Chapter 21

Two–Factor Studies–One Case Per Treatment

For various reasons (cost, time, material limitations, for instance), it is sometimes the case that only one case is observed per treatment. The one treatment case of a two–factor ANOVA is analyzed as a two–factor ANOVA with*out* an interaction term, where the error term is replaced by the interaction term. This makes sense if there is no interaction between the two factors. The Tukey test for additivity is used to test whether there is or is not a significant interaction term.

21.1 No–Interaction Model

SAS program: att4-21-1-prozac-onetreatANOVA

The (factor effects formulation of the) model, used for the one case per treatment situation, is

$$Y_{ij} = \mu_{\cdot\cdot} + \alpha_i + \beta_j + \varepsilon_{ij},$$

where

Y_{ij} is the value of the response variable
for the ith factor level of factor A
and the jth factor level of factor B

$\mu_{\cdot\cdot} = \frac{1}{ab}\sum_i \sum_j \mu_{ij}$ is the average of all cell mean parameters

$\alpha_i = \mu_{i\cdot} - \mu_{\cdot\cdot}$ main effect for factor A at $i$the level

$\beta_j = \mu_{\cdot j} - \mu_{\cdot\cdot}$ main effect for factor B at $j$the level

ε_{ij} are independent $N(0, \sigma^2)$

$i = 1, \ldots, r;\;\; j = 1, \ldots, n_i$

where, notice, in particular, there is *no* interaction term[1], $(\alpha\beta)_{ij}$ and, since there is only one (1) case per treatment, no k subscript (used to indicate the cases in each treatment). The following ANOVA table is used to make inferences on this model,

Source	Degrees of Freedom, df	Sum Of Squares, SS	Mean Squares, MS
Factor A	$a-1$	$SSA = b\sum(\overline{Y}_{i\cdot} - \overline{Y}_{\cdot\cdot})^2$	$MSTA = \frac{SSA}{a-1}$
Factor B	$b-1$	$SSB = a\sum(\overline{Y}_{\cdot j} - \overline{Y}_{\cdot\cdot})^2$	$MSTB = \frac{SSB}{b-1}$
Error (Interaction AB)	$(a-1)(b-1)$	$SSAB = \sum\sum(\overline{Y}_{ij} - \overline{Y}_{i\cdot} - \overline{Y}_{\cdot j} - \overline{Y}_{\cdot\cdot})^2$	$MSTAB = \frac{SSAB}{(a-1)(b-1)}$
Total	$ab-1$	$SSTO = \sum\sum(Y_{ijk} - \overline{Y}_{\cdot\cdot})^2$	

Exercise 21.1 (No–Interaction Model: Prozac Yield)
Consider the effect of different catalysts and blends on the Prozac yields from 10 milliliters of rice liquor.

	Factor B, blend →	1	2	3	4	5	
Factor A,	A	89	84	81	87	79	$\overline{Y}_{1\cdot} = 84$
catalyst	B	88	77	87	92	81	$\overline{Y}_{2\cdot} = 85$
	C	97	92	87	89	80	$\overline{Y}_{3\cdot} = 89$
	D	94	79	85	84	88	$\overline{Y}_{4\cdot} = 86$
		$\overline{Y}_{\cdot 1} = 92$	$\overline{Y}_{\cdot 2} = 83$	$\overline{Y}_{\cdot 3} = 85$	$\overline{Y}_{\cdot 4} = 88$	$\overline{Y}_{\cdot 5} = 82$	$\overline{Y}_{\cdot\cdot} = 86$

Notice, in particular, that there is only one case per treatment.

Source	df	SS	MS
Factor A (Catalyst)	3	70	23.33
Factor B (Blend)	4	264	66
Error (Interaction AB)	12	226	18.83
Total	19	560	

Test if either of the two main effects are significant at $\alpha = 0.05$.

1. *Preliminary Analysis: Treatment Means,* μ_{ij}
   Estimate the treatment mean μ_{ij} with

   $$\hat{\mu}_{ij} = \overline{Y}_{i\cdot} + \overline{Y}_{\cdot j} - \overline{Y}_{\cdot\cdot}$$

   rather than just Y_{ij}.

   (a) The treatment mean estimate of μ_{12} is

   $$\hat{\mu}_{12} = \overline{Y}_{1\cdot} + \overline{Y}_{\cdot 2} - \overline{Y}_{\cdot\cdot} = 84 + 83 - 86 =$$

   (choose one) **80 / 81 / 82**

[1] This model compares to the one *with* interaction,

$$Y_{ij} = \mu_{\cdot\cdot} + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ij},$$

(b) The treatment mean estimate of μ_{32} is

$$\hat{\mu}_{32} = \overline{Y}_{3\cdot} + \overline{Y}_{\cdot 2} - \overline{Y}_{\cdot\cdot} = 89 + 83 - 86 =$$

(choose one) **85** / **86** / **87**

(c) The treatment mean estimate of μ_{35} is

$$\hat{\mu}_{35} = \overline{Y}_{3\cdot} + \overline{Y}_{\cdot 5} - \overline{Y}_{\cdot\cdot} = 89 + 82 - 86 =$$

(choose one) **85** / **86** / **87**

(d) There are (choose one)
(choose one) **4** / **5** / **20**
treatment mean estimates.

2. *Treatment plots*
From SAS, the treatment means plots appear to show (choose one)

   (a) only factor A, the catalyst, is significant[2]
   (b) only factor B, the blend, is significant
   (c) only the AB (catalyst–blend) interaction is significant
   (d) catalyst, blend and catalyst–blend are all significant

3. *Implication of Significant Interaction*
Since the *interaction* appears to be significant, it (choose one) **does** / **does not** make sense to replace the error with the interaction factor[3] (and so be able to proceed with this analysis).

4. *Test Factor A (Catalyst), P–Value Versus Level of Significance.*

   (a) *Statement*
   The statement of the test is (check none, one or more):

      i. $H_0 : \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ versus $H_a : \alpha_1 \neq \alpha_2, \alpha_1 = \alpha_3$.
      ii. $H_0 : \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ versus $H_a : \alpha_1 \neq \alpha_3, \alpha_1 \neq \alpha_2$.
      iii. $H_0 : \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ versus
      $H_a :$ at least one $\alpha_i \neq 0,\ i = 1, 2, 3, 4$.
      iv. $H_0 : \mu_{1\cdot} = \mu_{2\cdot} = \mu_{3\cdot} = \mu_{4\cdot}$ versus
      $H_a :$ at least one $\mu_{i\cdot} \neq \mu_{j\cdot},\ i, j = 1, 2, 3, 4$.

[2] To say the catalyst is "significant" means the prozac yields are different for different catalysts or, equivalently, the prozac *effects* (differences between individual prozac yield and average prozac yield) are *not* all zero.

[3] We will look at the Tukey test for additivity in the next section.

(b) *Test*
Since the test statistic is $F = \frac{23.33}{18.83} = 1.24$, the p–value, with $a - 1 = 4 - 1 = 3$ and $ab = 4(3) = 12$ degrees of freedom, is given by

$$\text{p–value} = P(F \geq 1.24)$$

which equals (circle one) **0.00** / **0.34** / **0.43**.
(Use 2nd DISTR 9:Fcdf(1.24,E99,3,12).)
The level of significance is 0.05.

(c) *Conclusion*
Since the p–value, 0.34, is larger than the level of significance, 0.05, we (circle one) **accept** / **reject** the null hypothesis that the average Prozac yields for the four catalysts are the same.

5. *Test Factor B, P–Value Versus Level of Significance.*

(a) *Statement*
The statement of the test is (check none, one or more):

i. $H_0 : \beta_1 = \beta_2 = \beta_3 = 0$ versus $H_a : \beta_1 \neq \beta_2, \beta_1 = \beta_3$.
ii. $H_0 : \beta_1 = \beta_2 = \beta_3 = 0$ versus $H_a : \beta_1 \neq \beta_3, \beta_1 \neq \beta_2$.
iii. $H_0 : \beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = 0$ versus
$H_a :$ at least one $\beta_i \neq 0,\ i = 1, 2, 3$.
iv. $H_0 : \mu_{\cdot 1} = \mu_{\cdot 2} = \mu_{\cdot 3} = \mu_{\cdot 4} = \mu_{\cdot 5}$ versus
$H_a :$ at least one $\mu_{i\cdot} \neq \mu_{j\cdot},\ i, j = 1, 2, 3, 4, 5$.

(b) *Test*
Since the test statistic is $F = \frac{66}{18.83} = 3.50$, the p–value, with $b - 1 = 5 - 1 = 4$ and $ab = 4(3) = 12$ degrees of freedom, is given by

$$\text{p–value} = P(F \geq 3.50)$$

which equals (circle one) **0.04** / **0.35** / **0.51**.
(Use 2nd DISTR 9:Fcdf(3.50,E99,4,12).)
The level of significance is 0.05.

(c) *Conclusion*
Since the p–value, 0.04, is smaller than the level of significance, 0.05, we (circle one) **accept** / **reject** the null hypothesis that the average Prozac yields to the five different blends are the same.

6. *Kimball inequality*
An upper bound on the family level of significance is given by the Kimball inequality. Since $\alpha = 0.05$ for both factors A and B,

$$\alpha \leq 1 - (1 - 0.05)(1 - 0.05)$$

(choose one) **0.0975** / **0.1075** / **0.1175**

7. *Paired Bonferroni confidence intervals*
   Obtain confidence intervals for $D_1 = \mu_{\cdot 2} - \mu_{\cdot 1}$ and $D_2 = \mu_{2\cdot} - \mu_{1\cdot}$. Since[4]

$$\begin{aligned} s\left\{\hat{D}\right\} &= \sqrt{\frac{2MSE}{n}} \\ &= \sqrt{\frac{2(18.8333)}{2}} \\ &= 4.3397 \\ t(1-\alpha/2g;(a-1)(b-1)) &= t(1-0.05/(2(2));(4-1)(5-1)) \\ &= t(0.9875;12) \\ &= 2.5600 \end{aligned}$$

   then

pair	$\hat{D}$	$\hat{D} \pm t(1-\alpha/2g;(a-1)(b-1))s\left\{\hat{D}\right\}$
$D_1 = \mu_{\cdot 2} - \mu_{\cdot 1}$	-9	$-9 \pm 2.56(4.3397) = (-12.11, 10.11)$
$D_2 = \mu_{2\cdot} - \mu_{1\cdot}$	(a)	(b)

   where (a) = (choose one) $\mathbf{-1}$ / $\mathbf{0}$ / $\mathbf{1}$
   and (b) = (choose one) $\mathbf{(-10.11, 12.11)}$ / $\mathbf{(-19.11, 2.11)}$ / $\mathbf{(-20.11, 2.11)}$

8. *Regression version of model*
   The *factor effects* version of this two–factor ANOVA,

$$Y_{ij} = \mu_{\cdot\cdot} + \alpha_i + \beta_j + \varepsilon_{ij},$$

   can be written as a multiple *regression* model in the following way,

$$Y_{ijk} = \mu_{\cdot\cdot} + \alpha_1 X_{ij1} + \alpha_2 X_{ij2} + \alpha_3 X_{ij3} + \beta_1 X_{ij4} + \beta_2 X_{ij5} + \beta_3 X_{ij6} + \beta_4 X_{ij7} + \varepsilon_{ij}$$

   where $i = 1, 2, 3, 4$; $j = 1, 2, 3, 4, 5$ and where (*choose none, one or more!*)

   (i) variable 1

$$X_{ijk1} = \begin{cases} 1, & \text{if case from Factor A level 1} \\ -1, & \text{if case from Factor A level 4} \\ 0, & \text{otherwise,} \end{cases}$$

   (ii) variable 2

$$X_{ijk2} = \begin{cases} 1, & \text{if case from Factor A level 2} \\ -1, & \text{if case from Factor A level 4} \\ 0, & \text{otherwise,} \end{cases}$$

[4] The *MSE* can be found on the SAS output, $g = 2$ because we are calculating two confidence intervals, $a = 4$ levels of factor A (catalyst) and $b = 5$ levels of factor B (blend).

(iii) variable 3

$$X_{ijk3} = \begin{cases} 1, & \text{if case from Factor A level 3} \\ -1, & \text{if case from Factor A level 4} \\ 0, & \text{otherwise,} \end{cases}$$

(iv) variable 4

$$X_{ijk4} = \begin{cases} 1, & \text{if case from Factor B level 1} \\ -1, & \text{if case from Factor B level 5} \\ 0, & \text{otherwise,} \end{cases}$$

(v) variable 5

$$X_{ijk5} = \begin{cases} 1, & \text{if case from Factor B level 2} \\ -1, & \text{if case from Factor B level 5} \\ 0, & \text{otherwise,} \end{cases}$$

(vi) variable 6

$$X_{ijk6} = \begin{cases} 1, & \text{if case from Factor B level 3} \\ -1, & \text{if case from Factor B level 5} \\ 0, & \text{otherwise,} \end{cases}$$

(vii) variable 7

$$X_{ijk7} = \begin{cases} 1, & \text{if case from Factor B level 4} \\ -1, & \text{if case from Factor B level 5} \\ 0, & \text{otherwise,} \end{cases}$$

and so

$$\mathbf{Y} = \mathbf{X}\beta + \varepsilon$$

where[5]

$$\mathbf{Y} = \begin{bmatrix} 89 \\ 84 \\ \vdots \\ 84 \\ 88 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & -1 & -1 & -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \end{bmatrix}, \quad \beta = \begin{bmatrix} \mu_{..} \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix}$$

[5]Notice that the first column of 1s in matrix **X** corresponds to $\mu_{..}$

and so

$$\mathbf{X}\beta = \begin{bmatrix} \mu_{\cdot\cdot} + \alpha_1 + \beta_1 \\ \mu_{\cdot\cdot} + \alpha_1 + \beta_2 \\ \vdots \\ \mu_{\cdot\cdot} - \alpha_1 - \alpha_2 - \alpha_3 + \beta_3 \\ \mu_{\cdot\cdot} - \alpha_1 - \alpha_2 - \alpha_3 - \beta_1 - \beta_2 - \beta_3 - \beta_4 \end{bmatrix} = \begin{bmatrix} \mu_{11} \\ \mu_{12} \\ \vdots \\ \mu_{44} \\ \mu_{45} \end{bmatrix}$$

9. *Confidence interval for* $\hat{\mu}_{14}$
   Since

$$\hat{\mu}_{14} = \overline{Y}_{1\cdot} + \overline{Y}_{\cdot 4} - \overline{Y}_{\cdot\cdot} = 84 + 88 - 87 = 85$$

   and, from SAS[6],

$$s\{\hat{\mu}_{14}\} = MSE(\mathbf{X_h'}(\mathbf{X'X})^{-1}\mathbf{X_h}) = 2.74469$$

   and so a 95% confidence interval for $\hat{\mu}_{14}$ is[7]

$$\hat{\mu}_{14} \pm t(1-\alpha/2; (a-1)(b-1))s\{\hat{\mu}_{14}\} = 85 \pm t(0.995; 12)(2.74469)$$

   (choose one) **(76.63, 93.37)** / **(77.43, 92.57)** / **(79.43, 92.57)**

21.2 Tukey Test for Additivity

SAS program: att4-21-2-prozac-onetreat-Tukey

As demonstrated in the last section, one treatment case of a two–factor ANOVA has been analyzed as a two–factor ANOVA with*out* an interaction term, where the error term is replaced by the interaction term. This makes sense if there is no interaction between the two factors. The Tukey test for additivity is used to test whether there is or is not a significant interaction term. The (factor effects formulation of the) model assumed for the Tukey test, is

$$Y_{ij} = \mu_{\cdot\cdot} + \alpha_i + \beta_j + D(\alpha_i\beta_j) + \varepsilon_{ij},$$

where D is some constant.

[6]Notice that we are using the regression version of the model to determine $s\{\hat{\mu}_{14}\}$.

[7]Remember $t(0.995; 12)$ is determined using INVT.

Exercise 21.2 (Tukey Test for Additivity: Prozac Yield)
Consider the effect of different catalysts and blends on the Prozac yields from 10 milliliters of rice liquor.

	Factor B, blend →	1	2	3	4	5	
Factor A,	A	89	84	81	87	79	$\overline{Y}_{1\cdot} = 84$
catalyst	B	88	77	87	92	81	$\overline{Y}_{2\cdot} = 85$
	C	97	92	87	89	80	$\overline{Y}_{3\cdot} = 89$
	D	94	79	85	84	88	$\overline{Y}_{4\cdot} = 86$
		$\overline{Y}_{\cdot 1} = 92$	$\overline{Y}_{\cdot 2} = 83$	$\overline{Y}_{\cdot 3} = 85$	$\overline{Y}_{\cdot 4} = 88$	$\overline{Y}_{\cdot 5} = 82$	$\overline{Y}_{\cdot\cdot} = 86$

Source	*df*	*SS*	*MS*
Factor A (Catalyst)	3	70	23.33
Factor B (Blend)	4	264	66
Error (Interaction AB)	12	226	18.83
Total	19	560	

Test if the interaction is significant at $\alpha = 0.05$.

1. *Preliminary analysis*
   Since

$$\begin{aligned} SSAB^* &= \frac{\left(\sum_i \sum_j (\overline{Y}_{i\cdot} - \overline{Y}_{\cdot\cdot})(\overline{Y}_{\cdot j} - \overline{Y}_{\cdot\cdot}) Y_{ij}\right)^2}{\sum_i (\overline{Y}_{i\cdot} - \overline{Y}_{\cdot\cdot})^2 \sum_j (\overline{Y}_{\cdot j} - \overline{Y}_{\cdot\cdot})^2} \\ &= \frac{[(84-86)(92-86)89 + (84-86)(83-86)84 + \cdots + (86-86)(82-86)88]^2}{[(84-86)^2 + \cdots + (86-86)^2][(92-86)^2 + \cdots + (82-86)^2]} \\ &= \frac{(-7)^2}{[14]^2[46]^2} \\ &\approx 0.00011814 \end{aligned}$$

   and, using SAS for *SSTO*, *SSA* and *SSB*

$$\begin{aligned} SSRem^* &= SSTO - SSA - SSB - SSAB* \\ &= 560 - 70 - 264 - 0.00011814 \; = \end{aligned}$$

   (choose one) **−1 / 0 / 225.9998819**

2. *Test for interaction*

   (a) *Statement*
   The statement of the test is (check none, one or more):

      i. $H_0 : \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ versus $H_a : \; \alpha_1 \neq \alpha_2, \alpha_1 = \alpha_3$.
      ii. $H_0 : \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ versus $H_a : \; \alpha_1 \neq \alpha_3, \alpha_1 \neq \alpha_2$.
      iii. $H_0 : D = 0$ versus $H_a : \; D \neq 0$

iv. $H_0 : \mu_{1\cdot} = \mu_{2\cdot} = \mu_{3\cdot} = \mu_{4\cdot}$ versus
$H_a :$ at least one $\mu_{i\cdot} \neq \mu_{j\cdot}, \ i, j = 1, 2, 3, 4.$

(b) *Test*
the test statistic is

$$F^* = \frac{SSAB^*}{1} \div \frac{SSRem^*}{ab - a - b} = \frac{0.00011814}{1} \div \frac{225.9998819}{(4)(5) - 4 - 5} = 0.00000575$$

and the critical value is
and $F(1 - \alpha; 1, ab - a - b) = F(1 - 0.05; 1, 11) = 4.844$

(c) *Conclusion*
Since the test statistic, 0.00000575, is smaller than the critical value, 4.844, we (circle one) **accept** / **reject** the null hypothesis that *no* interactions are present (which seems to contradict our treatment means plot above)

Chapter 22

Two–Factor Studies–Unequal Sample Sizes and Unequal Treatment Importance

We look at two–factor ANOVA when there is either unequal sample sizes or the treatments are of unequal importance or both.

22.1 Unequal Sample Sizes

We look at the models and notation used to describe two–factor studies with unequal sample sizes[1]. Recall, the *factor effects* model,

$$Y_{ijk} = \mu_{..} + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk},$$

can be written as a *multiple regression* model,

$$\begin{aligned} Y_{ijk} &= \mu_{..} + \alpha_1 X_{ijk1} + \cdots + \alpha_{a-1} X_{ijk,a-1} + \beta_1 X_{ijk,a} + \cdots + \beta_{b-1} X_{ij,a+b-2} \\ &\quad + (\alpha\beta)_{11} X_{ijk1} X_{ijka} + \cdots + (\alpha\beta)_{a-1,b-1} X_{ijk,a-1} X_{ijk,a+b-2} + \varepsilon_{ijk} \end{aligned}$$

Although both models are equivalent, we will find that the regression model is often more useful when dealing with two–factor studies that have unequal sample sizes.

Exercise 22.1 (Notation for Unequal Sample Size Two Factor Studies) Consider the effect of different poisons and dosages of an antidote on the survival time of rats. For instance, the survival times of the three mice, subjected to poison I and a low dosage of the antidote, are 8.3, 7.2 and 7.5 minutes, respectively.

[1]There is another model called the *cell means* model,

$$Y_{ijk} = \mu_{ij} + \varepsilon_{ijk},$$

	Factor B, antidote →	low dosage	medium dosage	high dosage	row ave
Factor A,	I	8.3, 7.2, 7.5 ($\overline{Y}_{11\cdot} = 7.76$)	9.1, 9.4 ($\overline{Y}_{12\cdot} = 9.25$)	10.1, 11.1 ($\overline{Y}_{13\cdot} = 10.6$)	$\overline{Y}_{1\cdot\cdot} = 8.96$
poison	II	1.8, 3.8 ($\overline{Y}_{21\cdot} = 2.8$)	12.1 ($\overline{Y}_{22\cdot} = 12.1$)	15.1, 16.2, 14.4 ($\overline{Y}_{23\cdot} = 15.23$)	$\overline{Y}_{2\cdot\cdot} = 10.57$
	column average	$\overline{Y}_{\cdot 1\cdot} = 5.72$	$\overline{Y}_{\cdot 2\cdot} = 10.2$	$\overline{Y}_{\cdot 3\cdot} = 13.38$	$\overline{Y}_{\cdot\cdot\cdot} = 9.7$

1. *Notation for observations*

   (a) number, Factor A level 1, Factor level 2,
   $n_{12} =$ (circle one) **1 / 2 / 3**

   (b) number, Factor A level 2, Factor level 3,
   $n_{23} =$ (circle one) **1 / 2 / 3**

   (c) number, Factor A level 1,
   $n_{1\cdot} = \sum_j n_{1j} = n_{11} + n_{12} + n_{13} =$ (circle one) **5 / 6 / 7**

   (d) number, Factor B level 2,
   $n_{\cdot 2} = \sum_i n_{i2} = n_{12} + n_{22} =$ (circle one) **1 / 2 / 3**

   (e) total number,
   $n_T = \sum_i \sum_j n_{ij} =$ (circle one) **11 / 12 / 13**

   (f) number, Factor A level 1, Factor B level 1,
   $Y_{11\cdot} = \sum_{k=1}^{n_{11}} Y_{ijk} = \sum_{k=1}^{3} Y_{ijk} =$ (circle one) **22 / 23 / 24**

   (g) average, Factor A level 1, Factor B level 1,
   $\overline{Y}_{11\cdot} = \frac{1}{n_{11}} Y_{ij\cdot} =$ (circle one) **2.8 / 5.72 / 7.76**

2. *Regression version of factor effects model*
   In this example, the regression version of the factor effects model is[2]

$$\begin{aligned} Y_{ijk} &= \mu_{\cdot\cdot} + \alpha_1 X_{ijk1} + \beta_1 X_{ijk2} + \beta_2 X_{ijk3} \\ &+ (\alpha\beta)_{11} X_{ijk1} X_{ijk2} + (\alpha\beta)_{12} X_{ijk1} X_{ijk3} + \varepsilon_{ijk}, \end{aligned}$$

   where

$$X_{ijk1} = \begin{cases} 1, & \text{if case from Factor A level 1} \\ -1, & \text{if case from Factor A level 2} \end{cases}$$

$$X_{ijk2} = \begin{cases} 1, & \text{if case from Factor B level 1} \\ -1, & \text{if case from Factor B level 3} \\ 0, & \text{otherwise,} \end{cases}$$

$$X_{ijk3} = \begin{cases} 1, & \text{if case from Factor B level 2} \\ -1, & \text{if case from Factor B level 3} \\ 0, & \text{otherwise,} \end{cases}$$

   and, so, for a case from Factor A level 1 and Factor B level 1,
   where $(X_{ijk1}, X_{ijk2}, X_{ijk3}) = (1, 1, 0)$,

[2]Notice that only α_1 appears, not α_1 and α_2; similarly, notice that only β_1 and β_2 appear and not β_1, β_2 and β_3. This is because $\sum \alpha_i = 0$ and $\sum \beta_j = 0$.

and furthermore $(X_1X_2, X_1X_3) = ((1)(1), (1)(0)) = (1, 0)$, and so
(choose one)
$\boldsymbol{Y_{11k} = \mu_{..} + \alpha_1 + \beta_1 + (\alpha\beta)_{11} + (\alpha\beta)_{12} + \varepsilon_{11k}}$
$\boldsymbol{Y_{11k} = \mu_{..} + \alpha_1 + \beta_1 + (\alpha\beta)_{12} + \varepsilon_{11k}}$
$\boldsymbol{Y_{11k} = \mu_{..} + \alpha_1 + \beta_1 + (\alpha\beta)_{11} + \varepsilon_{11k}}$

and, so, for a case from Factor A level 1 and Factor B level 2,
where $(X_{ijk1}, X_{ijk2}, X_{ijk3}) = (1, 0, 1)$,
and furthermore $(X_1X_2, X_1X_3) = ((1)(0), (1)(1)) = (0, 1)$, and so
(choose one)
$\boldsymbol{Y_{12k} = \mu_{..} + \alpha_1 + \beta_2 + (\alpha\beta)_{11} + (\alpha\beta)_{11} + \varepsilon_{12k}}$
$\boldsymbol{Y_{12k} = \mu_{..} + \alpha_1 + \beta_2 + (\alpha\beta)_{12} + \varepsilon_{12k}}$
$\boldsymbol{Y_{12k} = \mu_{..} + \alpha_1 + \beta_1 + (\alpha\beta)_{12} + \varepsilon_{12k}}$

and, so, for a case from Factor A level 2 and Factor B level 1,
where $(X_{ijk1}, X_{ijk2}, X_{ijk3}) = (-1, 1, 0)$,
and furthermore $(X_1X_2, X_1X_3) = ((-1)(1), (-1)(0)) = (-1, 0)$, and so
(choose one)
$\boldsymbol{Y_{21k} = \mu_{..} - \alpha_1 + \beta_2 + (\alpha\beta)_{11} + (\alpha\beta)_{11} + \varepsilon_{21k}}$
$\boldsymbol{Y_{21k} = \mu_{..} - \alpha_1 + \beta_2 + (\alpha\beta)_{12} + \varepsilon_{21k}}$
$\boldsymbol{Y_{21k} = \mu_{..} - \alpha_1 + \beta_1 - (\alpha\beta)_{11} + \varepsilon_{21k}}$
which seems incorrect at first, but notice,
since $\sum_i \alpha_i = \alpha_1 + \alpha_2 = 0$, $-\alpha_1 = \alpha_2$
and since $\sum_i (\alpha\beta)_{i1} = (\alpha\beta)_{11} + (\alpha\beta)_{21} = 0$, $-(\alpha\beta)_{11} = (\alpha\beta)_{21}$
and so, in other words,
$Y_{21k} = \mu_{..} - \alpha_1 + \beta_1 - (\alpha\beta)_{11} + \varepsilon_{21k}$
can be rewritten as (choose one)
$\boldsymbol{Y_{21k} = \mu_{..} - \alpha_1 + \beta_2 + (\alpha\beta)_{11} + (\alpha\beta)_{11} + \varepsilon_{21k}}$
$\boldsymbol{Y_{21k} = \mu_{..} - \alpha_1 + \beta_2 + (\alpha\beta)_{12} + \varepsilon_{21k}}$
$\boldsymbol{Y_{21k} = \mu_{..} + \alpha_2 + \beta_1 + (\alpha\beta)_{21} + \varepsilon_{21k}}$

and, so, for a case from Factor A level 1 and Factor B level 3,
where $(X_{ijk1}, X_{ijk2}, X_{ijk3}) = (1, -1, -1)$,
and furthermore $(X_1X_2, X_1X_3) = ((1)(-1), (1)(-1)) = (-1, -1)$, and so
(choose one)
$\boldsymbol{Y_{13k} = \mu_{..} - \alpha_1 + \beta_2 + (\alpha\beta)_{11} + (\alpha\beta)_{11} + \varepsilon_{13k}}$
$\boldsymbol{Y_{13k} = \mu_{..} - \alpha_1 + \beta_2 + (\alpha\beta)_{12} + \varepsilon_{13k}}$
$\boldsymbol{Y_{13k} = \mu_{..} + \alpha_1 - \beta_1 - \beta_2 - (\alpha\beta)_{11} - (\alpha\beta)_{12} + \varepsilon_{13k}}$
which seems incorrect at first, but notice,
since $\sum_j \beta_j = \beta_1 + \beta_2 + \beta_3 = 0$, $-\beta_1 - \beta_2 = \beta_3$
and since $\sum_j (\alpha\beta)_{1j} = (\alpha\beta)_{11} + (\alpha\beta)_{12} + (\alpha\beta)_{12} = 0$,
$-(\alpha\beta)_{11} - (\alpha\beta)_{12} = (\alpha\beta)_{13}$

and so, in other words,
$Y_{13k} = \mu_{..} + \alpha_1 - \beta_1 - \beta_2 - (\alpha\beta)_{11} - (\alpha\beta)_{12} + \varepsilon_{13k}$
can be rewritten as (choose one)
$\mathbf{Y_{13k} = \mu_{..} - \alpha_1 + \beta_2 + (\alpha\beta)_{11} + (\alpha\beta)_{11} + \varepsilon_{21k}}$
$\mathbf{Y_{13k} = \mu_{..} - \alpha_1 + \beta_2 + (\alpha\beta)_{12} + \varepsilon_{21k}}$
$\mathbf{Y_{13k} = \mu_{..} + \alpha_1 + \beta_3 + (\alpha\beta)_{13} + \varepsilon_{13k}}$

3. *Matrix version of factor effects model*
The factor effects model,

$$\begin{aligned} Y_{ijk} &= \mu_{..} + \alpha_1 X_{ijk1} + \beta_1 X_{ijk2} + \beta_2 X_{ijk3} \\ &+ (\alpha\beta)_{11} X_{ijk1} X_{ijk2} + (\alpha\beta)_{12} X_{ijk1} X_{ijk3} + \varepsilon_{ijk}, \end{aligned}$$

can be written using matrix notation as

$$\mathbf{Y} = \mathbf{X}\beta + \varepsilon$$

where[3]

$$\mathbf{Y} = \begin{bmatrix} 8.3 \\ 7.2 \\ 7.5 \\ 9.1 \\ 9.4 \\ 10.1 \\ 11.1 \\ 1.8 \\ 3.8 \\ 12.1 \\ 15.1 \\ 16.2 \\ 14.4 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 0 & -1 & 0 \\ 1 & -1 & 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 1 & 0 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 \end{bmatrix}, \quad \beta = \begin{bmatrix} \mu_{.} \\ \alpha_1 \\ \beta_1 \\ \beta_2 \\ (\alpha\beta)_{11} \\ (\alpha\beta)_{12} \end{bmatrix}$$

[3]Notice that the first column in **X** is a column of 1s which corresponds to $\mu_{..}$.

and so

$$\mathbf{X}\beta = \begin{bmatrix} \mu. + \alpha_1 + \beta_1 + (\alpha\beta)_{11} \\ \mu. + \alpha_1 + \beta_1 + (\alpha\beta)_{11} \\ \mu. + \alpha_1 + \beta_1 + (\alpha\beta)_{11} \\ \mu. + \alpha_1 + \beta_2 + (\alpha\beta)_{12} \\ \mu. + \alpha_1 + \beta_2 + (\alpha\beta)_{12} \\ \mu. + \alpha_1 - \beta_1 - \beta_2 - (\alpha\beta)_{11} - (\alpha\beta)_{12} \\ \mu. + \alpha_1 - \beta_1 - \beta_2 - (\alpha\beta)_{11} - (\alpha\beta)_{12} \\ \mu. - \alpha_1 + \beta_1 - (\alpha\beta)_{11} \\ \mu. - \alpha_1 + \beta_1 - (\alpha\beta)_{11} \\ \mu. - \alpha_1 + \beta_2 - (\alpha\beta)_{12} \\ \mu. - \alpha_1 - \beta_1 - \beta_2 + (\alpha\beta)_{11} + (\alpha\beta)_{12} \\ \mu. - \alpha_1 - \beta_1 - \beta_2 + (\alpha\beta)_{11} + (\alpha\beta)_{12} \end{bmatrix}$$

where the six columns of **X** are (choose one)

(a) $(\mu_{..}, X_1, X_2, X_3, X_1X_2, X_1X_3)$

(b) $(\mu_{..}, X_1, X_2, X_3, X_2X_3, X_1X_3)$

(c) $(\mu_{..}, X_1, X_2, X_2X_3, X_1X_3)$

22.2 Use of Regression Approach for Testing Factor Effects when Sample Sizes are Unequal

SAS program: att4-22-2-rats-twoANOVA,regression

Exercise 22.2 (Use of Regression Approach for Testing Factor Effects when Sample Sizes are Unequal)
Consider the effect of different poisons and dosages of an antidote on the survival time of rats.

	Factor B, antidote →	low dosage	medium dosage	high dosage	row ave
Factor A,	I	8.3, 7.2, 7.5 ($\overline{Y}_{11}. = 7.76$)	9.1, 9.4 ($\overline{Y}_{12}. = 9.25$)	10.1, 11.1 ($\overline{Y}_{13}. = 10.6$)	$\overline{Y}_{1..} = 8.96$
poison	II	1.8, 3.8 ($\overline{Y}_{21}. = 2.8$)	12.1 ($\overline{Y}_{22}. = 12.1$)	15.1, 16.2, 14.4 ($\overline{Y}_{23}. = 15.23$)	$\overline{Y}_{2..} = 10.57$
	column average	$\overline{Y}_{.1.} = 5.72$	$\overline{Y}_{.2.} = 10.2$	$\overline{Y}_{.3.} = 13.38$	$\overline{Y}_{...} = 9.7$

Test if the either of the two main effects or the interaction effect are significant at $\alpha = 0.05$.

1. *Preliminary analysis: treatment mean plots*
   From SAS, either of the two treatment means plots (choose one) **does / does not** indicate interaction.

2. *Preliminary analysis: residuals*
   From SAS, the residual plot looks reasonably randomly scattered and so assumption of constant variance has been satisfied. In particular,
   e_{113} = (choose one) **−0.46667 / −0.16667 / 0.63333**

3. *Preliminary analysis: normal probability plot*
   From SAS, the normal probability plot looks reasonably linear and so assumption of normality (choose one) **has / has not** been satisfied.

4. *Test Interaction AB, P–Value Versus Level of Significance.*

   (a) *Statement*
   The statement of the test is (check none, one or more):

   i. H_0 : all $(\alpha\beta)_{ij} = 0$ versus H_a : not all $(\alpha\beta)_{ij} = 0$.
   ii. $H_0 : \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ versus H_a : $\alpha_1 \neq \alpha_3, \alpha_1 \neq \alpha_2$.
   iii. $H_0 : \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ versus
   H_a : at least one $\alpha_i \neq 0$, $i = 1, 2, 3, 4$.
   iv. $H_0 : \mu_{1\cdot} = \mu_{2\cdot} = \mu_{3\cdot} = \mu_{4\cdot}$ versus
   H_a : at least one $\mu_{i\cdot} \neq \mu_{j\cdot}$, $i, j = 1, 2, 3, 4$.

   (b) *Test*
   Using the SAS output, the test statistic is

$$F^* = \frac{SSE(R) - SSE(F)}{df_R - df_F} \div \frac{SSE(F)}{df_F} = \frac{63.580 - 4.838}{9 - 7} \div \frac{4.838}{7}$$

   (circle one) **42.49 / 357.69 / 3576.98** the p–value, with 2 and 7 degrees of freedom (given in the ANOVA table, is given by

$$\text{p–value} = P(F \geq 42.49)$$

   which equals (circle one) **0.00 / 0.34 / 0.43**.
   (Use 2nd DISTR 9:Fcdf(42.49,E99,2,7).)
   The level of significance is 0.05.

   (c) *Conclusion*
   Since the p–value, 0.00, is smaller than the level of significance, 0.05, we (circle one) **accept / reject** the null hypothesis that the interaction effect is zero.

   (d) In fact, since the interaction is not zero, we (choose one) **can / cannot** test the main effects (but we will anyway).

5. *Test Factor A (Poison), P–Value Versus Level of Significance.*

(a) *Statement*
The statement of the test is (check none, one or more):

i. $H_0 : \alpha_1 = \alpha_2 = 0$ versus $H_a : \alpha_1 \neq \alpha_2$.
ii. $H_0 : \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ versus $H_a : \alpha_1 \neq \alpha_3, \alpha_1 \neq \alpha_2$.
iii. $H_0 : \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ versus
$H_a :$ at least one $\alpha_i \neq 0,\ i = 1, 2, 3, 4$.
iv. $H_0 : \mu_{1\cdot} = \mu_{2\cdot} = \mu_{3\cdot} = \mu_{4\cdot}$ versus
$H_a :$ at least one $\mu_{i\cdot} \neq \mu_{j\cdot},\ i, j = 1, 2, 3, 4$.

(b) *Test*
Using the SAS output, the test statistic is

$$\begin{aligned} F^* &= \frac{SSE(R) - SSE(F)}{df_R - df_F} \div \frac{SSE(F)}{df_F} \\ &= \frac{7.001 - 4.838}{8 - 7} \div \frac{4.838}{7} \end{aligned}$$

(circle one) **3.13** / **357.69** / **3576.98** the p–value, with 1 and 7 degrees of freedom, is given by

$$\text{p–value} = P(F \geq 3.13)$$

which equals (circle one) **0.02** / **0.12** / **0.51**.
(Use 2nd DISTR 9:Fcdf(3.13,E99,1,7).)
The level of significance is 0.05.

(c) *Conclusion* Since the p–value, 0.12, is larger than the level of significance, 0.05, we (circle one) **accept** / **reject** the null hypothesis that the main Poison effect A is zero (although this conclusion is "muddied" by the significance of the interaction term).

6. *Test Factor B (Antidote), P–Value Versus Level of Significance.*

(a) *Statement*
The statement of the test is (check none, one or more):

i. $H_0 : \beta_1 = \beta_2 = \beta_3 = 0$ versus $H_a : \beta_i \neq \beta_j,\ i \neq j,\ i, j = 1, 2, 3$.
ii. $H_0 : \beta_1 = \beta_2 = \beta_3 = 0$ versus $H_a : \beta_1 \neq \beta_3, \beta_1 \neq \beta_2$.
iii. $H_0 : \beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = 0$ versus
$H_a :$ at least one $\beta_i \neq 0,\ i = 1, 2, 3$.
iv. $H_0 : \mu_{\cdot 1} = \mu_{\cdot 2} = \mu_{\cdot 3} = \mu_{\cdot 4} = \mu_{\cdot 5}$ versus
$H_a :$ at least one $\mu_{i\cdot} \neq \mu_{j\cdot},\ i, j = 1, 2, 3, 4, 5$.

(b) *Test*
Using the SAS output, the test statistic is

$$\begin{aligned} F^* &= \frac{SSE(R) - SSE(F)}{df_R - df_F} \div \frac{SSE(F)}{df_F} \\ &= \frac{151.861 - 4.838}{9 - 7} \div \frac{4.838}{7} \end{aligned}$$

(circle one) **106.36** / **357.69** / **3576.98** the p–value, with 2 and 7 degrees of freedom, is given by

$$\text{p–value} = P(F \geq 106.36)$$

which equals (circle one) **0.00** / **0.34** / **0.43**.
(Use 2nd DISTR 9:Fcdf(106.36,E99,2,7).)
The level of significance is 0.05.

(c) *Conclusion*
Since the p–value, 0.00, is smaller than the level of significance, 0.05, we (circle one) **accept** / **reject** the null hypothesis that the main Antidote effect is zero (although this conclusion is "muddied" by the significance of the interaction term).

7. *Different regression models used in analysis*
True / **False**
The different regression models used in the inference tests above are given below.

Full Model
$Y_{ijk} = \mu_{..} + \alpha_1 X_{ijk1} + \beta_1 X_{ijk2} + \beta_2 X_{ijk3} + (\alpha\beta)_{11} X_{ijk1} X_{ijk2} + (\alpha\beta)_{12} X_{ijk1} X_{ijk3} + \varepsilon_{ijk}$ $\hat{Y} = 9.608 - 0.436X_1 - 4.375X_2 + 1.067X_3 + 2.869X_1X_2 - 2.869X_1X_3$
Reduced Model, No Interaction
$Y_{ijk} = \mu_{..} + \alpha_1 X_{ijk1} + \beta_1 X_{ijk2} + \beta_2 X_{ijk3} + \varepsilon_{ijk}$ $\hat{Y} = 9.796 - 0.264X_1 - 4.023X_2 + 0.492X_3$
Reduced Model, No Factor A (Poison)
$Y_{ijk} = \mu_{..} + \beta_1 X_{ijk2} + \beta_2 X_{ijk3} + (\alpha\beta)_{11} X_{ijk1} X_{ijk2} + (\alpha\beta)_{12} X_{ijk1} X_{ijk3} + \varepsilon_{ijk}$ $\hat{Y} = 9.539 - 4.375X_2 + 0.929X_3 + 2.778X_1X_2 - 0.805X_1X_3$
Reduced Model, No Factor B (Antidote)
$Y_{ijk} = \mu_{..} + \alpha_1 X_{ijk1} + (\alpha\beta)_{11} X_{ijk1} X_{ijk2} + (\alpha\beta)_{12} X_{ijk1} X_{ijk3} + \varepsilon_{ijk}$ $\hat{Y} = 9.422 - 0.809X_1 + 2.405X_1X_2 - 0.198X_1X_3$

8. *Comparing regression ANOVA with the two–factor ANOVA*
From SAS, the two–factor ANOVA[4] is given by

[4]Use the type III sums of squares.

Source	*df*	*SS*	*MS*
Factor A (Poison)	1	2.162	2.162
Factor B (Antidote)	2	147.023	73.512
Interaction AB	2	58.741	29.371
Error	7	4.8383	0.6912
Total	12	212.100	

whereas the (full) regression ANOVA is given by

Source	*df*	*SS*	*MS*
Regression model	5	207.26167	41.45233
Error	7	4.8383	0.6912
Total	12	212.100	

The *total* sums of squares for the two main factors and the interaction but excluding the error, for the two–factor ANOVA (choose one) **equals** / **does not equal** the model sums of squares for the regression model ANOVA.

22.3 Inferences about Factor Effects when Sample Sizes Are Unequal

SAS program: att4-22-3-rats-unequaltreatment

We look at various inferences (tests and confidence intervals) related to two–factor studies with unequal sample sizes. As before, in the equal sample size case, when there is no interaction, inference is conducted on $\mu_{\cdot j}$ and $\mu_{i\cdot}$; otherwise, inference is conducted on μ_{ij}.

Type of Statistic	
Factor Level Mean A B	$\hat{\mu}_{i\cdot} = \frac{1}{b}\sum_j \overline{Y}_{ij\cdot}$, $s^2\{\hat{\mu}_{i\cdot}\} = \frac{1}{b^2} MSE \sum_j \frac{1}{n_{ij}}$ $\hat{\mu}_{\cdot j} = \frac{1}{a}\sum_i \overline{Y}_{ij\cdot}$, $s^2\{\hat{\mu}_{\cdot j}\} = \frac{1}{a^2} MSE \sum_i \frac{1}{n_{ij}}$
Pairwise Comparison of Factor Level Mean A B	$\hat{D} = \hat{\mu}_{i\cdot} - \hat{\mu}_{i'\cdot}$, $s^2\{\hat{D}\} = \frac{1}{b^2} MSE \sum_j \left(\frac{1}{n_{ij}} + \frac{1}{n_{i'j}}\right)$ $\hat{D} = \hat{\mu}_{\cdot j} - \hat{\mu}_{\cdot j'}$, $s^2\{\hat{D}\} = \frac{1}{a^2} MSE \sum_i \left(\frac{1}{n_{ij}} + \frac{1}{n_{ij'}}\right)$
Contrast or Linear Combination of Factor Level Means A B	$\hat{L} = \sum_i c_i \hat{\mu}_{i\cdot}$, $s^2\{\hat{L}\} = \frac{1}{b^2} MSE \sum_i c_i^2 \sum_j \frac{1}{n_{ij}}$ $\hat{L} = \sum_j c_j \hat{\mu}_{\cdot j}$, $s^2\{\hat{L}\} = \frac{1}{a^2} MSE \sum_j c_j^2 \sum_i \frac{1}{n_{ij}}$
Confidence Interval Multiple Single Estimate A B	$t(1-\alpha/2; n_T - ab)$ $t(1-\alpha/2; n_T - ab)$
Confidence Interval Multiple Multiple Comparisons A B	$t(1-\alpha/2g; n_T - ab)$, $\frac{1}{\sqrt{2}} q(1-\alpha; a, n_T - ab)$, $\sqrt{(a-1)F(1-\alpha; a-1, n_T - ab)}$ $t(1-\alpha/2g; n_T - ab)$, $\frac{1}{\sqrt{2}} q(1-\alpha; b, n_T - ab)$, $\sqrt{(b-1)F(1-\alpha; b-1, n_T - ab)}$
Treatment Mean	$\hat{\mu}_{ij} = \overline{Y}_{ij\cdot}$, $s^2\{\hat{\mu}_{ij}\} = MSE \frac{1}{n_{ij}}$
Pairwise Comparison of Treatment Means	$\hat{D} = \hat{\mu}_{ij\cdot} - \hat{\mu}_{i'j'\cdot}$, $s^2\{\hat{D}\} = MSE \left(\frac{1}{n_{ij}} + \frac{1}{n_{i'j'}}\right)$
Contrast of Linear Combination of Treatment Means	$\hat{L} = \sum_i \sum_j c_{ij} \overline{Y}_{ij\cdot}$, $s^2\{\hat{L}\} = MSE \sum_i \sum_j c_{ij}^2 \frac{1}{n_{ij}}$
Confidence Interval Multiple Single Estimate	$t(1-\alpha/2; n_T - ab)$
Confidence Interval Multiple Multiple Comparisons	$t(1-\alpha/2g; n_T - ab)$, $\frac{1}{\sqrt{2}} q(1-\alpha; ab, n_T - ab)$, $\sqrt{(ab-1)F(1-\alpha; ab-1, n_T - ab)}$

Exercise 22.3 (Inferences about Factor Effects when Sample Sizes Are Unequal) Consider the effect of different poisons and dosages of an antidote on the survival time of rats. Conduct inference on various individual and multiple linear combinations for treatment effects and treatment means at $\alpha = 0.05$.

	Factor B, antidote →	low dosage	medium dosage	high dosage	row ave
Factor A,	I	8.3, 7.2, 7.5 ($\overline{Y}_{11\cdot} = 7.76$)	9.1, 9.4 ($\overline{Y}_{12\cdot} = 9.25$)	10.1, 11.1 ($\overline{Y}_{13\cdot} = 10.6$)	$\overline{Y}_{1\cdot\cdot} = 8.96$
poison	II	1.8, 3.8 ($\overline{Y}_{21\cdot} = 2.8$)	12.1 ($\overline{Y}_{22\cdot} = 12.1$)	15.1, 16.2, 14.4 ($\overline{Y}_{23\cdot} = 15.23$)	$\overline{Y}_{2\cdot\cdot} = 10.57$
	column average	$\overline{Y}_{\cdot 1\cdot} = 5.72$	$\overline{Y}_{\cdot 2\cdot} = 10.2$	$\overline{Y}_{\cdot 3\cdot} = 13.38$	$\overline{Y}_{\cdot\cdot\cdot} = 9.7$

1. *Confidence Interval, Multiple Pairwise, Factor B, Bonferroni.*
   Since there are $b = 3$ levels of factor B (antidote), there are a total of $\frac{b(b-1)}{2} = \frac{3(2)}{2} = 3$ possible pairwise comparisons. This is not a large number of pairwise comparisons, but determine a Tukey multiple pairwise confidence interval anyway. In particular, determine a 95% confidence interval for $\mu_{\cdot 1} - \mu_{\cdot 2}$.

   (a) *Confidence interval*

Since

$$
\begin{aligned}
\hat{D} &= \hat{\mu}_{\cdot 1} - \hat{\mu}_{\cdot 2} \\
&= \frac{1}{a}\sum_i \overline{Y}_{i1} - \frac{1}{a}\sum_i \overline{Y}_{i2} \\
&= \frac{\overline{Y}_{11\cdot} + \overline{Y}_{21\cdot}}{2} - \frac{\overline{Y}_{12\cdot} + \overline{Y}_{22\cdot}}{2} \\
&= \frac{7.76 + 2.8}{2} - \frac{9.25 + 12.1}{2} \\
&= 5.28 - 10.675 \\
&= -5.395 \\
s\left\{\hat{D}\right\} &= \sqrt{\frac{1}{a^2} MSE \sum_i \left(\frac{1}{n_{ij}} + \frac{1}{n_{ij'}}\right)} \\
&= \sqrt{\frac{1}{2^2}(0.6912)\left(\left(\frac{1}{n_{11}} + \frac{1}{n_{12}}\right) + \left(\frac{1}{n_{21}} + \frac{1}{n_{22}}\right)\right)} \\
&= \sqrt{\frac{1}{2^2}(0.6912)\left(\left(\frac{1}{3} + \frac{1}{2}\right) + \left(\frac{1}{2} + \frac{1}{1}\right)\right)} \\
&= 0.635 \\
t(1 - \alpha/2g; n_T - ab) &= t(1 - 0.05/(2(3)); 13 - (2)(3)) \\
&= t(0.992; 7) \\
&= 3.127
\end{aligned}
$$

the confidence interval for $\mu_{\cdot 1} - \mu_{\cdot 2}$ is given by

$$
\hat{D} \pm t(1 - \alpha/2g; n_T - ab)s\left\{\hat{D}\right\} = -5.395 \pm (3.127)(0.635) =
$$

(choose one) **(−6.466, −2.494)** / **(−5.98, −2.98)** / **(−7.381, −3.409)**

(b) *Equal versus unequal sample size*
Notice that

$$
\begin{aligned}
\hat{D} &\neq \overline{Y}_{\cdot 1\cdot} - \overline{Y}_{\cdot 2\cdot} \\
&= 5.72 - 10.2 \\
&= -4.48
\end{aligned}
$$

which it would, if this was a two–factor study with (choose one) **equal** / **unequal** sample sizes.

(c) *Implication of significant interaction*
Since we know the interaction between the two factors is significant, it (choose one) **does** / **does not** make sense to calculate this confidence interval of two levels of the *same* factor B (antidote).

2. *Test, Treatment Mean,* μ_{23}
   Test if the survival subjected to factor A, level 2 and factor B, level 3, is greater than 15.1 units at $\alpha = 0.05$.

   (a) *Statement*
   The statement of the test is (check none, one or more):

   i. $H_0 : \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ versus $H_a : \alpha_1 \neq \alpha_2, \alpha_1 = \alpha_3$.
   ii. $H_0 : \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ versus $H_a : \alpha_1 \neq \alpha_3, \alpha_1 \neq \alpha_2$.
   iii. $H_0 : \mu_{23} = 15.1$ versus $H_a : \mu_{23} > 15.1$.
   iv. $H_0 : \mu_{1\cdot} = \mu_{2\cdot} = \mu_{3\cdot} = \mu_{4\cdot}$ versus
   $H_a :$ at least one $\mu_{i\cdot} \neq \mu_{j\cdot}$, $i, j = 1, 2, 3, 4$.

   (b) *Test*
   Since

$$\begin{aligned}
\hat{\mu}_{23} &= \overline{Y}_{23\cdot} \\
&= \frac{15.1 + 16.2 + 14.4}{3} \\
&= 15.23 \\
s\{\hat{\mu}_{23}\} &= \sqrt{\frac{MSE}{n_{ij}}} \\
&= \sqrt{\frac{0.6912}{3}} \\
&= 0.48
\end{aligned}$$

   the test statistic is

$$t^* = [\hat{\mu}_{23} - c]/s\{\hat{\mu}_{23}\} = [15.23 - 15.1]/0.48 = 0.271$$

   the p–value, with $n_T - ab = 13 - (2)(3) = 7$ degrees of freedom, is given by

$$\text{p–value} = P(t \geq 0.271)$$

   which equals (circle one) **0.00** / **0.37** / **0.40**.
   (Use 2nd DISTR $t$cdf(0.355,E99,7).)
   The level of significance is 0.05.

   (c) *Conclusion*
   Since the p–value, 0.40, is larger than the level of significance, 0.05, we (circle one) **accept** / **reject** the null hypothesis that the ROC, subjected to factor A, level 1 (0^o F), is equal to 15.1 units.

   (d) *Implication of significant interaction*
   Since we know the interaction between the two factors is significant, it (choose one) **does** / **does not** make sense to calculate this test of one cell mean, μ_{23}, as opposed to the test of one factor mean, $\mu_{1\cdot}$ or $\mu_{\cdot 3}$, say.

3. *Confidence Interval, Multiple Pairwise of Treatment Means, $\mu_{11} - \mu_{23}$, Tukey.* Since there are $a = 2$ levels of factor A (poison) and $b = 3$ levels of factor B (antidote), there are a total of $\frac{ab(ab-1)}{2} = \frac{6(5)}{2} = 15$ possible pairwise comparisons. This is not a large number of pairwise comparisons, but determine a Tukey multiple pairwise confidence interval anyway. In particular, determine a 95% confidence interval for $\mu_{11} - \mu_{23}$.

   (a) *Confidence interval*
   Since

$$
\begin{aligned}
\hat{D} &= \overline{Y}_{11\cdot} - \overline{Y}_{23\cdot} \\
&= 7.67 - 15.23 \\
&= -7.57 \\
s\left\{\hat{D}\right\} &= \sqrt{MSE\left(\frac{1}{n_{ij}} + \frac{1}{n_{ij'}}\right)} \\
&= \sqrt{0.6912\left(\frac{1}{n_{11}} + \frac{1}{n_{23}}\right)} \\
&= \sqrt{0.6912\left(\frac{1}{3} + \frac{1}{3}\right)} \\
&= 0.679 \\
\frac{1}{\sqrt{2}}q(1-\alpha; ab, n_T - ab) &= \frac{1}{\sqrt{2}}q(1 - 0.05; (3)(2), 13 - (3)(2)) \\
&= \frac{1}{\sqrt{2}}q(0.95; 6, 7) \\
&= 5.36
\end{aligned}
$$

   the confidence interval for $\mu_{11} - \mu_{23}$ is given by

$$
\hat{D} \pm \frac{1}{\sqrt{2}}q(1-\alpha; ab, n_T - ab)s\left\{\hat{D}\right\} = -7.57 \pm (5.36)(0.679) =
$$

   (choose one) **(−11.21, −3.93)** / **(−5.98, −2.98)** / **(−5.433, −3.527)**

   (b) *Implication of significant interaction*
   Since we know the interaction between the two factors is significant, it (choose one) **does** / **does not** make sense to calculate the confidence interval of the difference between two different cell means, $\mu_{11} - \mu_{23}$.

22.4 Empty Cells in Two–Factor Studies

SAS program: att4-22-4-rats-nocellsample

When one or more treatment cells are empty, a *complete* analysis of variance cannot be done. However, it is possible to do a *partial* analysis of variance which works around the empty cell.

Exercise 22.4 (Empty Cells in Two–Factor Studies)
Consider the effect of different poisons and dosages of an antidote on the survival time of rats. Notice not only the empty cell for factor A, level 2, and factor B, level 1 but also some of the row and column averages have been changed.

	Factor B, antidote →	low dosage	medium dosage	high dosage	row ave
Factor A,	I	8.3, 7.2, 7.5 ($\bar{Y}_{11\cdot} = 7.76$)	9.1, 9.4 ($\bar{Y}_{12\cdot} = 9.25$)	10.1, 11.1 ($\bar{Y}_{13\cdot} = 10.6$)	$\bar{Y}_{1\cdot\cdot} = 8.96$
poison	II	empty cell	12.1 ($\bar{Y}_{22\cdot} = 12.1$)	15.1, 16.2, 14.4 ($\bar{Y}_{23\cdot} = 15.23$)	$\bar{Y}_{2\cdot\cdot} = 14.45$
	column average	$\bar{Y}_{\cdot 1\cdot} = 7.76$	$\bar{Y}_{\cdot 2\cdot} = 10.2$	$\bar{Y}_{\cdot 3\cdot} = 13.38$	$\bar{Y}_{\cdot\cdot\cdot} = 9.7$

Test if the either of the two main effects or the interaction effect are significant at $\alpha = 0.05$.

1. *Preliminary analysis: treatment mean plots*
   Because of the empty (1,1) cell, we can only use the following treatment means to investigate whether or not there is interaction.

	Factor B, antidote →	low dosage	medium dosage	high dosage
Factor A,	I	not useable	μ_{21}	μ_{13}
poison	II	empty cell	μ_{22}	μ_{23}

   From SAS, it seems fairly clear both of the two treatment means plots (choose one) **do / do not** indicate interaction since the graphs are more or less parallel to one another.

2. *Test Interaction AB*

   (a) *Statement*
   The statement of the test is (check none, one or more):

   i. $H_0:$ all $(\alpha\beta)_{ij} = 0$ versus $H_a:$ not all $(\alpha\beta)_{ij} = 0$.
   ii. $H_0: \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ versus $H_a: \alpha_1 \neq \alpha_3, \alpha_1 \neq \alpha_2$.
   iii. $H_0: \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ versus
   $H_a:$ at least one $\alpha_i \neq 0,\ i = 1, 2, 3, 4$.
   iv. $H_0: \mu_{1\cdot} = \mu_{2\cdot} = \mu_{3\cdot} = \mu_{4\cdot}$ versus
   $H_a:$ at least one $\mu_{i\cdot} \neq \mu_{j\cdot},\ i, j = 1, 2, 3, 4$.

(b) *Test*
Using the SAS output, the test statistic[5] is

$$F^* = \frac{MSAB}{MSE^*} = \frac{1.36297}{2.191667/7}$$

(circle one) **2.49** / **4.35** / **6.98** the p–value, with 1 and 7 degrees of freedom (given in the ANOVA table, is given by

$$\text{p–value} = P(F \geq 4.35)$$

which equals (circle one) **0.08** / **0.34** / **0.43**.
The level of significance is 0.05.

(c) *Conclusion*
Since the p–value, 0.08, is larger than the level of significance, 0.05, we (circle one) **accept** / **reject** the null hypothesis that the interaction effect is zero (in spite of the treatment plots given above).

(d) In fact, since the interaction is *not* zero, we (choose one) **can** / **cannot** test the main effects (but we will anyway).

3. *Test Factor A (Poison), assuming no interaction*

(a) *Statement*
The statement of the test is (check none, one or more):
   i. $H_0 : \alpha_1 = \alpha_2 = 0$ versus $H_a : \alpha_1 \neq \alpha_2$.
   ii. $H_0 : \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ versus $H_a : \alpha_1 \neq \alpha_3, \alpha_1 \neq \alpha_2$.
   iii. $H_0 : \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ versus
   $H_a :$ at least one $\alpha_i \neq 0,\ i = 1, 2, 3, 4$.
   iv. $H_0 : \mu_{1\cdot} = \mu_{2\cdot} = \mu_{3\cdot} = \mu_{4\cdot}$ versus
   $H_a :$ at least one $\mu_{i\cdot} \neq \mu_{j\cdot},\ i, j = 1, 2, 3, 4$.

(b) *Test*
Using the SAS output, the test statistic is

$$F^* = \frac{SSE(R) - SSE(F)}{df_R - df_F} \div \frac{SSE(F)}{df_F} = \frac{4.2518333 - 4.20130952}{8 - 7} \div \frac{4.20130952}{7}$$

[5] Notice that *MSE** is based *all* observations and, as a consequence, is based *not* on 4 degrees of freedom, as indicated in the SAS output, but $4 + 3 = 7$ degrees of freedom, where the three (3) is added because of the three (3) observations given in the (1,1) cell.

(circle one) **0.84** / **357.69** / **3576.98** the p–value, with 1 and 7 degrees of freedom, is given by

$$\text{p–value} = P(F \geq 0.84)$$

which equals (circle one) **0.02** / **0.12** / **0.78**.
The level of significance is 0.05.

(c) *Conclusion* Since the p–value, 0.78, is larger than the level of significance, 0.05, we (circle one) **accept** / **reject** the null hypothesis that the main Poison effect A is zero.

4. *Test Factor B (Antidote), assuming no interaction*

(a) *Statement*
The statement of the test is (check none, one or more):

i. $H_0 : \beta_1 = \beta_2 = \beta_3 = 0$ versus $H_a : \beta_i \neq \beta_j, \; i \neq j, \; i,j = 1,2,3.$

ii. $H_0 : \beta_1 = \beta_2 = \beta_3 = 0$ versus $H_a : \beta_1 \neq \beta_3, \beta_1 \neq \beta_2.$

iii. $H_0 : \beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = 0$ versus
$H_a :$ at least one $\beta_i \neq 0, \; i = 1,2,3.$

iv. $H_0 : \mu_{\cdot 1} = \mu_{\cdot 2} = \mu_{\cdot 3} = \mu_{\cdot 4} = \mu_{\cdot 5}$ versus
$H_a :$ at least one $\mu_{i\cdot} \neq \mu_{j\cdot}, \; i,j = 1,2,3,4,5.$

(b) *Test*
Using the SAS output, the test statistic is

$$\begin{aligned} F^* &= \frac{SSE(R) - SSE(F)}{df_R - df_F} \div \frac{SSE(F)}{df_F} \\ &= \frac{20.76714286 - 4.20130952}{9 - 7} \div \frac{4.20130952}{7} \end{aligned}$$

(circle one) **13.80** / **17.74** / **76.98** the p–value, with 2 and 7 degrees of freedom, is given by

$$\text{p–value} = P(F \geq 13.8)$$

which equals (circle one) **0.004** / **0.034** / **0.043**.
The level of significance is 0.05.

(c) *Conclusion*
Since the p–value, 0.004, is smaller than the level of significance, 0.05, we (circle one) **accept** / **reject** the null hypothesis that the main Antidote effect is zero.

22.5 ANOVA Inferences when Treatment Means Are of Unequal Importance

SAS program: att4-22-5-rats-twoANOVA,unequal

Two–factor studies which involve treatment means of unequal importance are essentially questions about linear combinations (contrasts)[6]. Even though the two–factor studies considered here have *un*equal sample sizes, inference about treatments means of unequal importance are dealt with essentially in the same way as in two–factor studies with equal sample sizes. We look at the special case of testing effects when the weights are proportional to sample size.

Exercise 22.5 (ANOVA Inferences when Treatment Means Are of Unequal Importance)
Consider the effect of different poisons and dosages of an antidote on the survival time of rats.

	Factor B, antidote →	low dosage	medium dosage	high dosage	row ave
Factor A,	I	8.3, 7.2, 7.5 ($\overline{Y}_{11\cdot} = 7.76$)	9.1, 9.4 ($\overline{Y}_{12\cdot} = 9.25$)	10.1, 11.1 ($\overline{Y}_{13\cdot} = 10.6$)	$\overline{Y}_{1\cdot\cdot} = 8.96$
poison	II	1.8, 3.8 ($\overline{Y}_{21\cdot} = 2.8$)	12.1 ($\overline{Y}_{22\cdot} = 12.1$)	15.1, 16.2, 14.4 ($\overline{Y}_{23\cdot} = 15.23$)	$\overline{Y}_{2\cdot\cdot} = 10.57$
	column average	$\overline{Y}_{\cdot 1\cdot} = 5.72$	$\overline{Y}_{\cdot 2\cdot} = 10.2$	$\overline{Y}_{\cdot 3\cdot} = 13.38$	$\overline{Y}_{\cdot\cdot\cdot} = 9.7$

Test for factor A (poison) effects when it is assumed the weights for the treatment means, μ_{ij}, are proportional to the treatment sample sizes, n_{ij}, at $\alpha = 0.05$.

1. *Statement*
   Let $L_1 = \frac{3}{6}\mu_{11} + \frac{2}{6}\mu_{12} + \frac{2}{6}\mu_{13}$
   and $L_2 = \frac{2}{6}\mu_{11} + \frac{1}{6}\mu_{12} + \frac{3}{6}\mu_{13}$ The statement of the test is (check none, one or more):

   (a) $H_0 : L_1 - L_2 = 0$ versus $H_1 : L_1 - L_2 < 0$.

   (b) $H_0 : L_1 - L_2 = 0$ versus $H_1 : L_1 - L_2 > 0$.

   (c) $H_0 : L_1 - L_2 = 0$ versus $H_1 : L_1 - L_2 \neq 0$.

2. *Test*
   From SAS, the test statistic is

   $$F^* = \frac{SSE(R) - SSE(F)}{df_R - df_F} \div \frac{SSE(F)}{df_F} = \frac{SSA}{a-1} \div \frac{SSE(F)}{df_F} = \frac{0.6911905}{2-1} \div \frac{4.83833}{7} =$$

   which is (circle one) **0.93 / 1.00 / 1.65**.
   also, $F(1-\alpha; df_R - df_F, df_F) = F(1 - 0.05; 1, 7) = 5.59$

[6]In section 2 of this chapter, we looked at inference for differences, $\hat{D}$, which is a special case of a linear contrast.

3. *Conclusion.*
   Since the test statistic, 1.00, is smaller than the critical value, 5.59, we (circle one) **accept** / **reject** the null hypothesis that the factor A (poison) effect is zero when it is assumed the weights for the treatment means, μ_{ij}, are proportional to the treatment sample sizes, n_{ij}

22.6 Statistical Computing Packages

We continue to use SAS.