \ln 27$
ii. **True** / **False** $(2x)\ln 21 = \ln 27$
iii. **True** / **False** $2x = \frac{\ln 27}{\ln 21}$
iv. and so $x =$ (circle one or more) (i) $\mathbf{\frac{1}{2}\frac{\ln 27}{\ln 21}}$ (ii) **0.541** (iii) **0.675**.

(f) Solve $21^{2x} = 27$ for x.

i. **True** / **False** $\log_{10} 21^{2x} = \log_{10} 27$
ii. **True** / **False** $(2x)\log_{10} 21 = \log_{10} 27$
iii. **True** / **False** $2x = \frac{\log_{10} 27}{\log_{10} 21}$
iv. and so $x =$ (circle one or more) (i) $\mathbf{\frac{1}{2}\frac{\log_{10} 27}{\log_{10} 21}}$ (ii) **0.541** (iii) **0.675**.

(g) Solve $e^{2x} = 27$ for x.

i. **True** / **False** $\ln e^{2x} = \ln 27$
ii. **True** / **False** $(2x)\ln e = \ln 27$
iii. **True** / **False** $2x = \frac{\ln e}{\ln 27}$
iv. and so $x =$ (circle one or more) (i) $\mathbf{\frac{1}{2} \cdot \ln e}$ (ii) $\mathbf{\frac{1}{2} \cdot \frac{1}{\ln 27}}$ (iii) **0.164**

(h) Solve $e^{-0.02x} = 12$ for x.
$x =$ (circle one or more) (i) $\mathbf{-\frac{1}{0.02} \cdot \ln e}$ (ii) $\mathbf{-\frac{\ln 12}{0.02}}$ (iii) **−16.423**
Since $\ln e^{-0.02x} = \ln 12$, then $-0.02x = \ln 12$, and so $x = \frac{\ln 12}{-0.02} =$?

7. *Compound interest: number of interest periods, $n = mt$*

(a) If $A = 700$, $P = 15$, $r = 0.08$ interest compounded yearly
Since $A = P\left(1 + \frac{r}{m}\right)^{mt}$, $700 = 15\left(1 + \frac{0.08}{1}\right)^{mt}$ or $(1 + 0.08)^{mt} = \frac{700}{15}$
or taking natural logs of both sides,
$\ln(1 + 0.08)^{mt} = \ln \frac{700}{15}$ or $mt \ln(1 + 0.08) = \ln \frac{700}{15}$
or $n = mt = \frac{\ln \frac{700}{15}}{\ln 1.08} \approx$ **48** / **50** / **52**.
Calculator: ln(700/15)/ ln(1.08)

(b) If $A = 700$, $P = 15$, $r = 0.08$ interest compounded monthly
Since $A = P\left(1 + \frac{r}{m}\right)^{mt}$, $700 = 15\left(1 + \frac{0.08}{12}\right)^{mt}$ or $\left(1 + \frac{0.08}{12}\right)^{mt} = \frac{700}{15}$
or taking natural logs of both sides,
$\ln\left(1 + \frac{0.08}{12}\right)^{mt} = \ln \frac{700}{15}$ or $12t \ln\left(1 + \frac{0.08}{12}\right) = \ln \frac{700}{15}$
or $n = 12t = \frac{\ln \frac{700}{15}}{\ln\left(1 + \frac{0.08}{12}\right)} \approx$ **563** / **578** / **589**.
Calculator: ln(700/15)/ ln(1 + 0.08/12)

(c) *Doubling time for compound interest: rule of 70, 72.*
True / **False** If $m = 1$, then $A = P\left(1 + \frac{r}{m}\right)^{mt} = P(1+r)^t$, and time to double principal when $P = 1$ is given by $2 = (1+r)^t$ or

$$t = \frac{\ln 2}{\ln(1+r)} \approx \frac{\ln 2}{r} \approx \frac{0.693}{r}, \quad \text{if } r \text{ is small}$$

Specifically, if $0.001 \leq r < 0.05$, doubling time is $t \approx \frac{70}{100r}$, or, if $0.05 \leq r < 0.12$, doubling time is $t \approx \frac{72}{100r}$.

2.6 Applications: Growth and Decay; Mathematics of Finance

For y_0 amount present at time $t = 0$, let amount present at time t be

$$y = y_0 e^{kt}.$$

If $k > 0$, then k is a *growth constant* and y is an exponential growth function (used in bacterial growth, for example); if $k < 0$, then k is a *decay constant* and y is an exponential decay function (used in radioactive decay, for example). In addition to this *unbounded* model, the *limited growth model* is given by

$$y = L - (L - y_0)e^{kt}$$

where $k < 0$ and L is a limit to growth.

Also, *effective rate for compound interest* is

$$r_E = \left(a + \frac{r}{m}\right)^m - 1$$

which becomes $r_E = e^r - 1$ if interest is compounded continuously.

Exercise 6.6 (Applications: Growth and Decay; Mathematics of Finance)

1. *Biological growth function, k and t known.* How many cells will there be after 10 hours, if there are an initial 5000 cells and growth rate $k = 0.02$?

   (a) The y in $y = y_0 e^{kt}$ describes the
   (i) **cell growth rate** (ii) **initial cell count** (iii) **cell count**

   (b) The y_0 describes initial cell count given by (i) **4** (ii) **10** ((iii) **5000**

   (c) so at $t = 10$, $y = 5000e^{0.02(10)} \approx$ (i) **6004** (ii) **6053** (iii) **6107**

2. *Population decay, k and t known.* How many people will there be after 10 years, if there are an initial population of 5000 people and the population decays at a rate of $k = -0.02$?

   (a) The y in $y = y_0 e^{kt}$ describes the
   (i) **population decay rate** (ii) **initial population** (iii) **population**.

   (b) The y_0 describes initial population given by (i) $\mathbf{-0.02}$ (ii) **10** (iii) **5000**.

   (c) at $t = 10$, $y = 5000e^{-0.02(10)} \approx$ (i) **4094** (ii) **4154** (iii) **4254**.

3. *Biological growth, k unknown and t known.* If cells in a bacterial culture divide every 4 hours (cell count *doubles* every 4 hours), how many cells will there be after 10 hours, if there are an initial 5000 cells?

   (a) Since $y_0 = 5000$, $y =$ (i) $\mathbf{5000e^{5000t}}$ (ii) $\mathbf{5000e^{kt}}$ (iii) $\boldsymbol{ke^{kt}}$

   (b) (i) **True** (ii) **False** At $t = 0$, initial cell count is $P = 5000e^{k(0)} = 5000$.

   (c) Since cell count *doubles* every $t = 4$ hours, at $t = 4$, four hours after initial count of 5000 cells, there must be (i) **2500** (ii) **25000** (iii) **10000** cells. In other words, $y = 5000e^{k(4)} = 10000$.

   (d) Rewrite $5000e^{k(4)} = 10000$ as $e^{4k} = 2$. Taking natural logarithms of both sides, $\ln e^{4k} = \ln 2$ or $4k = \ln 2$ and so $k \approx$ (i) **0.173** (ii) **0.456**

   (e) At $t = 10$, $y = 5000e^{0.173(10)} \approx$ (i) $\mathbf{28,204}$ (ii) $\mathbf{29,204}$ (iii) $\mathbf{30,204}$

4. *Population growth, k unknown and t known.* A population doubles every 25 years. If the population was 10,000 in 2000, what will it be in 2100?

   (a) $y = y_0 e^{kt} =$ (i) $\mathbf{10000e^{5000t}}$ (ii) $\mathbf{10000e^{kt}}$ (iii) $\boldsymbol{ke^{kt}}$.

   (b) (i) **True** (ii) **False** At $t = 0$, initial population is $y = 10000e^{k(0)} = 10000$.

   (c) Since population *doubles* every $t = 25$ years, at $t = 25$, 25 years after initial count of 10000, there must be (i) **2500** (ii) **25000** (iii) **20000** people. In other words, $P = 10000e^{k(25)} = 20000$.

   (d) Rewrite $10000e^{k(25)} = 20000$ as $e^{25k} = 2$. Taking natural logarithms of both sides, $25k = \ln 2$ and so $k = \frac{\ln 2}{25} \approx$ (i) **0.0173** (ii) **0.0277**.
   Notice $k = \frac{\ln 2}{25} = \frac{0.6931}{25} \approx \frac{69.31}{2500} \approx \frac{70}{2500}$ and so population doubling is an example of the *rule of 70.*

   (e) So, at $t = 2100 - 2000 = 100$,
   $y = 10000e^{0.0277(100)} \approx$ (i) $\mathbf{158,587}$ (ii) $\mathbf{159,587}$ (iii) $\mathbf{200,587}$.

5. *Radioactive decay, k unknown and t known.* If the half–life of blueberrium is 745 hours, how much of 23 grams of blueberrium will be left after 1045 hours?

   (a) Since $y_0 = 23$, $P =$ (i) $\mathbf{23e^{23t}}$ (ii) $\mathbf{23e^{-kt}}$ (iii) $\boldsymbol{ke^{-kt}}$.

   (b) (i) **True** (iii) **False** At $t = 0$, $P = 23e^{-k(0)} = 23$.

(c) Since blueberrium decreases to one–half its original amount every $t = 745$ hours, original 23 grams decreases to 11.5 grams and so, at $t = 745$, $y = 23e^{-k(745)} =$ (i) **23** (ii) **11.5** (iii) **5.75**.

(d) Rewrite $23e^{-k(745)} = 11.5$ as $e^{-745k} = 0.5$, so $\ln e^{-745k} = \ln 0.5$, or $-745k = \ln 0.5$ and so $k = \frac{\ln 0.5}{-745} \approx$ (i) **0.0009303** (ii) **0.00234**.

(e) So, at $t = 1045$, $y = 23e^{-0.0009303(1045)} \approx$ (i) **7.7** (ii) **8.7** (iii) **9.7**.

6. *Population growth, k known and t unknown.* How many years will it take before a town hits a size of 1000 people, if there are an initial 400 inhabitants and $k = 0.0177$?

(a) $y = y_0e^{kt} =$ (i) $\mathbf{400e^{0.0177t}}$ (ii) $\mathbf{0.0177e^{400t}}$ (iii) $\mathbf{400e^{400t}}$.

(b) (i) **True** (ii) **False** At $t = 0$, $y = 400e^{k(0)} = 400$.

(c) Since we are interested in when the town size hits 1000, let $y = 400e^{0.0177t} = 1000$, or $e^{0.0177t} = 2.5$, so $\ln e^{0.0177t} = \ln 2.5$ or $0.0177t = \ln 2.5$ and so $t = \frac{\ln 2.5}{0.0177} \approx$ (i) **51.8** (ii) **83.2** (iii) **93.8** years.

7. *Carbon dating, k and t unknown.* How old is a wood building if 34% of carbon 14 is left in the wood? Assume the half–life of carbon 14 is 5570 years; that is, assume 50% of the original 100% of carbon 14 is left in the wood 5570 years after a tree dies (and is used to construct the building).

(a) Since $y_0 = 100\%$, $y =$ (i) $\mathbf{100e^{-100t}}$ (ii) $\mathbf{100e^{-kt}}$ (iii) $\mathbf{ke^{-100t}}$.

(b) (i) **True** (ii) **False** At $t = 0$, $P = 100e^{-k(0)} = 100$.

(c) Since carbon 14 decreases to 50% its original amount every $t = 5570$ years, the original 100% decreases to 50% and so, at $t = 5570$, $y = 100e^{-k(5570)} =$ (i) **25** (ii) **50** (iii) **75**.

(d) Rewrite $100e^{-k(5570)} = 50$ as $e^{-5570k} = 0.5$, so $\ln e^{-5570k} = \ln 0.5$ or $-5570k = \ln 0.5$ and so $k \approx$ (i) **0.0009303** (ii) **0.0001244**.

(e) So, since 34% of the original 100% of carbon 14 remains, $y = 100e^{-0.0001244t} =$ (i) **34** (ii) **55** (iii) **73**.

(f) Rewrite $100e^{-0.0001244t} = 34$ as $e^{-0.0001244t} = 0.34$, so $\ln e^{-0.0001244t} = \ln 0.34$ or $-0.0001244t = \ln 0.34$ and so $t \approx$ (i) **7669.3** (ii) **8672.1** (iii) **9669.3**.

8. *Graphs of limited growth functions, $y = L - (L - y_0)e^{kt}$.*

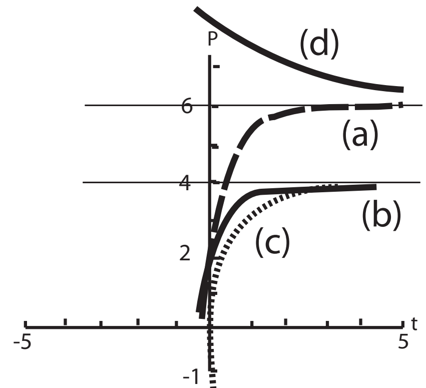

Figure 2.14 (Various limited growth models)

(Calculator: Use WINDOW -1 4 1 -1 8 1 1)

(a) Logarithm function $f(x) = 4 - 4e^{-3t}$ corresponds to
graph (i) **(a)** (ii) **(b)** (iii) **(c)** (iv) **(d)**
with y-intercept $L - y_0 = 4 - 4 =$ (i) **0** (ii) **2** (iii) **4**

(b) Limited growth function $f(x) = 4 - 2e^{-3t}$ corresponds to
graph (i) **(a)** (ii) **(b)** (iii) **(c)** (iv) **(d)**
with y-intercept $L - y_0 = 4 - 2 =$ (i) **0** (ii) **2** (iii) **4**

(c) Logarithm function $f(x) = 6 - 4e^{-3t}$ corresponds to
graph (i) **(a)** (ii) **(b)** (iii) **(c)** (iv) **(d)**
with y-intercept $L - y_0 = 6 - 4 =$ (i) **0** (ii) **2** (iii) **4**

(d) If limit is greater than initial value, $L > y_0$, L is a(n) (i) **upper** (ii) **lower** limit, otherwise it is a lower limit.

9. *Little green men, k known and t unknown.* Little green men can only grow to a maximum height of 4.2 feet tall. If Tim (an alien) is now 2.4 feet tall, how tall will he be 5 years from now, if $k = -0.4$? Assume initial height is $y_0 = 0$.

(a) The y in $y = L - (L - y_0)e^{kt}$ describes the
(i) **growth rate** (ii) **height** (iii) **height** (iv) **maximum height**
of little green men given by (i) **2.4** (ii) **3.7** (iii) **4.2** feet tall.

(b) The L in $y = L - (L - y_0)e^{kt}$ describes the
(i) **growth rate** (ii) **height** (iii) **height** (iv) **maximum height**
of little green men given by (i) **2.4** (ii) **3.7** (iii) **4.2** feet tall.

(c) Since $y = 2.4$, $y = 4.2 - 4.2e^{-0.4t} = 2.4$ or $4.2e^{-0.4t} = 1.8$ or $e^{-0.4t} = \frac{1.8}{4.2}$ or $\ln e^{-0.4t} = \ln\left(\frac{1.8}{4.2}\right)$, so $-0.4t = \frac{\ln(1.8/4.2)}{-0.4}$, so $t \approx$ (i) **1.9** (ii) **2.1** (iii) **2.5**

(d) so, 5 years from now, in other words, at $t = 2.1 + 5 = 7.1$ years,
$y = L - (L - y_0)e^{kt} = 4.2 - 4.2e^{-0.4\times 7.1} \approx$ (i) **3.8** (ii) **4.0** (iii) **4.2**.

10. *Advertising Car Parts, k and t unknown.* After 10 days, 40% of 24000 viewers of a local TV station had seen an advertisement on car parts. How long must the advertisement air to reach 80% of the station's viewers? Use the limited growth model.

   (a) Since no one sees the advertisement before it airs, the initial number of viewers must be $y_0 =$ (i) **0** (ii) **9600** (iii) **19200** (iv) **24000**.

   (b) Since maximum number of viewers is 24,000,
   $L =$ (i) **0** (ii) **9600** (iii) **19200** (iv) **24000**.

   (c) Since 40% of viewers ($0.4(24000) = 9600$ viewers) see the ad after $t = 10$ days, $y = 24000 - 24000e^{k(10)} =$ (i) **0** (ii) **9600** (iii) **19200** (iv) **24000**.

   (d) Rewrite $24000 - 24000e^{k(10)} = 9600$ as $e^{10k} = \frac{14400}{24000}$, or $10k = \ln\left(\frac{14400}{24000}\right)$ and so $k \approx$ (i) $\mathbf{-0.05108}$ (ii) $\mathbf{-0.08244}$.

   (e) Since we are interested at what time t the advertisement reaches 80% of the station's viewers ($0.8(24000) = 19200$ viewers),
   $y = 24000 - 24000e^{-0.05108t} =$ (i) **0** (ii) **9600** (iii) **19200** (iv) **24000**.

   (f) Rewrite $24000 - 24000e^{-0.05108t} = 19200$ as $e^{-0.05108k} = \frac{4800}{24000}$, or $-0.05108t = \ln\left(\frac{4800}{24000}\right)$ and so $t \approx$ (i) **29.5** (ii) **30.5** (iii) **31.5** days.

11. $r_E = \left(a + \frac{r}{m}\right)^m - 1.$
Which is larger: 10% compounded monthly or 10.2% compounded quarterly?

   (a) After 1 year, \$1 invested 10% compounded *monthly*, $A = P\left(1 + \frac{r}{m}\right)^{mt} =$
   $1\left(1 + \frac{0.10}{12}\right)^{12(1)} \approx$ (i) **1.084713** (ii) **1.094713** (iii) **1.104713**
   Calculator: $1 * (1 + 0.10/12) \wedge (12)$
   so *interest* earned in one year is this amount subtract \$1,
   $r_E = \left(1 + \frac{r}{m}\right)^m - 1 \approx$ (i) **0.1047** (ii) **0.1147** (iii) **0.1247** or 10.47%
   Calculator: $(1 + 0.10/12) \wedge (12) - 1$

   (b) After 1 year, \$1 invested 10.2% compounded *quarterly*, $A = P\left(1 + \frac{r}{m}\right)^{mt} =$
   $1\left(1 + \frac{0.102}{4}\right)^{4(1)} \approx$ (i) **1.085968** (ii) **1.095968** (iii) **1.105968**
   Calculator: $(1 + 0.102/4) \wedge (4)$
   so *interest* earned in one year,
   $r_E = \left(1 + \frac{r}{m}\right)^m - 1 =$ (i) **0.10597** (ii) **0.11597** (iii) **0.12597** or 10.60%
   Calculator: $(1 + 0.102/4) \wedge (4) - 1$

   (c) Consequently, 10% compounded monthly (r_E: 10.47%) is
   (i) **less** (ii) **more** than 10.2% compounded quarterly (r_E: 10.60%).