Chapter 8

Further Techniques and Applications of Integration

8.1 Integration by Parts

We look at an integration technique called *integration by parts*, where

$$\int u\,dv = uv - \int v\,du,$$

which often simplifies complicated expressions. The integration by parts integration technique is related to the product rule in differentiation. Standard and column methods are used to integrate by parts. Try the method of substitution and other techniques before trying integration by parts or try mixing these previous methods with the integration by parts. Integration by parts only works if a number of conditions are satisfied: integrand can be written as $u \cdot dv$, dv can be integrated to get v, u can be differentiated to get du and $\int v\,du$ can be found. Sometimes, online programs, such as at the link

http://integrals.wolfram.com,

or the following tables (found in Appendices of text) are required to solve integrals.

1. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$

2. $\int e^{kx}\,dx = \frac{1}{k} \cdot e^{kx} + C$

3. $\int \frac{a}{x}, dx = a \ln|x| + C$

4. $\int \ln|ax|\,dx = x(\ln|ax| - 1) + C$

5. $\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln|x + \sqrt{x^2 + a^2}| + C$

6. $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln|x + \sqrt{x^2 - a^2}| + C$

7. $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a} \cdot \ln\left|\frac{a+x}{a-x}\right| + C,\ a \neq 0$

8. $\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a} \cdot \ln\left|\frac{x-a}{x+a}\right| + C,\ a \neq 0$

9. $\int \frac{1}{x\sqrt{a^2-x^2}}\,dx = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right| + C,\ 0 < x < a$

10. $\int \frac{1}{x\sqrt{a^2+x^2}}\,dx = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2+x^2}}{x}\right| + C,\ a \neq 0$

11. $\int \frac{x}{ax+b}\,dx = \frac{x}{a} - \frac{b}{a^2}\ln|ax+b| + C,\ a \neq 0$

12. $\int \frac{x}{(ax+b)^2}\,dx = \frac{b}{a^2(ax+b)} + \frac{1}{a^2} \cdot \ln|ax+b| + C,\ a \neq 0$

13. $\int \frac{1}{x(ax+b)}\,dx = \frac{1}{b} \cdot \ln\left|\frac{x}{ax+b}\right| + C,\ b \neq 0$

14. $\int \frac{1}{x(ax+b)^2}\,dx = \frac{1}{b(ax+b)} + \frac{1}{b} \cdot \ln\left|\frac{x}{ax+b}\right| + C,\ b \neq 0$

15. $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2} \cdot \ln\left|x+\sqrt{x^2+a^2}\right| + C$

16. $\int x^n \ln x\,dx = x^{n+1}\left[\frac{\ln|x|}{n+1} - \frac{1}{(n+1)^2}\right] + C,\ n \neq -1$

17. $\int x^n e^{ax}\,dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \cdot \int x^{n-1}e^{ax} + C,\ a \neq 0$

Exercise 8.1 (Integration by Parts)

1. *Find* $\int xe^x\,dx$.

   (a) *Method 1.* Let

   $$\int xe^x\,dx = \int u\,dv$$

   and then *guess* $u = x$ and $dv = e^x\,dx$,

   so $\underset{\text{differentiate } u = x}{du} =$ (i) $\boldsymbol{2\,(3x+5)\,(3)\,dx}$ (ii) $\boldsymbol{(8x+4)\ dx}$ (iii) $\boldsymbol{dx}$

   and $\underset{\text{integrate } dv = e^x\,dx}{v} =$ (i) $\boldsymbol{e^x}$ (ii) $\boldsymbol{4 \cdot \frac{1}{3}x^3 + 5x}$ (iii) $\boldsymbol{2x+4}$.

   so, summarizing,

   $$\begin{array}{ll} u = x & v = e^x \\ du = dx & dv = e^x\,dx \end{array}$$

and so $uv - \int v\,du =$
(i) $\boldsymbol{(x)\,(e^x) - \int (3x+5)\,3\,dx}$
(ii) $\boldsymbol{(x)\,(e^x) - \int (x^2+4x)\,(2x+4)\ dx}$
(iii) $\boldsymbol{(x)\,(e^x) - \int (e^x)\ dx}$

but $\int (e^x)\ dx = e^x + C$ and so, combining, $uv - \int v\,du =$
(i) $\boldsymbol{(x)\,(e^x) - (3x+5)\,3 + C}$
(ii) $\boldsymbol{(x)\,(e^x) - (x^2+4x)\,(2x+4) + C}$
(iii) $\boldsymbol{(x)\,(e^x) - e^x + C}$

(b) *Method 2.* Let

$$\int xe^x\,dx = \int u\,dv$$

where we guess $u = x$ and $dv = e^x\,dx$,
and construct following table (D is differentiation, I is integration),

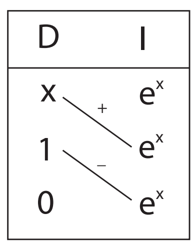

Figure 8.1 (Column integration, $\int xe^x\,dx$)

in column D, *successive differentiation* of guess $u = f(x) = x$ gives
guess $f(x) = x$, then $\frac{d}{dx}x = 1 \cdot x^{1-1} = 1$ and finally $\frac{d}{dx}1 = 0$ (1 is a constant)

in column I, *successive integration* of guess $dv = f(x) = e^x$ gives
guess $f(x) = e^x$, then $\int e^x\,dx = e^x$ and finally $\int e^x\,dx = e^x$

then, multiplying items in columns as indicated (with indicated signs)

$$\int xe^x\,dx = +x \cdot e^x - 1 \cdot e^x + C =$$

(i) $\boldsymbol{(x)\,(e^x) - (3x+5)\,3 + C}$
(ii) $\boldsymbol{(x)\,(e^x) - (x^2+4x)\,(2x+4) + C}$
(iii) $\boldsymbol{(x)\,(e^x) - e^x + C}$

2. *Find* $\int (3x+5)e^x\,dx$.

(a) *Method 1.* Let

$$\int (3x+5)e^x\,dx = \int u\,dv$$

then guess $u = 3x + 5$ and $dv = e^x\,dx$.

so $du =$ (i) $\mathbf{2\,(3x+5)\,(3)\,dx}$ (ii) $\mathbf{(8x+4)\ dx}$ (iii) $\mathbf{3\,dx}$
differentiate $u = 3x + 5$

$v =$ (i) $\boldsymbol{e^x}$ (ii) $\mathbf{4 \cdot \frac{1}{3}x^3 + 5x}$ (iii) $\mathbf{2x + 4}$
integrate $dv = e^x\,dx$

so, summarizing,

$$\begin{array}{ll} u = 3x+5 & v = e^x \\ du = 3\,dx & dv = e^x\,dx \end{array}$$

and so $uv - \int v\,du =$
(i) $\mathbf{(3x+5)\,(e^x) - \int (3x+5)\,3\,dx}$
(ii) $\mathbf{(3x+5)\,(e^x) - \int (x^2+4x)\,(2x+4)\ dx}$
(iii) $\mathbf{(3x+5)\,(e^x) - \int (e^x)\,3\,dx}$

but $\int (e^x)\,3\,dx = 3e^x + C$ and so, combining, $uv - \int v\,du =$
(i) $\mathbf{(3x+5)\,(e^x) - (3x+5)\,3 + C}$
(ii) $\mathbf{(3x+5)\,(e^x) - (x^2+4x)\,(2x+4) + C}$
(iii) $\mathbf{(3x+5)\,(e^x) - 3e^x + C}$

(b) *Method 2.* Let

$$\int (3x+5)e^x\,dx = \int u\,dv$$

where we guess $u = 3x + 5$ and $dv = e^x\,dx$,

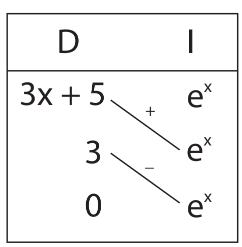

Figure 8.2 (Column integration, $\int(3x+5)e^x\,dx$)

in column D, *successive differentiation* of guess $u = f(x) = x$ gives
guess $f(x) = 3x+5$, then $\frac{d}{dx}(3x+5) = 3 \cdot x^{1-1} + 0 = 3$ and finally $\frac{d}{dx}1 = 0$

in column I, *successive integration* of guess $dv = f(x) = e^x$ gives
guess $f(x) = e^x$, then $\int e^x\,dx = e^x$ and finally $\int e^x\,dx = e^x$

then, multiplying items in columns as indicated (with indicated signs)

$$\int (3x+5)e^x\,dx = +(3x+5) \cdot e^x - 3 \cdot e^x + C =$$

(i) $\boldsymbol{(3x+5)\,(e^x) - (3x+5)\,3 + C}$
(ii) $\boldsymbol{(3x+5)\,(e^x) - (x^2+4x)\,(2x+4) + C}$
(iii) $\boldsymbol{(3x+5)\,(e^x) - 3e^x + C}$

3. *Find* $\int x^2 e^x\,dx$, *repeated integration by parts.*

   (a) *Method 1.*

      i. *First Integration By Parts.* Let

$$\int x^2 e^x\,dx = \int u\,dv$$

where we guess $u = x^2$ and $dv = e^x\,dx$.
Differentiating u,
$du =$ (i) $\boldsymbol{2\,(3x+5)\,(3)\,dx}$ (ii) $\boldsymbol{(8x+4)\ dx}$ (iii) $\boldsymbol{2x\,dx}$
and integrating dv, $v =$ (i) $\boldsymbol{e^x}$ (ii) $\boldsymbol{4\cdot\frac{1}{3}x^3+5x}$ (iii) $\boldsymbol{2x+4}$
so, summarizing,

$$\begin{array}{ll} u = x^2 & v = e^x \\ du = 2x\,dx & dv = e^x\,dx \end{array}$$

and so $uv - \int v\,du =$
(i) $\boldsymbol{(x^2)\,(e^x) - \int (3x+5)\,3\,dx}$
(ii) $\boldsymbol{(x^2)\,(e^x) - \int (x^2+4x)\,(2x+4)\ dx}$
(iii) $\boldsymbol{(x^2)\,(e^x) - \int (e^x)\,2x\,dx}$

      ii. *Second Integration By Parts.* Let

$$\int 2xe^x\,dx = \int u\,dv$$

where we guess $u = 2x$ and $dv = e^x\,dx$.
Differentiating u, $du =$
(i) $\boldsymbol{(3x+5)\ dx}$ (ii)$\boldsymbol{2\,(3x+5)\,(3)\ dx}$ (iii) $\boldsymbol{2\,dx}$
integrating dv, $v =$ (i) $\boldsymbol{\frac{1}{3}x^3+4\cdot\frac{1}{2}x^2}$ (ii) $\boldsymbol{2\cdot\frac{1}{2}x^2+4x}$ (iii) $\boldsymbol{e^x}$
so, summarizing,

$$\begin{array}{ll} u = 2x & v = e^x \\ du = 2\,dx & dv = e^x\,dx \end{array}$$

and so $uv - \int v\,du =$
(i) $\boldsymbol{(2x)\,(e^x) - \int (3x+5)\,3\,dx}$
(ii) $\boldsymbol{(2x)\,(e^x) - \int (x^2+4x)\,(12x+30)\ dx}$
(iii) $\boldsymbol{(2x)\,(e^x) - \int (e^x)\,2\,dx}$
where $\int (e^x)\,2\,dx =$ (choose one or more)
(i) $\boldsymbol{2\int (e^x)\ dx}$ (ii) $\boldsymbol{2e^x + C}$ (iii) $\boldsymbol{x^4+8x^3+C}$

and so $(2x)(e^x) - \int (e^x) 2\, dx =$
(i) $\boldsymbol{(2x)(e^x) - (3x+5) + C}$
(ii) $\boldsymbol{(2x)(e^x) - (x^4 + 8x^3) + C}$
(iii) $\boldsymbol{(2x)(e^x) - 2e^x + C}$

iii. *Combining Two Integrations.*
and so $(x^2)(e^x) - \int (e^x) 2x\, dx =$
(i) $\boldsymbol{x^2e^x - 2xe^x + 2e^x + C}$
(ii) $\boldsymbol{x^2e^x - 4xe^x + 4e^x + C}$
(iii) $\boldsymbol{x^2e^x - 2xe^x + 4e^x + C}$

(b) *Method 2.* Let

$$\int x^2 e^x\, dx = \int u\, dv$$

where we guess $u = x^2$ and $dv = e^x\, dx$,

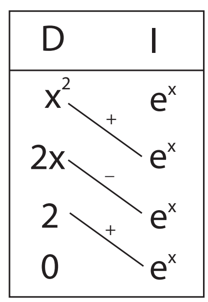

Figure 8.3 (Column integration, $\int x^2 e^x\, dx$)

in column D, successive differentiation of guess $u = f(x) = x^2$ gives
guess $f(x) = x^2$, then $\frac{d}{dx}x^2 = 2 \cdot x^{2-1} = 2x$, $\frac{d}{dx}2x = 2$ and finally $\frac{d}{dx}2 = 0$

in column I, successive integration of guess $dv = f(x) = e^x$ gives
guess $f(x) = e^x$, then $\int e^x\, dx = e^x$ and $\int e^x\, dx = e^x$ and $\int e^x\, dx = e^x$

then, multiplying items in columns as indicated (with indicated signs)

$$\int x^2 e^x\, dx = x^2 \cdot e^x - 2x \cdot e^x + 2 \cdot e^x + C =$$

(i) $\boldsymbol{x^2e^x - 2xe^x + 2e^x + C}$
(ii) $\boldsymbol{x^2e^x - 4xe^x + 4e^x + C}$
(iii) $\boldsymbol{x^2e^x - 2xe^x + 4e^x + C}$

4. *Find* $\int \ln x\, dx$.

(a) *Method 1.* Let

$$\int \ln x \, dx = \int u \, dv$$

then guess $u = \ln x$ and $dv = dx$.

differentiating u,
$du =$ (i) $\mathbf{2\,(3x+5)\,(3)\,dx}$ (ii) $\mathbf{(8x+4)\;dx}$ (iii) $\boldsymbol{x^{-1}\,dx}$
and integrating dv,
$v =$ (i) $\boldsymbol{e^x}$ (ii) $\boldsymbol{4 \cdot \frac{1}{3}x^3 + 5x}$ (iii) $\boldsymbol{x}$

so, summarizing,

$$\begin{array}{ll} u = \ln x & v = x \\ du = x^{-1}\,dx & dv = dx \end{array}$$

and so $uv - \int v\,du =$
(i) $\boldsymbol{(\ln x)\,(x) - \int x \cdot x^{-1}\,dx}$
(ii) $\boldsymbol{(\ln x)\,(x) - \int (x^2+4x)\,(2x+4)\;dx}$
(iii) $\boldsymbol{(\ln x)\,(x) - \int (e^x)\;dx}$

but $\int x \cdot x^{-1}\,dx = \int\, dx = x + C$ and so, combining, $uv - \int v\,du =$
(i) $\boldsymbol{(\ln x)\,(x) - x + C}$
(ii) $\boldsymbol{(\ln x)\,(x) - \int (x^2+4x)\,(2x+4)\;dx}$
(iii) $\boldsymbol{(\ln x)\,(x) - \int (e^x)\;dx}$

(b) *Method 2 variation for* $\ln x$. Let

$$\int \ln x \, dx = \int u \, dv$$

where we guess $u = \ln x$ and $dv = dx$,

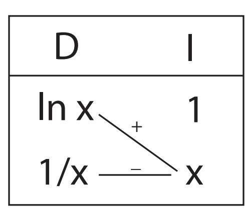

Figure 8.4 (Column integration, $\int \ln x\,dx$)

in column D, successive differentiation of guess $u = f(x) = x$ gives
guess $f(x) = \ln x$, then $\frac{d}{dx} \ln x = \frac{1}{x}$
stop here because natural logarithm, $\ln x$, is gone

in column I, successive integration of guess $dv = f(x) = 1$ gives
guess $f(x) = 1$, then $\int 1\,dx = x$

then, multiplying items in columns as indicated (with indicated signs)
also, horizontal line in table indicates an integration of, in this case, $-\frac{1}{x} \cdot x$

$$\int \ln x \, dx = \ln x \cdot x - \int \frac{1}{x} \cdot x \, dx + C = \ln x \cdot x - \int 1 \, dx + C =$$

(i) $\boldsymbol{(\ln x)\,(x) - x + C}$
(ii) $\boldsymbol{(\ln x)\,(x) - \int (x^2 + 4x)\,(2x+4)\ dx}$
(iii) $\boldsymbol{(\ln x)\,(x) - \int (e^x)\ dx}$

5. *Find* $\int_0^3 x^3 \ln 2x \, dx$. Let

$$\int x^3 \ln 2x \, dx = \int u \, dv$$

where we guess $u = \ln 2x$ and $dv = x^3 dx$,

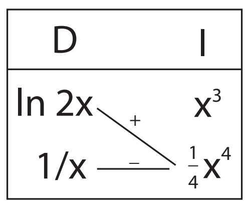

Figure 8.5 (Column integration, $\int x^3 \ln x \, dx$)

in column D, successive differentiation of guess $u = f(x) = x$ gives
guess $f(x) = \ln 2x$, then $\frac{d}{dx} \ln 2x = \frac{1}{2x} \cdot 2 =$ (i) $\boldsymbol{\frac{1}{x}}$ (ii) $\boldsymbol{x}$
stop here because natural logarithm, $\ln x$, is gone

in column I, successive integration of guess $dv = f(x) = x^3$ gives
guess $f(x) = x^3$, then $\int x^3 \, dx = \frac{1}{3+1} x^{3+1} =$ (i) $\boldsymbol{\frac{4}{x^4}}$ (ii) $\boldsymbol{\frac{x^4}{4}}$

then, multiplying items in columns as indicated (with indicated signs)
also, horizontal line in table indicates an integration of, in this case, $-\frac{1}{x} \cdot \frac{1}{4}x^4$

$$\int x^3 \ln 2x \, dx = \ln 2x \cdot \frac{1}{4}x^4 - \int \frac{1}{x} \cdot \frac{1}{4}x^4 \, dx + C = \frac{1}{4}x^4 \ln 2x - \frac{1}{4}\int x^3 \, dx + C =$$

(i) $\boldsymbol{\frac{1}{4}x^4 \ln x + \frac{1}{16}x^4 + C}$ (ii) $\boldsymbol{\frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C}$ (iii) $\boldsymbol{\frac{1}{4}x^4 \ln x - \frac{1}{4}x^4 + C}$

so

$$\int_0^3 x^3 \ln x \, dx = \left[\frac{1}{4}x^4 \ln 2x - \frac{1}{16}x^4\right]_0^3 \approx$$

(i) **25.30** (ii) **31.22** (iii) **37.30**

6. *Find* $\int x^3 e^{-3x}\, dx$. Let
$$\int x^3 e^{-3x}\, dx = \int u\, dv$$
where we guess $u = x^3$ and $dv = e^{-3x} dx$,

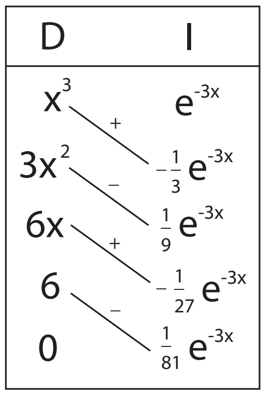

Figure 8.6 (Column integration, $\int x^3 e^{-3x}\, dx$)

so multiplying items in columns as indicated (with indicated signs)

$$\int x^3 e^{-3x}\, dx = x^3 \cdot \left(-\frac{e^{-3x}}{3}\right) - 3x^2 \cdot \left(\frac{e^{-3x}}{9}\right) + 6x \cdot \left(-\frac{e^{-3x}}{27}\right) - 6 \cdot \left(\frac{e^{-3x}}{81}\right) + C =$$

(i) $\boldsymbol{e^{-3x}\left(\frac{x^3}{3} + \frac{3x^2}{9} + \frac{6x}{27} + \frac{6}{81}\right) + C}$
(ii) $\boldsymbol{3e^{-3x}\left(-\frac{x^3}{3} - \frac{3x^2}{9} - \frac{6x}{27} - \frac{6}{81}\right) + C}$
(iii) $\boldsymbol{e^{-3x}\left(-\frac{x^3}{3} - \frac{3x^2}{9} - \frac{6x}{27} - \frac{6}{81}\right) + C}$

7. *Find* $\int \frac{1}{x(4-3x)^2}\, dx$, *using integration table.*

From formula 14 above,
$$\int \frac{1}{x(ax+b)^2}\, dx = \frac{1}{b(ax+b)} + \frac{1}{b} \ln\left|\frac{x}{ax+b}\right| + C$$
and since
$$\int \frac{1}{x(4-3x)^2}\, dx = \int \frac{1}{x(-3x+4)^2}\, dx$$
where $a = -3$ and $b =$ (i) $\mathbf{-3}$ (ii) $\mathbf{2}$ (iii) $\mathbf{4}$

then
$$\int \frac{1}{x(4-3x)^2}\, dx =$$

(i) $\frac{1}{-3(4x-3)} + \frac{1}{-3} \ln\left(\frac{x}{4x-3}\right) + C$
(ii) $\frac{1}{4(-3x+4)} + \frac{1}{-3} \ln\left(\frac{x}{-3x+4}\right) + C$
(iii) $\frac{1}{4(-3x+4)} + \frac{1}{4} \ln\left(\frac{x}{-3x+4}\right) + C$.

8. *Find* $\int x^{-1}(5-x^2)^{-1/2}\, dx$, *using integration table*

   From formula 9 above,

$$\int \frac{1}{x\sqrt{a^2-x^2}}\, dx = -\frac{1}{a} \ln\left(\frac{a+\sqrt{a^2-x^2}}{x}\right) + C, 0 < x < a$$

   and since

$$\int x^{-1}\left(5-x^2\right)^{-1/2} dx = \int \frac{1}{x\sqrt{\left(\sqrt{5}\right)^2 - x^2}}\, dx$$

   then $a =$ (i) -5 (ii) $\sqrt{5}$ (iii) $-\sqrt{5}$
   a cannot be $-\sqrt{5}$ because a must be positive, $a > 0$

   then

$$\int x^{-1}\left(5-x^2\right)^{-1/2} dx =$$

   (i) $-\frac{1}{\sqrt{5}} \ln\left(\frac{\sqrt{5}+\sqrt{\sqrt{5}-x^2}}{x}\right) + C$
   (ii) $-\frac{1}{\sqrt{5}} \ln\left(\frac{\sqrt{5}+\sqrt{5-x^2}}{x}\right) + C$
   (iii) $-\frac{1}{5} \ln\left(\frac{5+\sqrt{5-x^2}}{x}\right) + C$

9. *Find* $\int \frac{1}{17-x^2}\, dx$, *using integration table*

   From formula (i) **7** (ii) **8** (iii) **9**

   where $a =$ (i) x^2 (ii) $\sqrt{17}$ (iii) 17

   and so

$$\int \frac{1}{17-x^2}\, dx = \frac{1}{2a} \cdot \ln\left|\frac{a+x}{a-x}\right| + C =$$

   (i) $\frac{1}{2\sqrt{5}} \ln\left|\frac{\sqrt{5}+x}{\sqrt{5}-x}\right| + C$
   (ii) $\frac{1}{2\sqrt{17}} \ln\left|\frac{\sqrt{17}+x}{\sqrt{17}-x}\right| + C$
   (iii) $\frac{1}{2(17)} \ln\left|\frac{17+x}{17-x}\right| + C$.

10. Find

$$\int_2^8 u\,dv$$

if

$$\int_2^8 v\,du = 15$$

and

x	$u(x)$	$v(x)$
2	7	4
8	-6	4

The answer is

$$\int_2^8 u\,dv = [uv]_2^8 - \int_2^8 v\,du = [(-6)(4) - (7)(4)] - 15 =$$

(i) $\mathbf{-65}$ (ii) $\mathbf{-66}$ (iii) $\mathbf{-67}$

8.2 Volume and Average Value

The *volume of a solid of revolution* is obtained by rotating the area under nonnegative $f(x)$ over $[a, b]$ on x-axis:

$$V = \lim_{\Delta \to 0} \sum_{i=1}^{n} \pi [f(x_i)]^2 \Delta x = \int_a^b \pi [f(x)]^2\,dx$$

The *average value of a function* $f(x)$ on interval $[a, b]$ is

$$\frac{1}{b-a} \int_a^b f(x)\,dx$$

Exercise 8.2 (Volume and Average Value)

1. *Volumes of cylinders*

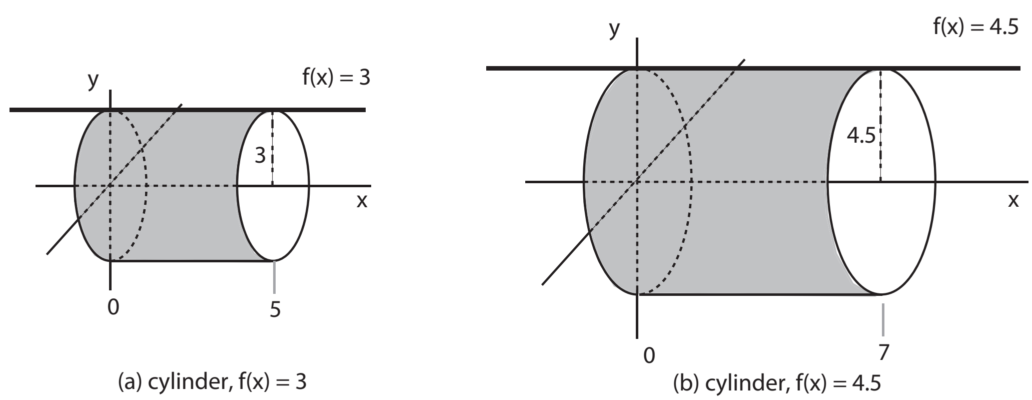

Figure 8.7 (Volumes of cylinders)

(a) *Figure(a): $f(x) = 3$, cylinder with radius 3 and length 5.* Volume of a solid of revolution obtained by rotating area under $f(x) = 3$ over $[0, 5]$:

$$\begin{aligned}\int_a^b \pi[f(x)]^2\,dx &= \int_0^5 \pi[3]^2\,dx \\ &= \int_0^5 \pi[9]\,dx \\ &= [9\pi x]_{x=0}^{x=5} \\ &= 9\pi(5) - 9\pi(0) \ =\end{aligned}$$

(i) $\boldsymbol{9\pi}$ (ii) $\boldsymbol{18\pi}$ (iii) $\boldsymbol{45\pi}$

Also, geometrically, volume of cylinder area of base × length $= \pi r^2 \times l = \pi(3)^2 \times 5 = 45\pi$;

also notice length of 5 implies limits of integration from 0 to 5.

(b) *Figure(b): $f(x) = 4.5$, cylinder with radius 4.5 and length 7.*

$$\begin{aligned}\int_a^b \pi[f(x)]^2\,dx &= \int_0^7 \pi[4.5]^2\,dx \\ &= \int_0^7 \pi[20.25]\,dx \\ &= [20.25\pi x]_{x=0}^{x=7} \\ &= 20.25\pi(7) - 20.25\pi(0) \ =\end{aligned}$$

(i) $\boldsymbol{90.25\pi}$ (ii) $\boldsymbol{133.25\pi}$ (iii) $\boldsymbol{141.75\pi}$

(c) $f(x) = 8$, $-3 \leq x \leq 12$.

$$\begin{aligned}\int_a^b \pi[f(x)]^2\,dx &= \int_{-3}^{12} \pi[8]^2\,dx \\ &= \int_{-3}^{12} \pi[64]\,dx \\ &= [64\pi x]_{x=-3}^{x=12} \\ &= 64\pi(12) - 64\pi(-3) \ =\end{aligned}$$

(i) $\boldsymbol{960\pi}$ (ii) $\boldsymbol{1050\pi}$ (iii) $\boldsymbol{1200\pi}$

2. *Volumes of cones*

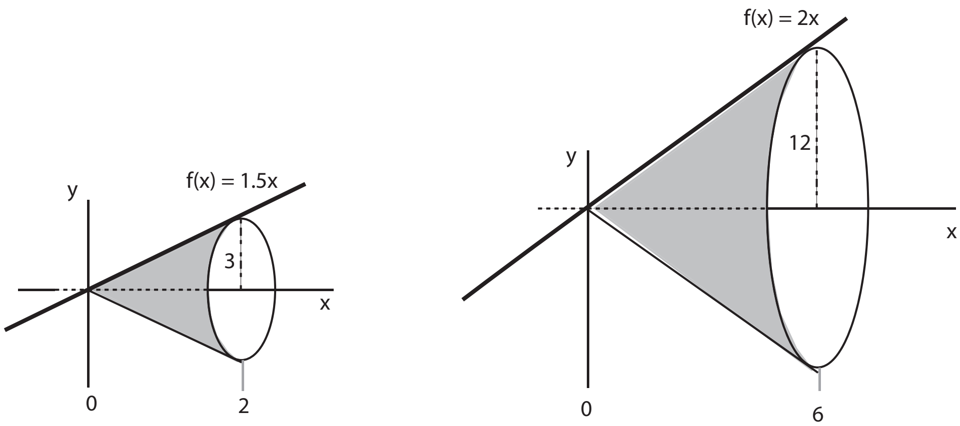

Figure 8.8 (Volumes of cones)

(a) *Figure(a): $f(x) = 1.5x$, cone with radius 3 and length 2.* Volume of a solid of revolution obtained by rotating area under $f(x) = 1.5x$ over $[0, 2]$:

$$\begin{aligned}
\int_a^b \pi[f(x)]^2\,dx &= \int_0^2 \pi[1.5x]^2\,dx \\
&= \int_0^2 \left[\pi(1.5)^2x^2\right]\,dx \\
&= \int_0^2 \left[\pi(2.25)x^2\right]\,dx \\
&= \left[\pi(2.25)\frac{1}{3}x^3\right]_0^2 \\
&= \frac{2.25\pi}{3}(2)^3 - \frac{2.25\pi}{3}(0)^3 =
\end{aligned}$$

(i) $\boldsymbol{3\pi}$ (ii) $\boldsymbol{6\pi}$ (iii) $\boldsymbol{9\pi}$

Geometrically, volume of cone is $\frac{1}{3}\pi r^2 l = \frac{1}{3}\pi(3)^2(2) = 6\pi$;

notice radius, $r = 3$, required for this formula, but not volume of a solid of revolution integration.

(b) *Figure(b): $f(x) = 2x$, cone with length 6.*

$$\begin{aligned}
\int_a^b \pi[f(x)]^2\,dx &= \int_0^6 \pi[2x]^2\,dx \\
&= \int_0^6 \left[\pi(2)^2x^2\right]\,dx \\
&= \int_0^6 \left[\pi(4)x^2\right]\,dx \\
&= \left[\pi(4)\frac{1}{3}x^3\right]_0^6 \\
&= \frac{4\pi}{3}(6)^3 - \frac{4\pi}{3}(0)^3 =
\end{aligned}$$

(i) $\boldsymbol{286\pi}$ (ii) $\boldsymbol{287\pi}$ (iii) $\boldsymbol{288\pi}$

(c) $f(x) = 3x$, $[0, 10]$.

$$\begin{aligned}\int_a^b \pi[f(x)]^2\,dx &= \int_0^{10} \pi[3x]^2\,dx \\ &= \int_0^{10} \left[\pi(3)^2x^2\right]\,dx \\ &= \int_0^{10} \left[\pi(9)x^2\right]\,dx \\ &= \left[\pi(9)\frac{1}{3}x^3\right]_0^{10} \\ &= \frac{9\pi}{3}(10)^3 - \frac{9\pi}{3}(0)^3 \;=\end{aligned}$$

(i) $\boldsymbol{1000\pi}$ (ii) $\boldsymbol{2000\pi}$ (iii) $\boldsymbol{3000\pi}$

(d) $f(x) = -3x$, $[0, 10]$.

Volume of a solid of revolution integration *not* used because
(i) $f(x) = -3x$ is **positive** over [0,10]
(ii) $f(x) = -3x$ is **nonnegative** over [0,10]
(iii) $f(x) = -3x$ is **nonpositive** over [0,10]
although, it is possible to figure out volume: $\left|\int_0^{10} \pi[-3x]^2\,dx\right| = \left|\int_0^{10} \left[\pi(3)^2x^2\right]\,dx\right| = 3000\pi$
similar to calculation of *area*, break integration up into subregions according to zeroes of function

3. *More volume of a solid of revolution integrations.*

(a) $f(x) = \sqrt{x}$, $[0, 9]$.

$$\begin{aligned}\int_a^b \pi[f(x)]^2\,dx &= \int_0^{9} \pi\left[\sqrt{x}\right]^2\,dx \\ &= \int_0^{9} \pi\,[x]\,dx \\ &= \int_0^{9} [\pi x]\,dx \\ &= \left[\pi\frac{1}{2}x^2\right]_0^{9} \\ &= \frac{\pi}{2}(9)^2 - \frac{\pi}{2}(0)^2 \;=\end{aligned}$$

(i) $\boldsymbol{\frac{18}{2}\pi}$ (ii) $\boldsymbol{\frac{81}{2}\pi}$ (iii) $\boldsymbol{\frac{115}{2}\pi}$

(b) $f(x) = \frac{1}{\sqrt{x}}$, $[1, 9]$.

$$\begin{aligned} \int_a^b \pi [f(x)]^2 \, dx &= \int_1^9 \pi \left[\frac{1}{x^{1/2}} \right]^2 dx \\ &= \int_1^9 \pi \left[x^{-1} \right] dx \\ &= \left[\pi \ln x \right]_1^9 \\ &= \pi \ln 9 - \pi \ln 1 \; = \end{aligned}$$

(i) $\boldsymbol{9\pi}$ (ii) $\boldsymbol{\pi \ln 9}$ (iii) $\boldsymbol{\pi \ln 8}$

recall, $\ln 1 = 0$

(c) $f(x) = x^2 + 5$, $[0, 5]$.

$$\begin{aligned} \int_a^b \pi [f(x)]^2 \, dx &= \int_0^5 \pi \left[x^2 + 5 \right]^2 dx \\ &= \int_0^5 \pi \left[x^4 + 10x^2 + 25 \right] dx \\ &= \pi \left[\frac{1}{5} x^5 + \frac{10}{3} x^3 + 25x \right]_0^5 \; = \end{aligned}$$

(i) $\boldsymbol{\frac{3500}{3}\pi}$ (ii) $\boldsymbol{\frac{4500}{3}\pi}$ (iii) $\boldsymbol{\frac{5500}{3}\pi}$

4. Find *average value* of $f(x) = x^2$ over [-3,4].

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(x) \, dx &= \frac{1}{4 - (-3)} \int_{-3}^4 x^2 \, dx \\ &= \frac{1}{7} \left[\frac{1}{3} x^3 \right]_{x=-3}^{x=4} \\ &= \frac{1}{7} \left(\frac{1}{3} (4)^3 - \frac{1}{3} (-3)^3 \right) = \end{aligned}$$

(i) **4.3** (ii) **30.3** (iii) **31.3**

MATH 9:fnInt($X^2, X, -3, 4$) ENTER, THEN DIVIDE BY 7!
OR MATH 9:fnInt($\frac{X^2}{7}, X, -3, 4$) ENTER

5. Find average value of $f(x) = x^2 - x^{-1}$ over [1,5].

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(x) \, dx &= \frac{1}{5-1} \int_1^5 \left(x^2 - x^{-1} \right) dx \\ &= \frac{1}{4} \left[\frac{1}{3} x^3 - \ln x \right]_{x=1}^{x=5} = \end{aligned}$$

(i) **7.94** (ii) **9.93** (iii) **39.72**

MATH 9:fnInt($X^2 + X^{-1}, X, 1, 5$) ENTER, then divide by 4.

6. Find average value of $f(x) = x^2 + x^{-1}$ over [1,15].

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)\,dx &= \frac{1}{15-1}\int_1^{15}\left(x^2+x^{-1}\right)\,dx \\ &= \frac{1}{14}\left[\frac{1}{3}x^3 + \ln x\right]_{x=1}^{x=15} =\end{aligned}$$

(i) **10.74** (ii) **30.33** (iii) **80.53**

7. Find average value of $f(x) = e^{3x}$ over [1,3].

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)\,dx &= \frac{1}{3-1}\int_1^{3}\left(e^{3x}\right)\,dx \\ &= \frac{1}{2}\left[\frac{1}{3}e^{3x}\right]_{x=1}^{x=3} =\end{aligned}$$

(i) **1074.33** (ii) **1347.17** (iii) **2694.33**

8. *Understanding average value, y_{av}.*

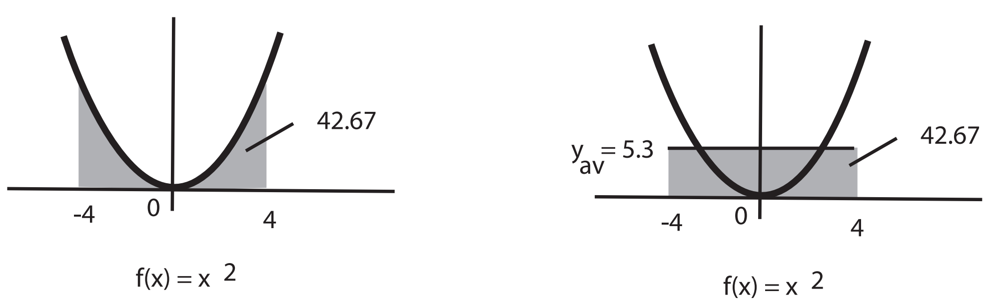

Figure 8.9 (Average value of $f(x) = x^2$)

(a) Find the average value of $f(x) = x^2$ over $[-4, 4]$.

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)\,dx &= \frac{1}{4-(-4)}\int_{-4}^{4} x^2\,dx \\ &= \frac{1}{8}\left[\frac{1}{3}x^3\right]_{x=-4}^{x=4} \\ &= \frac{1}{8}\left(\frac{1}{3}(4)^3 - \frac{1}{3}(-4)^3\right) =\end{aligned}$$

(i) **4.3** (ii) **5.3** (iii) **31.3**

MATH 9:fnInt($X^2, X, -4, 4$) ENTER, then divide by 8.

(b) The area under the $f(x) = x^2$ over $[-4, 4]$ is (i) the **same as** (ii) **different from** the area under $f(x) = 5.3$ over the $[-4, 4]$. It is as though the parabola in the graph on the left above, which is holding back the gray–shaded "water", is punctured. The level of which the water settles at, in the graph on the right above, is the average value, y_{av}.