Chapter 5

Control Charts For Variables

We look at control charts for variables (as opposed to attributes).

5.1 Introduction

We look at three types of sets of control charts:

- the $\bar{x}$ (mean) and the R (range) charts.
- the $\bar{x}$ (mean) and the S (standard deviation) charts.
- for individual observations and the R (range) charts.

We will use these control charts to determine a variety of things, but mostly to determine the probability of detecting outliers or not, of determining the capability of a process and of determining normality.

5.2 Control Charts For $\bar{x}$ and R

We look at the use of the $\bar{x}$ (mean) and R (range) charts in quality control processes. We also calculate process capability, C_p, and two versions of *average run length*:

1. ARL_0: average run length, if the process actually is *in* control[1]
2. ARL_1: average run length, if the process actually is *out* of control[2]

Exercise 5.1 (Control Charts For $\bar{x}$ and R)

[1]We want this to be big.
[2]We want this to be small.

1. *Ice Cream Treat Weight, Raw Data*
   Ice cream treats are weighed (in ounces), four at a time ($n = 4$).

sample	x_1	x_2	x_3	x_4
1	7	9	12	14
2	9	5	5	12
3	7	10	11	7
4	8	9	5	14
5	7	8	11	12
6	12	11	9	10
7	16	9	10	7
8	6	5	11	3
9	8	7	10	14
10	15	17	9	14
11	7	12	13	17
12	5	11	8	12
13	16	10	13	14
14	8	13	9	13
15	11	8	11	16
16	15	11	10	14
17	10	9	11	10
18	16	8	11	12
19	8	7	10	13
20	15	17	10	15

   SAS program: att4-5-2-icecream-xrchart

   (a) *$\bar{x}$ chart and R chart*
   From SAS, both the $\bar{x}$ chart and R chart for the ice cream weight is (choose one) **in / out of** statistical control because there are no out-of-control signals, runs, trends, or cycles.

   (b) *(LCL, UCL) for $\bar{x}$ chart and R chart*
   From the SAS graph the average of all averages is,
   $\bar{\bar{x}}$ = (choose one) **10.45 / 10.55 / 10.65**
   The upper control limit and the lower control limit for $\bar{x}$ chart are,

$$\bar{\bar{x}} \pm 3\hat{\sigma}/\sqrt{n} = (LCL, UCL) =$$

   (choose one)

   i. $(5.78, 15.32)$
   ii. $(5.68, 14.32)$
   iii. $(6.78, 15.32)$

Also, from SAS, $\bar{R}$ = (choose one) **6.6 / 7.6 / 8.6**
The upper control limit and the lower control limit for the R chart are, (LCL, UCL) = (choose one)

i. $(5.5, 14.9)$
ii. $(0, 14.9)$
iii. $(-14.9, 14.9)$

(c) *Natural tolerance limits, control limits and specification limits*

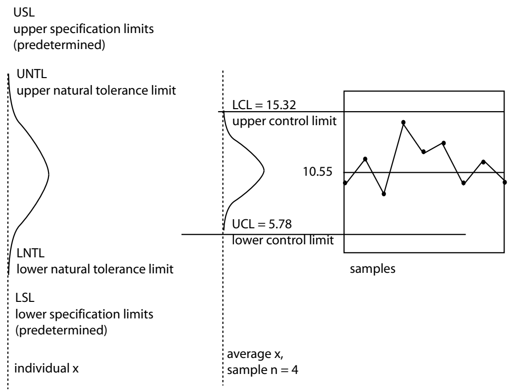

Figure 5.1 (Natural tolerance limits, control limits and specification limits)

On the one hand, the control limits for the $\bar{x}$ chart are,
$\bar{\bar{x}} \pm 3\hat{\sigma}/\sqrt{n} = (LCL, UCL) = (5.78, 15.32)$,
whereas, on the other hand, the *natural tolerance limits* for an *individual* x are

$$(UNTL, LNTL) = \bar{\bar{x}} \pm 3\hat{\sigma} = 10.55 \pm 3(3.18154) =$$

(choose one[3])

i. $(1.01, 20.09)$
ii. $(0.68, 24.32)$
iii. $(-1.02, 25.32)$

and the *specification limits* are (arbitrarily) predetermined limits set as "goals" to be achieved by the ice cream treat process in this case.

(d) *More process standard deviation*
In addition to reading the process standard deviation $\hat{\sigma}$ from SAS, we could also use $d_2 = 2.059$ from Appendix VI, page 761 of the text, and so determine the process standard deviation to be

$$\hat{\sigma} = \bar{R}/d_2 \approx 6.55/2.059 \approx$$

(choose one) **3.18 / 4.34 / 5.66**

[3]The process standard deviation $\hat{\sigma}$ can be found on the SAS printed output.

(e) *Process standard deviation versus standard error*
True / **False**
The process standard deviation $\hat{\sigma} = 3.18$ does not equal the standard deviation of the mean, $\hat{\sigma}_{\bar{x}} = \frac{\hat{\sigma}}{\sqrt{n}} = \frac{3.18}{\sqrt{4}} = 1.59$

(f) *Process Capability*
The process capability, C_p, is a statistic used to measure if the natural tolerance limits (not the control limits!) meet the specification limits or not. For example, if we (arbitrarily) set the specification limits to be

$$(USL, LSL) = (6, 15)$$

and, from above, since we determined the natural tolerance limits to be

$$(UNTL, LNTL) = (1.01, 20.09)$$

then the process capability is, in this case,

$$\begin{aligned} \hat{C}_p &= \frac{USL - LSL}{UNTL - LNTL} \\ &= \frac{15 - 6}{20.09 - 1.01} = \end{aligned}$$

(choose one) **0.47** / **2.12** / **5.66**
which indicates the "natural" process limits are a well outside the "prescribed" specification limits (which is "bad"); in other words, the process is *not* able to meet specifications.

(g) *More process capability*
If we (arbitrarily) set the specification limits to be

$$(USL, LSL) = (0, 21)$$

the process capability is
$C_p = \frac{USL - LSL}{UNTL - LNTL} = \frac{21 - 0}{20.09 - 1.01} =$ (choose one) **0.47** / **1.10** / **5.66**
In other words, the process (choose one) **is** / **is not** able to meet specifications.

(h) *More process capability*
True / **False**

$$\begin{aligned} \hat{C}_p &= \frac{USL - LSL}{UNTL - LNTL} \\ &= \frac{USL - LSL}{6\hat{\sigma}} \\ &= \frac{USL - LSL}{6(\bar{R}/d_2)} \\ &\approx \frac{15 - 6}{6(6.6/2.059)} \\ &\approx 0.47 \end{aligned}$$

(i) *Normal probability plot*
The SAS normal probability plot is more or less linear and so indicates the weights of the ice cream treats (choose one) **are** / **are not** more or less normally distributed.

(j) *New samples in process control?*
Five new samples from the ice cream process are plotted on the *previously* determined control charts above.

sample	x_1	x_2	x_3	x_4
1	17	19	22	24
2	19	14	16	21
3	16	20	21	17
4	19	18	15	24
5	17	18	21	22

From SAS, it appears the ice cream process has gone out of control and, more specifically, has shifted (choose one) **downwards** / **upwards** although the range of the process still appears to be in control.

(k) *Probability in control if there is a shift in mean?*
The probability the process remains in control if the mean shifts from 10.55 to 12 ounces, is equal to the probability $\bar{x}$ falls between the upper control limit and the lower control limit,

$$\begin{aligned} P\,(\text{in control}) &= P\,(\text{LCL } \leq \bar{x} \leq \text{ UCL}) \\ &= P(5.78 \leq \bar{x} \leq 15.32; \mu = 12, \hat{\sigma}_{\bar{x}} = 1.59) \; = \end{aligned}$$

(choose one) **0.47** / **0.78** / **0.98**
Assuming normality, it is 2nd DISTR normalcdf(5.78, 15.32, 12, 1.59)

(l) *More probability in control if there is a shift in mean?*
The probability the process remains in control if the mean shifts to 4 ounces, is equal to the probability $\bar{x}$ falls between the upper control limit and the lower control limit,

$$\begin{aligned} P\,(\text{in control}) &= P\,(\text{LCL } \leq \bar{x} \leq \text{ UCL}) \\ &= P(5.78 \leq \bar{x} \leq 15.32; \mu = 4, \hat{\sigma}_{\bar{x}} = 1.59) \; = \end{aligned}$$

(choose one) **0.13** / **0.68** / **0.87**
Hint: 2nd DISTR normalcdf(5.78, 15.32, 4, 1.59)

(m) *Different ways of saying the same thing*
The probability the process remains in control if the mean shifts to 4 ounces, is another way of saying (choose none, one or more)

i. the probability of *not* detecting a mean shift to 4 ounces.

ii. the probability of *not* detecting the first sample average after the shift out of control, if the mean shifts to 4 ounces.
iii. the probability of a type II error, β, or the probability of accidently rejecting the alternative hypothesis of a mean shift to 4 ounces.
iv. the probability of *not* detecting the *second* sample average after the shift out of control, if the mean shifts to 4 ounces.
v. the probability of *not* detecting the *third* sample average after the shift out of control, if the mean shifts to 4 ounces.

(n) *Probability of not detecting first sample average out of control*
The probability *not* detecting first sample average out of control if the mean shifts to 4 ounces is
$P(5.78 \leq \bar{x} \leq 15.32; \mu = 4, \hat{\sigma}_{\bar{x}} = 1.59) =$ (choose one) **0.13** / **0.68** / **0.87**

(o) *Probability of not detecting third sample average out of control*
The probability *not* detecting *third* sample average out of control if the mean shifts to 4 ounces is
$P(5.78 \leq \bar{x} \leq 15.32; \mu = 4, \hat{\sigma}_{\bar{x}} = 1.59)^3 = (0.13)^3 =$
(choose one) **0.002** / **0.068** / **0.87**

(p) *Probability of detecting mean shift*
The probability *of* detecting a mean shift to 4 ounces is
$1 - P(5.78 \leq \bar{x} \leq 15.32; \mu = 4, \hat{\sigma}_{\bar{x}} = 1.59) =$
(choose one) **0.13** / **0.68** / **0.87**

2. *Ice Cream Treat Weight, Summary Data*[4]
Samples of $n = 4$ units are taken from an ice cream treat process every hour. After 25 samples have been collected, we find $\bar{\bar{x}} = 10$ ounces and $\bar{R} = 7$ ounces.

(a) *Control limits*
The control limits for $\bar{x}$ are

$$\bar{\bar{x}} \pm A_2\bar{R} = 10 \pm 0.729(7) =$$

(choose one)
i. $(4.90, 15.10)$
ii. $(5.68, 14.32)$
iii. $(6.78, 15.32)$

The (estimated) control limits for R are

$$(D_3\bar{R}, D_4\bar{R}) = (0(7), 2.282(7)) =$$

(choose one)

[4]Since summary data is given, we cannot use SAS to help us in this question and must use Appendix VI on page 761 of the text.

i. $(0, 15.10)$
ii. $(0, 14.32)$
iii. $(0, 15.98)$

(b) *Process standard deviation*
The (estimated) process standard deviation is

$$\hat{\sigma} = \bar{R}/d_2 \approx 7/2.059 \approx$$

(choose one) **2.13** / **3.40** / **4.87**

(c) *Process capability*
If the specification limits are (3, 17), the (estimated) process capability is

$$\begin{aligned} \hat{C}_p &= \frac{USL - LSL}{6\hat{\sigma}} \\ &= \frac{USL - LSL}{6(\bar{R}/d_2)} \\ &= \frac{17-3}{6(7/2.059)} \approx \end{aligned}$$

(choose one) **0.13** / **0.40** / **0.69**
which indicates the "natural" process limits are (choose one) **inside** / **outside** the "prescribed" specification limits; the process is not able to meet specifications.

(d) *Probability of not detecting mean process shift*
The probability of *not* detecting a mean process shift from $\mu = 10$ to $\mu = 14$ is equal to the probability $\bar{x}$ falls between the upper control limit and the lower control limit when $\mu = 14$,

$$\begin{aligned} P\,(\text{in control}) &= P\,(\text{LCL } \leq \bar{x} \leq \text{ UCL}) \\ &= P(4.90 \leq \bar{x} \leq 15.10; \mu = 14, \hat{\sigma}_{\bar{x}} \approx 3.40/\sqrt{4}) = \end{aligned}$$

(choose one) **0.13** / **0.40** / **0.74**
2nd DISTR normalcdf(4.9, 15.1, 14, $3.4/\sqrt{4}$)

(e) *Review: Average Run Length (ARL)*
Since $\mu \approx \bar{\bar{x}} = 10$ and $\hat{\sigma} = 3.4$, the probability the process is *in* control is,

$$\begin{aligned} P\,(\text{LCL } \leq \bar{x} \leq \text{ UCL}) &= P(4.9 \leq \bar{x} \leq 15.1; \mu = 10, \hat{\sigma}_{\bar{x}} \approx 3.40/\sqrt{4}) \\ &= 0.9973 \end{aligned}$$

2nd DISTR normalcdf(4.9, 15.1, 10, $3.4/\sqrt{4}$)
and so the ARL_0 is

$$\text{ARL}_0 = \frac{1}{1 - 0.9973} =$$

(choose one) **1.13** / **2.40** / **370.37**

(f) *Control limits, using $\alpha = 0.03$ rather than σ*
If $\alpha = 0.03$, the control limits for $\bar{x}$ are

$$\bar{\bar{x}} \pm z_{0.03/2}\hat{\sigma}_{\bar{x}} = 10 \pm 2.17\left(\frac{3.4}{\sqrt{4}}\right)$$

(choose one)

i. $(4.90, 15.10)$
ii. $(5.68, 14.32)$
iii. $(6.31, 13.69)$

Hint: $z_{0.03/2} = 2.17$, using invNorm(0.985)

5.3 Control Charts For $\bar{x}$ and S

In this section, we look at control charts[5] for $\bar{x}$ and S

Exercise 5.2 (Control Charts For $\bar{x}$ and S: Ice Cream Treat Weight)
Ice cream treats are weighed (in ounces), four at a time ($n = 4$).

sample	x_1	x_2	x_3	x_4
1	7	9	12	14
2	9	5	5	12
3	7	10	11	7
4	8	9	5	14
5	7	8	11	12
6	12	11	9	10
7	16	9	10	7
8	6	5	11	3
9	8	7	10	14
10	15	17	9	14
11	7	12	13	17
12	5	11	8	12
13	16	10	13	14
14	8	13	9	13
15	11	8	11	16
16	15	11	10	14
17	10	9	11	10
18	16	8	11	12
19	8	7	10	13
20	15	17	10	15

[5] Notice, we are looking at a control chart for S, for standard deviation, and not R, for range, as it was done previously.

SAS program: att4-5-3-icecream-xschart

1. *$\bar{x}$ chart and S chart*
   From SAS, both the $\bar{x}$ chart and S chart for the ice cream weight is (choose one) **in** / **out of** statistical control because there are no out-of-control signals, runs, trends, or cycles.

2. *(LCL, UCL) for $\bar{x}$ chart and S chart*
   From the SAS graph, $\bar{\bar{x}}$ = (choose one) **10.45** / **10.55** / **10.65**
   The upper control limit and the lower control limit for $\bar{x}$ chart are,

   $$\bar{\bar{x}} \pm 3\hat{\sigma}/\sqrt{n} = (LCL, UCL) =$$

   (choose one)

   (a) $(5.78, 15.32)$
   (b) $(5.86, 15.24)$
   (c) $(6.78, 15.32)$

   From SAS, $\bar{S}$ = (choose one) **2.9** / **7.6** / **8.6**
   The upper control limit and the lower control limit for the S chart are, (LCL, UCL) = (choose one)

   (a) $(5.5, 14.9)$
   (b) $(0, 14.9)$
   (c) $(0, 6.5)$

3. *More (LCL, UCL)*
   The control limits for the $\bar{x}$ chart and S charts,
   $(LCL, UCL) = (5.86, 15.24)$
   is (choose one) **the same as** / **different from**
   the control limits for the $\bar{x}$ chart and R charts,
   $(LCL, UCL) = (5.78, 15.32)$
   because the process standard deviation, $\hat{\sigma}$ is calculated differently in the two cases.

4. *Process standard deviation*
   In addition to reading the process standard deviation $\hat{\sigma}$ from SAS, we could also use $c_4 = 0.913$ from Appendix VI, page 761 of the text, and so determine the process standard deviation to be

   $$\hat{\sigma} = \bar{S}/c_4 \approx 2.9/0.9213 \approx$$

   (choose one) **3.15** / **4.34** / **5.66**

5. *Process capability*
   If we (arbitrarily) set the specification limits to be

$$(USL, LSL) = (0, 21)$$

   and since the *natural tolerance limits* for an *individual* x are

$$(UNTL, LNTL) = \bar{\bar{x}} \pm 3\hat{\sigma} = 10.55 \pm 3(3.12691) = (1.169, 19.931)$$

   the process capability is
   $C_p = \frac{USL-LSL}{UNTL-LNTL} = \frac{21-0}{19.931-1.169} =$ (choose one) **0.47** / **1.12** / **5.66**
   In other words, the process (choose one) **is** / **is not** able to meet specifications.

6. *More process capability*
   True / **False**

$$\begin{aligned}\hat{C}_p &= \frac{USL - LSL}{UNTL - LNTL} \\ &= \frac{USL - LSL}{6\hat{\sigma}} \\ &= \frac{USL - LSL}{6(\bar{S}/c_4)} \\ &\approx \frac{21 - 0}{6(2.9/0.9213)} \\ &\approx 1.11\end{aligned}$$

7. *Probability in control if there is a shift in mean?*
   The probability the process remains in control if the mean shifts from 10.55 to 17 ounces, is equal to the probability $\bar{x}$ falls between the upper control limit and the lower control limit,

$$\begin{aligned}P\,(\text{in control}) &= P\,(\text{LCL } \le \bar{x} \le \text{ UCL}) \\ &= P(5.86 \le \bar{x} \le 15.24; \mu = 17, \hat{\sigma}_{\bar{x}} \approx 3.12691/\sqrt{4}) \;=\end{aligned}$$

   (choose one) **0.13** / **0.78** / **0.98**
   Assuming normality, it is 2nd DISTR normalcdf(5.86, 15.24, 17, 1.56)

8. *Probability of not detecting first sample average out of control*
   The probability *not* detecting first sample average out of control if the mean shifts from 10.55 to 17 ounces is
   $P(5.86 \le \bar{x} \le 15.24; \mu = 17, \hat{\sigma}_{\bar{x}} = 1.56) =$ (choose one) **0.13** / **0.68** / **0.87**

5.4 The Shewhart Control Chart For Individual Measurements

In this section, we look at Shewhart control charts for individual measurements.

Exercise 5.3 (Shewhart control charts for individual measurements)
Thirty (30) *individual* ice cream treats are weighed (in ounces).

5.4	8.6	9.5	4.0	0.9	5.2	5.5
2.8	5.3	6.3	3.9	9.8	6.9	9.8
5.1	8.4	1.0	1.2	7.1	5.7	0.6
1.0	3.0	6.0	7.2	8.0	5.9	0.0
7.9	3.4					

SAS program: att4-5-4-icecream-irchart

1. *Individual measurement chart and moving range MR chart*
   From SAS, both the individual chart and *moving range* MR chart for the ice cream weight is (choose one) **in** / **out of** statistical control because there are no out-of-control signals, runs, trends, or cycles.

2. *Moving range, MR_i*
   Since the weight of the first ice cream treat is $x_0 = 5.4$ and the weight of the second ice cream treat is $x_1 = 8.6$, the first moving range (based on $n = 2$ observations) is
   $$\text{MR}_1 = |x_1 - x_0| = |8.6 - 5.4| = 3.2$$
   and so the second moving range is
   $$\text{MR}_2 = |x_2 - x_1| = |9.5 - 8.6| =$$
   (choose one) **0.5** / **0.9** / **1.6**
   The average of all moving ranges is denoted $\overline{\text{MR}}$.

3. (LCL, UCL) *for individual measurement chart*
   From the SAS graph, $\bar{\bar{x}}$ = (choose one) **5.2** / **10.5** / **10.6**
   The upper control limit and the lower control limit for the individual measurement chart are,
   $$\begin{aligned} \bar{\bar{x}} \pm 3\hat{\sigma} &= \bar{\bar{x}} \pm 3\bar{\bar{R}}/d_2 \\ &= \bar{\bar{x}} \pm 3\overline{\text{MR}}/d_2 \\ &= 5.2 \pm 3(3.22069/1.128) \\ &= (LCL, UCL) = \end{aligned}$$
   (choose one)

(a) $(-5.4, 11.7)$
(b) $(-3.4, 13.7)$
(c) $(-1.4, 15.7)$

4. *(LCL, UCL) for moving range chart*
   From the SAS graph, $\bar{R} =$ (choose one) **3.22 / 5.52 / 10.6**
   The upper control limit and the lower control limit for the moving range chart are, (choose one)

   (a) $(0, 10.52)$
   (b) $(0, 11.72)$
   (c) $(0, 15.75)$

5. *Difficulty with moving range charts*
   True / False
   Since the moving range chart is calculated using two consecutive observations in the individual measurements chart, the two charts are highly dependent on one another. For example, a sharp increase between two values in the individual chart will cause a sharp increase in the range chart. This dependence between charts worries some analysts enough to not use the moving range chart.

5.5 Summary of Procedures For $\bar{x}$, R and S Charts

This is covered in the previous three sections.

5.6 Applications of Variables Control Charts

There a number of interesting examples given in this section.