# Chapter 3

# The Derivative

The slope of the tangent line to a curve is called the *derivative*. Derivatives are defined and graphed after first discussing limits, continuity and rates of change.

## 3.1 Limits

The *limit* $L$ of function $f(x)$ as $x$ approaches (but does not equal) $a$ (from both sides of $a$) is written

$$\lim_{x \to a} f(x) = L$$

where $a$ and $L$ are both real numbers and where values of $f(x)$ approach (and perhaps equal) $L$. The limit of $L$ as $x$ approaches $a$ *does not exist* if

- as $x$ approaches $a$ from both sides, $f(x)$ approaches either positive (denoted $\lim_{x \to a} f(x) = \infty$) or negative infinity (denoted $\lim_{x \to a} f(x) = -\infty$), or
- as $x$ approaches $a$ from one side, $f(x)$ approaches positive infinity, but as $x$ approaches $a$ from the other side $f(x)$ approaches negative infinity or vis-versa,
- $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = M$, where $L \neq M$

If $a$, $A$, $B$ are real numbers, $f$ and $g$ are functions and

$$\lim_{x \to a} f(x) = A, \qquad \lim_{x \to a} g(x) = B,$$

then

1. If $k$ is a constant, $\lim_{x \to a} k = k$ and $\lim_{x \to a}[k \cdot f(x)] = k \cdot \lim_{x \to a} f(x) = k \cdot A$
2. $\lim_{x \to a}[f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = A \pm B$
3. $\lim_{x \to a}[f(x) \cdot g(x)] = [\lim_{x \to a} f(x)] \cdot [\lim_{x \to a} g(x)] = A \cdot B$
4. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{A}{B}, if B \neq 0$

5. If $p(x)$ is a polynomial, then $\lim_{x\to a} p(x) = p(a)$
6. For any real $k$, $\lim_{x\to a}[f(x)]^k = [\lim_{x\to a} f(x)]^k = A^k$, provided limit exists
7. $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ if $f(x) = g(x)$ for all $x \neq a$
8. For any real number $b > 0$, $\lim_{x\to a} b^{f(x)} = b^{[\lim_{x\to a} f(x)]} = b^A$
9. For any real number $b$ where $0 < b < 1$ or $b > 1$,
   $\lim_{x\to a}[\log_b f(x)] = \log_b[\lim_{x\to a} f(x)] = \log_b A$, if $A > 0$

Limits *at infinity* for $f(x) = \frac{p(x)}{q(x)}, q(x) \neq 0$, such as $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$, determined by

- dividing $p(x)$ and $q(x)$ by *highest power of* $q(x)$ *(not* $p(x)$*!)*
- then, for positive real $n$, using

$$\lim_{x\to\infty} \frac{1}{x^n} = 0, \qquad \lim_{x\to-\infty} \frac{1}{x^n} = 0$$

  where if $x < 0$, $x^n$ does not necessarily always exist (for example, for $n = \frac{1}{2}$) and so limit also does not exist in these cases

## Exercise 2.1 (Limits)

1. *Limits for function* $f(x) = \frac{(2x-3)(0.5x+1)}{2x-3}$.

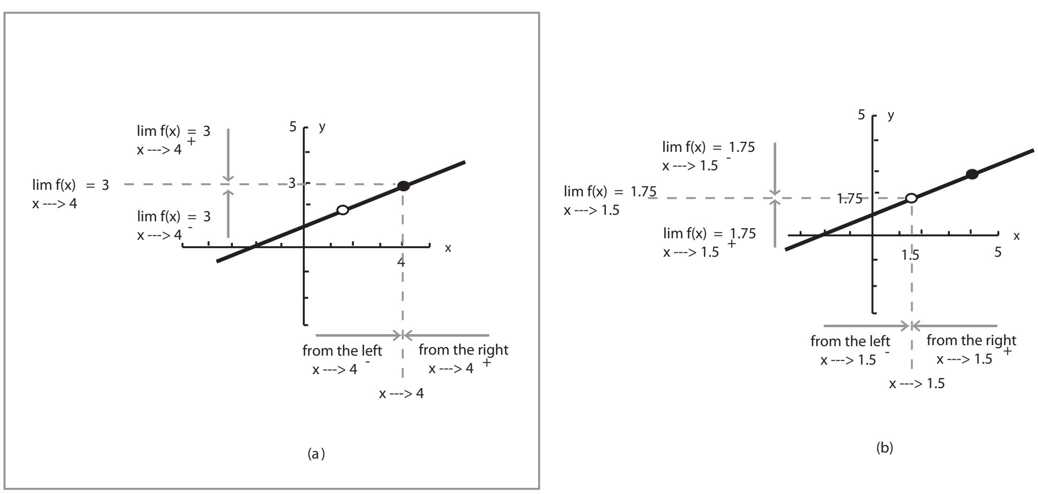



Figure 3.1 (Limits of $f(x) = \frac{(2x-3)(0.5x+1)}{2x-3}$)

(Type $y = \frac{(2x-3)(0.5x+1)}{2x-3}$ into Y=, then WINDOW -5 5 1 -5 5 1, then GRAPH; use WINDOW 1.45 1.55 1 1.72 1.78 1 with dotted line to see the removable discontinuity.)

(a) *Limit as $x \to 4$, Figure (a).*

i. *left limit, by table.* Complete the following table.

| $x \to$ | 3.9 | 3.99 | 3.999 | 3.9999 | 3.99999 |
|---|---|---|---|---|---|
| $f(x) = \frac{(2x-3)(0.5x+1)}{2x-3} \to$ | 2.95 | 2.995 | 2.9995 | 2.99995 | ______ |

(Type 2nd TBLSET 1 1 Ask Auto, then 2nd TABLE 3.9 ENTER 3.99 ENTER and so on.)

So, as $x$ approaches (but does not equal) 4 *from the left*,
$\lim_{x \to 4^-} f(x) =$ (i) **3** (ii) **4** (iii) **5**.

ii. *right limit, by table.* Complete the following table.

| 4.00001 | 4.0001 | 4.001 | 4.01 | 4.1 | $\leftarrow x$ |
|---|---|---|---|---|---|
| ______ | 3.0005 | 3.005 | 3.05 | 3.05 | $\leftarrow f(x) = \frac{(2x-3)(0.5x+1)}{2x-3}$ |

(Type 2nd TBLSET 1 1 Ask Auto, then 2nd TABLE 4.1 ENTER 4.01 ENTER and so on.)

So, as $x$ approaches (but does not equal) 4 *from the right*,
$\lim_{x \to 4^+} f(x) =$ (i) **3** (ii) **4** (iii) **5**.

iii. As $x$ approaches (but does not equal) 4 *from either left or the right*, so $\lim_{x \to 4} f(x) =$ (i) **3** (ii) **4** (iii) **5**.

iv. The value of the function at $x = 4$ is given by
$f(4) = \frac{(2(4)-3)(0.5(4)+1)}{2(4)-3} =$ (i) **3** (ii) **4** (iii) **5**.

v. *Limit* $L = \lim_{x \to 4} f(x)$ (i) **exists** (ii) **does not exist** (it equals 3), and *value* $f(4)$ (i) **exists** (ii) **does not exist** (it equals 3), so limit $L = 3$ is (i) **equal to** (ii) **different from** value $f(4) = 3$.

(b) *Limit as $x \to 1.5$, Figure (b).*

i. *left limit, by table.*

| $x \to$ | 1.4 | 1.49 | 1.499 | 1.4999 | 1.49999 |
|---|---|---|---|---|---|
| $f(x) = \frac{(2x-3)(0.5x+1)}{2x-3} \to$ | 1.7 | 1.745 | 1.7495 | 1.75 | ______ |

(Type 2nd TBLSET 1 1 Ask Auto, then 2nd TABLE 1.4 ENTER 1.49 ENTER and so on.)

so $\lim_{x \to 1.5^-} f(x) =$ (i) **1.50** (ii) **1.75** (iii) **1.80**.

ii. *right limit, by table.*

| 1.50001 | 1.5001 | 1.501 | 1.51 | 1.6 | $\leftarrow x$ |
|---|---|---|---|---|---|
| ______ | 1.7501 | 1.7505 | 1.755 | 1.8 | $\leftarrow f(x) = \frac{(2x-3)(0.5x+1)}{2x-3}$ |

(Type 2nd TBLSET 1 1 Ask Auto, then 2nd TABLE 1.6 ENTER 1.51 ENTER and so on.)

so $\lim_{x \to 1.5^+} f(x) =$ (i) **1.50** (ii) **1.75** (iii) **1.80**.

iii. so $\lim_{x \to 1.5} f(x) =$ (i) **1.50** (ii) **1.75** (iii) **1.80**.

iv. The value of the function at $x = 1.5$ is given by
$f(1.5) = \frac{(2(1.5)-3)(0.5(1.5)+1)}{2(1.5)-3} =$ (i) **3** (ii) **4** (iii) **5** (iv) **does not exist**.

v. Limit $\lim_{x \to 1.5} f(x)$ (i) **exists** (ii) **does not exist** (it equals 1.75), but value $f(1.5)$ (i) **exists** (ii) **does not exist**.

2. *Limits for function $f(x) = \frac{x^2+x-6}{x^2-4}$.*

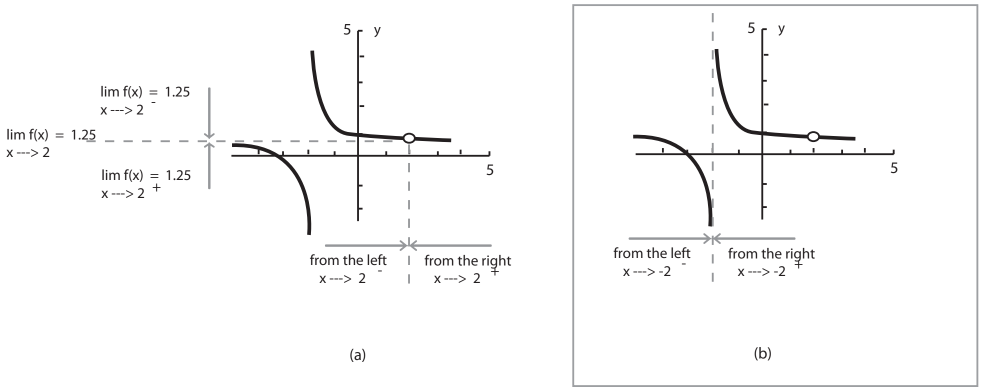



Figure 3.2 (Limits of $f(x) = \frac{x^2+x-6}{x^2-4}$)

(Type $y = \frac{x^2+x-6}{x^2-4}$ into Y=, then WINDOW -5 5 1 -5 5 1, then GRAPH.)

(a) *Figure (a), limit as $x \to 2$.*

i. *left limit, by table.*

| $x \to$ | 1.9 | 1.99 | 1.999 | 1.9999 | 1.99999 |
|---|---|---|---|---|---|
| $f(x) \to$ | 1.2564 | 1.2506 | 1.2501 | 1.25 | 1.25 |

(Type 2nd TBLSET 1 1 Ask Auto, then 2nd TABLE 1.9 ENTER and so on.)

so $\lim_{x \to 2^-} f(x) =$ (i) **0.50** (ii) **0.75** (iii) **1.25**.

ii. *right limit, by table.*

| 2.00001 | 2.0001 | 2.001 | 2.01 | 2.1 | $\leftarrow x$ |
|---|---|---|---|---|---|
| 1.25 | 1.25 | 1.2499 | 1.2494 | 1.2439 | $\leftarrow f(x)$ |

(Type 2nd TBLSET 1 1 Ask Auto, then 2nd TABLE 2.1 ENTER and so on.)

so $\lim_{x \to 2^+} f(x) =$ (i) **0.50** (ii) **0.75** (iii) **1.25**.

iii. so $\lim_{x \to 2} f(x) =$ (i) **0.50** (ii) **0.75** (iii) **1.25**.

iv. also, $f(2) = \frac{(2)^2+(2)-6}{(2)^2-4} =$ (i) **does not exist** (ii) **0** (iii) **3**

v. so $\lim_{x \to 2} f(x)$ (i) **exists** (ii) **does not exist** (it equals 1.25).
but $f(2)$ (i) **exists** (ii) **does not exist** (removable discontinuity).

(b) *Figure (b), limit as $x \to -2$.*

i. *left limit, by table.*

| $x \to$ | -2.1 | -2.01 | -2.001 | -2.0001 | -2.00001 |
|---|---|---|---|---|---|
| $f(x) \to$ | -9 | -99 | -999 | -9999 | -99999 |

(Type 2nd TBLSET 1 1 Ask Auto, then 2nd TABLE -2.1 ENTER and so on.)

so $\lim_{x \to -2^-} f(x) =$ (i) **0** (ii) $-\boldsymbol{\infty}$ (iii) $\boldsymbol{\infty}$.

ii. *right limit, by table.*

| -1.99999 | -1.9999 | -1.999 | -1.99 | -1.9 | $\leftarrow x$ |
|---|---|---|---|---|---|
| 100001 | 10001 | 1001 | 101 | 11 | $\leftarrow f(x)$ |

(Type 2nd TBLSET 1 1 Ask Auto, then 2nd TABLE -1.9 ENTER and so on.)

so $\lim_{x \to -2^+} f(x) =$ (i) **0** (ii) $-\boldsymbol{\infty}$ (iii) $\boldsymbol{\infty}$.

iii. so $\lim_{x\to -2} f(x) =$ (i) **exists** (ii) **does not exist**.

iv. also, $f(-2) = \frac{(-2)^2+(-2)-6}{(-2)^2-4} =$ (i) **does not exist** (ii) **0** (iii) **3**

v. so $\lim_{x\to -2} f(x)$ (i) **exists** (ii) **does not exist** and $f(-2)$ (i) **exists** (ii) **does not exist**.

3. *Limits for* $f(x) = \frac{1}{x^2}$ *and* $f(x) = \frac{1}{x}$.

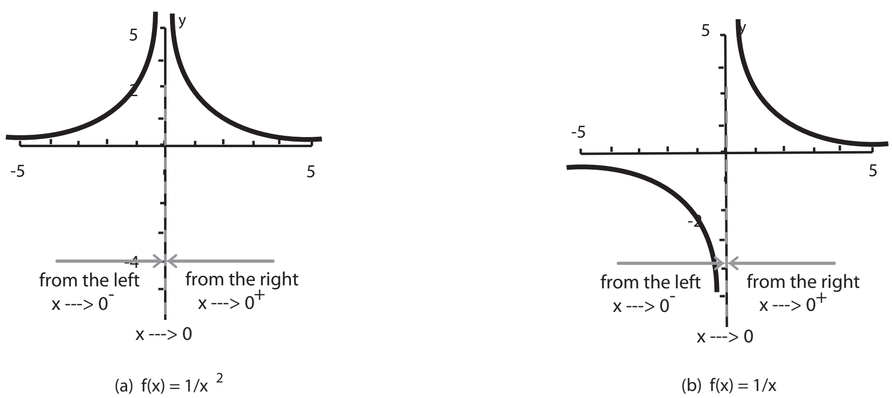



Figure 3.3 (Limits for $f(x) = \frac{1}{x^2}$ and $f(x) = \frac{1}{x}$)

(Type $y = \frac{1}{x^2}$ into $Y_1 =$ and $y = \frac{1}{x}$ into $Y_2 =$, then WINDOW -5 5 1 -5 5 1, then GRAPH.)

(a) *Function (a).* $f(x) = \frac{1}{x^2}$.

i. *left limit, by table.*

| $x \to$ | -0.1 | -0.01 | -0.001 | -0.0001 | -0.00001 |
|---|---|---|---|---|---|
| $f(x) \to$ | $10^2$ | $10^4$ | $10^6$ | $10^8$ | $10^{10}$ |

(Type 2nd TBLSET 1 1 Ask Auto, then 2nd TABLE -0.1 ENTER and so on.)

so $\lim_{x\to 0^-} \frac{1}{x^2} =$ (i) **0** (ii) $-\infty$ (iii) $\infty$.

ii. *right limit, by table.*

| 0.00001 | 0.0001 | 0.001 | 0.01 | 0.1 | $\leftarrow x$ |
|---|---|---|---|---|---|
| $10^{10}$ | $10^8$ | $10^6$ | $10^4$ | $10^2$ | $\leftarrow f(x)$ |

(Type 2nd TBLSET 1 1 Ask Auto, then 2nd TABLE 0.1 ENTER and so on.)

so $\lim_{x\to 0^+} \frac{1}{x^2} =$ (i) **0** (ii) $-\infty$ (iii) $\infty$.

iii. so $\lim_{x\to 0} f(x) =$ (i) **0** (ii) $-\infty$ (iii) $\infty$.

iv. also, $f(0) = \frac{1}{0^2} =$ (i) **does not exist** (ii) **0** (iii) **3**

v. so $\lim_{x\to 0} f(x) = \infty$, so (i) **exists** (ii) **does not exist** and $f(0)$ (i) **exists** (ii) **does not exist**.

(b) *Function (b).* $f(x) = \frac{1}{x}$.

i. *left limit, by table.*

| $x \to$ | -0.1 | -0.01 | -0.001 | -0.0001 | -0.00001 |
|---|---|---|---|---|---|
| $f(x) \to$ | $-10^1$ | $-10^2$ | $-10^3$ | $-10^4$ | $-10^5$ |

(Type 2nd TBLSET 1 1 Ask Auto, then 2nd TABLE -0.1 ENTER and so on.)
so $\lim_{x\to 0^-} \frac{1}{x}$ = (i) **0** (ii) $-\infty$ (iii) $\infty$.

ii. *right limit, by table.*

| | | | | | |
|---|---|---|---|---|---|
| 0.00001 | 0.0001 | 0.001 | 0.01 | 0.1 | $\leftarrow x$ |
| $10^5$ | $10^4$ | $10^3$ | $10^2$ | $10^1$ | $\leftarrow f(x)$ |

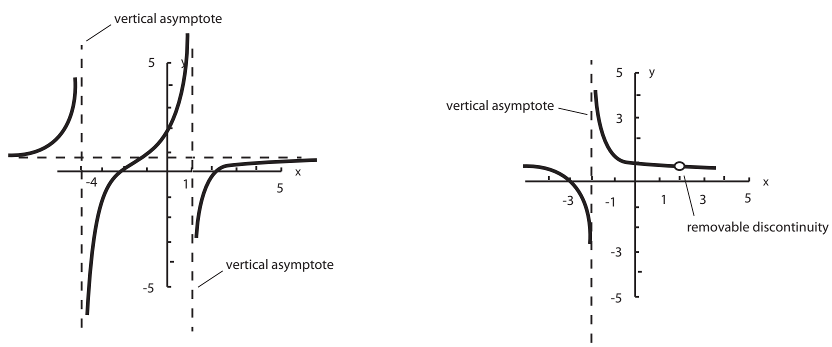
(Type 2nd TBLSET 1 1 Ask Auto, then 2nd TABLE 0.1 ENTER and so on.)
so $\lim_{x\to 0^+} \frac{1}{x}$ = (i) **0** (ii) $-\infty$ (iii) $\infty$.

iii. so $\lim_{x\to 0} f(x)$ = (i) **does not exist** (ii) $-\infty$ (iii) $\infty$.

iv. also, $f(0) = \frac{1}{0}$ = (i) **does not exist** (ii) **0** (iii) **3**

v. so $\lim_{x\to 0} f(x)$ (i) **exists** (ii) **does not exist**
and $f(0)$ (i) **exists** (ii) **does not exist**.

4. *Limits with vertical asymptotes and removable discontinuities.*




(a) f(x) = (x + 3)(x - 2) /((x + 4)(x - 1)) (b) f(x) = (x + 3)(x - 2)/((x + 2)(x - 2))

Figure 3.4 (More limits)

(a) *Function (a).* $f(x) = \frac{(x+3)(x-2)}{(x+4)(x-1)}$.

i. $\lim_{x\to -4^-} \frac{(x+3)(x-2)}{(x+4)(x-1)}$ = (i) **0** (ii) **1** (iii) $\infty$ (iv) $-\infty$.
(Hint: Look at graph of function (a).)

ii. $\lim_{x\to -4^+} \frac{(x+3)(x-2)}{(x+4)(x-1)}$ = (i) **0** (ii) **1** (iii) $\infty$ (iv) $-\infty$.

iii. so vertical asymptote at (i) (i) $\boldsymbol{x = -4}$ (ii) $\boldsymbol{x = 0}$ (iii) $\boldsymbol{x = 1}$

iv. $\lim_{x\to 1^-} \frac{(x+3)(x-2)}{(x+4)(x-1)}$ = (i) **0** (ii) **1** (iii) $\infty$ (iv) $-\infty$.

v. $\lim_{x\to 1^+} \frac{(x+3)(x-2)}{(x+4)(x-1)}$ = (i) **0** (ii) **1** (iii) $\infty$ (iv) $-\infty$.

vi. so vertical asymptote at (i) (i) $\boldsymbol{x = -4}$ (ii) $\boldsymbol{x = 0}$ (iii) $\boldsymbol{x = 1}$

vii. **True** / **False** Vertical asymptotes occur when the denominator of a (simplified) rational function equals zero.

(b) *Function (b).* $f(x) = \frac{(x+3)(x-2)}{(x+2)(x-2)}$.

i. *simplifying* $f(x)$.

$$\frac{(x+3)(x-2)}{(x+2)(x-2)} = \frac{(x+3)}{(x+2)}, x \neq 2$$

where (i) $\boldsymbol{x \neq -2}$ (ii) $\boldsymbol{x \neq 0}$ (iii) $\boldsymbol{x \neq 2}$.

ii. $\lim_{x \to -2^-} \frac{(x+3)}{(x+2)} =$ (i) **0** (ii) **1.5** (iii) $\boldsymbol{\infty}$ (iv) $\boldsymbol{-\infty}$.
(Hint: Look at graph in function (b).)

iii. $\lim_{x \to -2^+} \frac{(x+3)}{(x+2)} =$ (i) **0** (ii) **1.5** (iii) $\boldsymbol{\infty}$ (iv) $\boldsymbol{-\infty}$.

iv. so vertical asymptote at (i) $\boldsymbol{x = -2}$ (ii) $\boldsymbol{x = 0}$ (iii) $\boldsymbol{x = 1.5}$

v. $\lim_{x \to 2} \frac{(x+3)}{(x+2)} =$ (i) **0** (ii) **1.25** (iii) $\boldsymbol{\infty}$ (iv) $\boldsymbol{-\infty}$.
(Look at graph of function (b))

vi. **True** / **False** "Potential" vertical asymptote at $x = 2$ "cancelled out" in *simplified* rational function and, as a consequence, appears, instead, as a removable discontinuity.

5. *More limits with vertical asymptotes and removable discontinuities.*

(a) Function $f(x) = \frac{(x-2)(x+2)(x-5)}{(x-2)(x+2)(x-1)}$
has vertical asymptote(s) at (circle none, one or more)
(i) $\boldsymbol{x = -2}$ (ii) $\boldsymbol{x = 1}$ (iii) $\boldsymbol{x = 2}$ (iv) **none**
and removable discontinuities at (circle none, one or more)
(i) $\boldsymbol{x = -2}$ (ii) $\boldsymbol{x = 1}$ (iii) $\boldsymbol{x = 2}$ (iv) **none**

(b) Function $f(x) = \frac{(x-2)(x+3)(x-5)}{(x-2)(x+2)(x-1)}$
has vertical asymptote(s) at (circle none, one or more)
(i) $\boldsymbol{x = -2}$ (ii) $\boldsymbol{x = 1}$ (iii) $\boldsymbol{x = 2}$ (iv) **none**
and removable discontinuities at (circle none, one or more)
(i) $\boldsymbol{x = -2}$ (ii) $\boldsymbol{x = 1}$ (iii) $\boldsymbol{x = 2}$ (iv) **none**

(c) Function $f(x) = \frac{x^2+x-6}{(x-2)(x+2)(x-1)}$
has vertical asymptote(s) at (circle none, one or more)
(i) $\boldsymbol{x = -2}$ (ii) $\boldsymbol{x = 1}$ (iii) $\boldsymbol{x = 2}$ (iv) **none**
and removable discontinuities at (circle none, one or more)
(i) $\boldsymbol{x = -2}$ (ii) $\boldsymbol{x = 1}$ (iii) $\boldsymbol{x = 2}$ (iv) **none**

(d) Function $f(x) = \frac{x^2+x-6}{(x+4)(x-1)(x+5)}$
has vertical asymptote(s) at (circle none, one or more)
(i) $\boldsymbol{x = -4}$ (ii) $\boldsymbol{x = -5}$ (iii) $\boldsymbol{x = 1}$ (iv) **none**
and removable discontinuities at (circle none, one or more)
(i) $\boldsymbol{x = -4}$ (ii) $\boldsymbol{x = -5}$ (iii) $\boldsymbol{x = 1}$ (iv) **none**

(e) Function $f(x) = \frac{x^2+x-6}{\left(x+\frac{3}{2}\right)\left(x-\sqrt{3}\right)}$
has vertical asymptote(s) at (circle none, one or more)

(i) $\boldsymbol{x = -\frac{3}{2}}$ (ii) $\boldsymbol{x = 0}$ (iii) $\boldsymbol{x = \sqrt{3}}$ (iv) **none**
and removable discontinuities at (circle none, one or more)
(i) $\boldsymbol{x = -\frac{3}{2}}$ (ii) $\boldsymbol{x = 0}$ (iii) $\boldsymbol{x = \sqrt{3}}$ (iv) **none**

(f) **True** / **False** Function $f(x) = \frac{x-3}{x^2-5x+7}$, has no vertical asymptote because $x^2 - 5x + 7$ has no (real) zeros.
(Sketch $x^2 - 3x + 7$ and notice that this function does not cross the $x$–axis.)

6. *Limits for piecewise functions.*

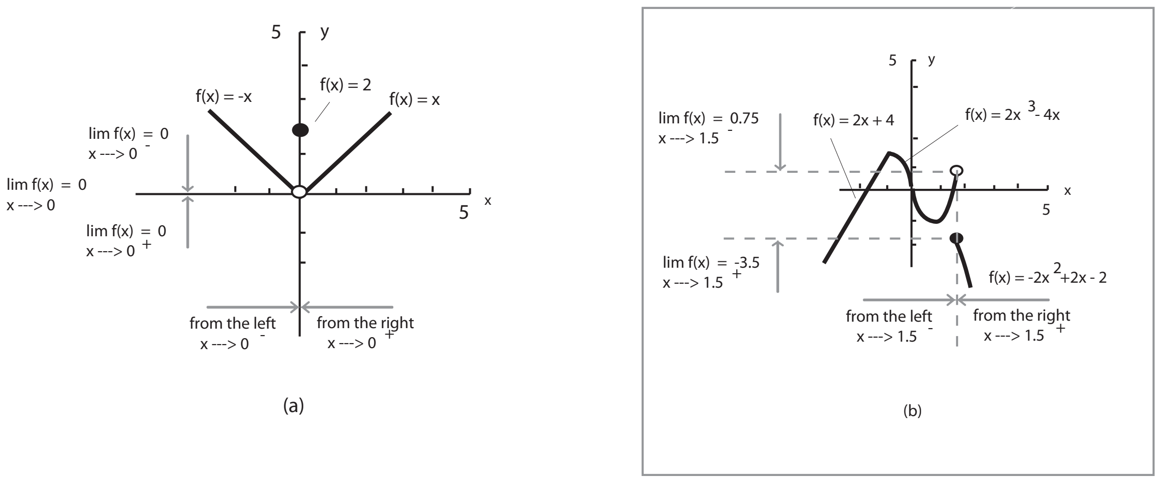



Figure 3.5 (Limits for piecewise functions)

(use WINDOW -5 5 1 -5 5 1, use dotted line, use 2nd TEST for in/equalities,
For (a), type $y = (-x)(x < 0) + (2)(x = 0) + (x)(x > 0)$ into Y=
For (b), type $y = (2x + 4)(X \leq -1) + (2x^3 - 4x)(x > -1)(x < 1.5) + (-2x^2 + 2x - 2)(x \geq 1.5)$ into Y=)

(a) *Figure (a), limit as $x \to 0$.*

$$f(x) = \begin{cases} x & \text{if } x > 0 \\ 2 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

i. *left limit, by table.*

| $x \to$ | -0.1 | -0.01 | -0.001 | -0.0001 | -0.00001 |
|---|---|---|---|---|---|
| $f(x) \to$ | 0.1 | 0.01 | 0.001 | 0.0001 | 0.00001 |

(Type 2nd TBLSET 1 1 Ask Auto, then 2nd TABLE -0.1 ENTER and so on.)
$\lim_{x\to 0^-} f(x) =$ (i) **0** (ii) $-\infty$ (iii) $\infty$.

ii. *right limit, by table.*

| | | | | | |
|---|---|---|---|---|---|
| 0.00001 | 0.0001 | 0.001 | 0.01 | 0.1 | $\leftarrow x$ |
| 0.00001 | 0.0001 | 0.001 | 0.01 | 0.1 | $\leftarrow f(x)$ |

(Type 2nd TBLSET 1 1 Ask Auto, then 2nd TABLE 0.1 ENTER and so on.)

iii. $\lim_{x \to 0^+} f(x) =$ (i) **0** (ii) $-\infty$ (iii) $\infty$.

iv. so $\lim_{x \to 0} f(x) =$ (i) **0** (ii) $-\infty$ (iii) $\infty$.

v. also, $f(0) =$ (i) **0** (ii) **1** (iii) **2**

vi. so $\lim_{x \to 0} f(x)$ (i) **exists** (ii) **does not exist** (it is 0)
and $f(0)$ (i) **exists** (ii) **does not exist** (it is 2)
and limit $L = 0$ is (i) **equal to** (ii) **different from** value $f(0) = 2$.

(b) *Figure (b), limit as* $x \to 1.5$.

$$f(x) = \begin{cases} 2x + 4 & \text{if } x \leq -1 \\ 2x^3 - 4x & \text{if } -1 < x < 1.5 \\ -2x^2 + 2x - 2 & \text{if } 1.5 \leq x, \end{cases}$$

i. *left limit, by table.*

| $x \to$ | 1.4 | 1.49 | 1.499 | 1.4999 | 1.49999 |
|---|---|---|---|---|---|
| $f(x) \to$ | -0.112 | -0.6559 | 0.74051 | 0.74905 | 0.74991 |

(Type 2nd TBLSET 1 1 Ask Auto, then 2nd TABLE 1.4 ENTER 1.49 ENTER and so on.)

$\lim_{x \to 1.5^-} f(x) =$ (i) **0.75** (ii) **1.75** (iii) **2.75**.

ii. *right limit, by table.*

| 1.50001 | 1.5001 | 1.501 | 1.51 | 1.6 | $\leftarrow x$ |
|---|---|---|---|---|---|
| -3.5 | -3.5 | -3.504 | -3.54 | -3.92 | $\leftarrow f(x)$ |

(Type 2nd TBLSET 1 1 Ask Auto, then 2nd TABLE 1.6 ENTER 1.51 ENTER and so on.)

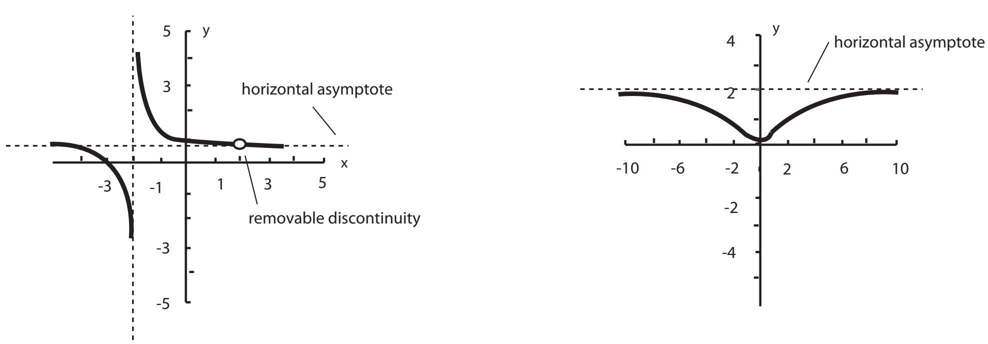
$\lim_{x \to 1.5^+} f(x) =$ (i) $\mathbf{-1.5}$ (ii) $\mathbf{-2.5}$ (iii) $\mathbf{-3.5}$.

iii. so $\lim_{x \to 1.5} f(x) =$ (i) **does not exist** (ii) **1.75** (iii) **1.80**.

iv. and $f(1.5) =$ (i) $\mathbf{-1.5}$ (ii) $\mathbf{-2.5}$ (iii) $\mathbf{-3.5}$.

v. Limit $\lim_{x \to 1.5} f(x)$ (i) **exists** (ii) **does not exist**,
but value $f(1.5)$ (i) **exists** (ii) **does not exist** (it is -3.5).

7. *Limits at infinity (horizontal asymptotes).*




(a) f(x) = (x² + x - 6) / (x² - 4)

(b) f(x) = (2x² + 2) / (x² + 5)

Figure 3.6 (Limits at infinity)

(a) *Function (a).* $f(x) = \frac{x^2+x-6}{x^2-4}$.

i. *limit at positive infinity, by table.*

| $x \rightarrow$ | 10 | 100 | 1000 | 10000 | 100000 |
|---|---|---|---|---|---|
| $f(x) \rightarrow$ | 1.0833 | 1.0098 | 1.001 | 1.0001 | 1.00001 |

(Type 2nd TBLSET 1 1 Ask Auto, then 2nd TABLE 10 ENTER and so on.)

so $\lim_{x\to\infty} \frac{x^2+x-6}{x^2-4} =$ (i) **0** (ii) **1** (iii) **2**.

ii. *limit at positive infinity, by algebra.*

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^2+x-6}{x^2-4} &= \lim_{x\to\infty} \frac{\frac{x^2}{x^2}+\frac{x}{x^2}-\frac{6}{x^2}}{\frac{x^2}{x^2}-\frac{4}{x^2}} \\
&= \lim_{x\to\infty} \frac{1+\frac{1}{x}-\frac{6}{x^2}}{1-\frac{4}{x^2}} \\
&= \frac{1+0-0}{1-0} =
\end{aligned}$$

(i) **0** (ii) **1** (iii) **2**.

iii. *limit at negative infinity, by table.*

| -100000 | -10000 | -1000 | -100 | -10 | $\leftarrow x$ |
|---|---|---|---|---|---|
| 0.99999 | 0.9999 | 0.999 | 0.9898 | 0.875 | $\leftarrow f(x)$ |

(Type 2nd TBLSET 1 1 Ask Auto, then 2nd TABLE -10 ENTER and so on.)

so $\lim_{x\to-\infty} \frac{x^2+x-6}{x^2-4} =$ (i) **0** (ii) **1** (iii) **2**.

iv. *limit at negative infinity, by algebra.*

$$\begin{aligned}
\lim_{x\to-\infty} \frac{x^2+x-6}{x^2-4} &= \lim_{x\to-\infty} \frac{\frac{x^2}{x^2}+\frac{x}{x^2}-\frac{6}{x^2}}{\frac{x^2}{x^2}-\frac{4}{x^2}} \\
&= \lim_{x\to-\infty} \frac{1+\frac{1}{x}-\frac{6}{x^2}}{1-\frac{4}{x^2}} \\
&= \frac{1+0-0}{1-0} =
\end{aligned}$$

(i) **0** (ii) **1** (iii) **2**.

v. so horizontal asymptote at (i) $\boldsymbol{y=0}$ (ii) $\boldsymbol{y=1}$ (iii) $\boldsymbol{y=2}$

(b) *Function (b).* $f(x) = \frac{2x^2+2}{x^2+5}$.

i. *limit at positive infinity.*

$$\begin{aligned}
\lim_{x\to\infty} \frac{2x^2+2}{x^2+5} &= \lim_{x\to\infty} \frac{\frac{2x^2}{x^2}+\frac{2}{x^2}}{\frac{x^2}{x^2}+\frac{5}{x^2}} \\
&= \lim_{x\to\infty} \frac{2+\frac{2}{x^2}}{1+\frac{5}{x^2}} =
\end{aligned}$$

(i) **0** (ii) **1** (iii) **2**.

ii. *limit at negative infinity.*

$$\lim_{x\to-\infty}\frac{2x^2+2}{x^2+5} = \lim_{x\to-\infty}\frac{\frac{2x^2}{x^2}+\frac{2}{x^2}}{\frac{x^2}{x^2}+\frac{5}{x^2}} = \lim_{x\to-\infty}\frac{2+\frac{2}{x^2}}{1+\frac{5}{x^2}} =$$

(i) **0** (ii) **1** (iii) **2**.

iii. so horizontal asymptote at (i) $\boldsymbol{y=0}$ (ii) $\boldsymbol{y=1}$ (iii) $\boldsymbol{y=2}$

8. *More limits at infinity*

(a) *limit at positive infinity.*

$$\lim_{x\to\infty}\frac{2x^2+x-6}{x^2-4} = \lim_{x\to\infty}\frac{\frac{2x^2}{x^2}+\frac{x}{x^2}-\frac{6}{x^2}}{\frac{x^2}{x^2}-\frac{4}{x^2}} = \lim_{x\to\infty}\frac{2+\frac{1}{x}-\frac{6}{x^2}}{1-\frac{4}{x^2}} =$$

(i) **0** (ii) **1** (iii) **2**

(b) *limit at positive infinity.*

$$\lim_{x\to\infty}\frac{2x+2}{x^2+5} = \lim_{x\to\infty}\frac{\frac{2x}{x^2}+\frac{2}{x^2}}{\frac{x^2}{x^2}+\frac{5}{x^2}} = \lim_{x\to\infty}\frac{\frac{1}{x}+\frac{2}{x^2}}{1+\frac{5}{x^2}} =$$

(i) **0** (ii) **1** (iii) **2**.

(c) *limit at positive infinity.*

$$\lim_{x\to\infty}\frac{-2x^2+x-6}{5x^2-4} = \lim_{x\to\infty}\frac{\frac{-2x^2}{x^2}+\frac{x}{x^2}-\frac{6}{x^2}}{\frac{5x^2}{x^2}-\frac{4}{x^2}} = \lim_{x\to\infty}\frac{-2+\frac{1}{x}-\frac{6}{x^2}}{5-\frac{4}{x^2}} =$$

(i) $\boldsymbol{-\frac{2}{5}}$ (ii) $\boldsymbol{-2}$ (iii) $\boldsymbol{-5}$.

(d) *limit at negative infinity.*

$$\lim_{x\to-\infty}\frac{-2x^2+x-6}{5x^2-4} = \lim_{x\to-\infty}\frac{\frac{-2x^2}{x^2}+\frac{x}{x^2}-\frac{6}{x^2}}{\frac{5x^2}{x^2}-\frac{4}{x^2}} = \lim_{x\to-\infty}\frac{-2+\frac{1}{x}-\frac{6}{x^2}}{5-\frac{4}{x^2}} =$$

(i) $\boldsymbol{-\frac{2}{5}}$ (ii) $\boldsymbol{-2}$ (iii) $\boldsymbol{-5}$.

(e) *limit at positive infinity.*

$$\lim_{x\to\infty} \frac{2x^2+2}{x+5} = \lim_{x\to\infty} \frac{\frac{2x^2}{x}+\frac{2}{x}}{\frac{x}{x}+\frac{5}{x}} \quad \text{divide by highest power of } x \text{ in denominator}$$

$$= \lim_{x\to\infty} \frac{2x+\frac{2}{x}}{1+\frac{5}{x}} =$$

(i) **0** (ii) **1** (iii) $\boldsymbol{\infty}$ (iv) **does not exist**.

(The function $\frac{2x^2+2}{x+5}$ does not have a horizontal asymptote.)

(f) *limit at positive infinity.*

$$\lim_{x\to\infty} \frac{x^3+x^2-6}{-x^2+1} = \lim_{x\to\infty} \frac{\frac{x^3}{x^2}+\frac{x^2}{x^2}-\frac{6}{x^2}}{-\frac{x^2}{x^2}+\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{x+1-\frac{6}{x^2}}{-1+\frac{1}{x^2}} =$$

(i) **0** (ii) $\boldsymbol{-\infty}$ (iii) $\boldsymbol{\infty}$ (iv) **does not exist**.

(Notice the negative in the denominator.)

9. *Oblique asymptotes.*

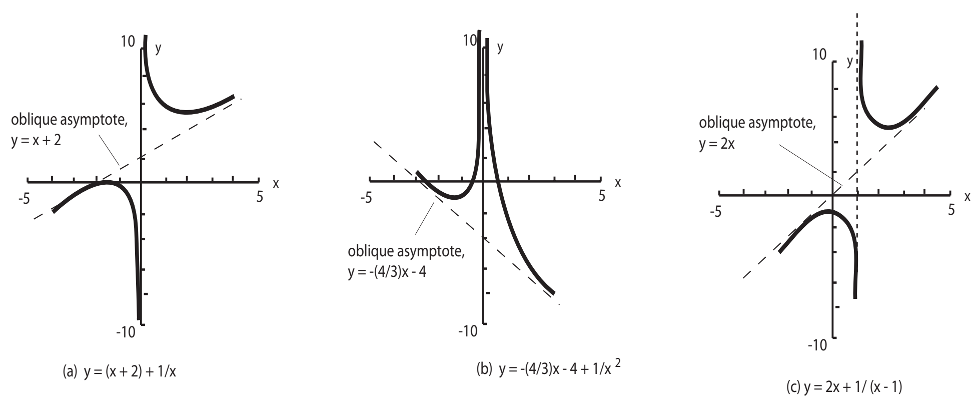



Figure 3.7 (Oblique asymptotes)

(a) *Figure (a).* $f(x) = (x+2) + \frac{1}{x}$.

i. $\lim_{x\to\infty}\left((x+2)+\frac{1}{x}\right) =$ (i) **0** (ii) **1** (iii) $\boldsymbol{\infty}$ (iv) $\boldsymbol{-\infty}$.

ii. $\lim_{x\to-\infty}\left((x+2)+\frac{1}{x}\right) =$ (i) **0** (ii) **1** (iii) $\boldsymbol{\infty}$ (iv) $\boldsymbol{-\infty}$.

iii. oblique asymptote at (i) $\boldsymbol{y = x+2}$ (ii) $\boldsymbol{y = -\frac{4}{3}x - 4}$ (iii) $\boldsymbol{y = 2x}$

iv. vertical asymptote at (i) $\boldsymbol{x = -1}$ (ii) $\boldsymbol{x = 0}$ (iii) $\boldsymbol{x = 1}$

(b) *Figure (b).* $f(x) = \left(-\frac{4}{3}x - 4\right) + \frac{1}{x^2}$.

i. $\lim_{x\to\infty}\left(\left(-\frac{4}{3}x-4\right)+\frac{1}{x^2}\right) =$ (i) **0** (ii) **1** (iii) $\boldsymbol{\infty}$ (iv) $\boldsymbol{-\infty}$.

ii. $\lim_{x\to-\infty}\left(\left(-\frac{4}{3}x-4\right)+\frac{1}{x^2}\right)=$ (i) $\mathbf{0}$ (ii) $\mathbf{1}$ (iii) $\boldsymbol{\infty}$ (iv) $\boldsymbol{-\infty}$.

iii. oblique asymptote at (i) $\boldsymbol{y=x+2}$ (ii) $\boldsymbol{y=-\frac{4}{3}x-4}$ (iii) $\boldsymbol{y=2x}$

iv. vertical asymptote at (i) $\boldsymbol{x=-1}$ (ii) $\boldsymbol{x=0}$ (iii) $\boldsymbol{x=1}$

(c) *Figure (c).* $f(x)=\frac{2x^2-2x+1}{x-1}$.

i. *function with disguised oblique asymptote.*

$$\begin{aligned}\frac{2x^2-2x+1}{x-1} &= \frac{2x(x-1)+1}{x-1}\\ &= \frac{2x(x-1)}{x-1}+\frac{1}{x-1} =\end{aligned}$$

(i) $\boldsymbol{2x+\frac{1}{x}}$ (ii) $\boldsymbol{2x+\frac{1}{x-1}}$

ii. $\lim_{x\to\infty}\left(2x+\frac{1}{x-1}\right)=$ (i) $\mathbf{0}$ (ii) $\mathbf{1}$ (iii) $\boldsymbol{\infty}$ (iv) $\boldsymbol{-\infty}$.

iii. $\lim_{x\to-\infty}\left(2x+\frac{1}{x-1}\right)=$ (i) $\mathbf{0}$ (ii) $\mathbf{1}$ (iii) $\boldsymbol{\infty}$ (iv) $\boldsymbol{-\infty}$.

iv. oblique asymptote at (i) $\boldsymbol{y=x+2}$ (ii) $\boldsymbol{y=-\frac{4}{3}x-4}$ (iii) $\boldsymbol{y=2x}$

v. vertical asymptote at (i) $\boldsymbol{x=-1}$ (ii) $\boldsymbol{x=0}$ (iii) $\boldsymbol{x=1}$

10. *Working with limits.* Let

$$\lim_{x\to5} f(x)=2 \qquad \lim_{x\to5} g(x)=16$$

(a) $\lim_{x\to5}[f(x)+g(x)]=$ (i) $\mathbf{18}$ (ii) $\mathbf{16}$ (iii) $\mathbf{32}$.

(b) $\lim_{x\to5}[f(x)\cdot g(x)]=$ (i) $\mathbf{16}$ (ii) $\mathbf{18}$. (iii) $\mathbf{32}$

(c) $\lim_{x\to5}\frac{g(x)}{f(x)}=$ (i) $\mathbf{32}$ (ii) $\mathbf{8}$ (iii) $\mathbf{18}$.

(d) $\lim_{x\to5}[f(x)]^4=$ (i) $\mathbf{16}$ (ii) $\mathbf{32}$ (iii) $\mathbf{18}$.

(e) $\lim_{x\to5}\sqrt{g(x)}=$ (i) $\mathbf{16}$ (ii) $\mathbf{18}$. (iii) $\mathbf{4}$

(f) $\lim_{x\to5}4^{[f(x)]}=$ (i) $\mathbf{32}$ (ii) $\mathbf{18}$. (iii) $\mathbf{16}$

(g) $\lim_{x\to5}[\log_2 f(x)]=$ (i) $\mathbf{1}$ (ii) $\mathbf{2}$ (iii) $\mathbf{16}$.

(h) $\lim_{x\to5}16=$ (i) $\mathbf{16}$ (ii) $\mathbf{5}$ (iii) $\mathbf{32}$.

(i) $\lim_{x\to5}\sqrt{\frac{f(x)+g(x)}{2}}=$ (i) $\mathbf{1}$ (ii) $\mathbf{2}$ (iii) $\mathbf{3}$.

11. *More working with limits.*

(a) $\lim_{x\to2}[x]^3=$ (i) $\mathbf{3^2}$ (ii) $\mathbf{2^3}$ (iii) $\mathbf{5}$.

(b) $\lim_{h\to0}[h]^3=$ (i) $\mathbf{0}$ (ii) $\mathbf{3}$ (iii) $\boldsymbol{h^3}$.

(c) $\lim_{x\to3}[3x^4-2h^2]=$ (i) $\mathbf{4^5}$ (ii) $\boldsymbol{-2(3)^2}$ (iii) $\boldsymbol{3^5-2h^2}$.

(d) $\lim_{x\to-2}[h\cdot x^2]=$ (i) $\boldsymbol{(h)^4}$ (ii) $\boldsymbol{(-2)^2}$ (iii) $\boldsymbol{h(-2)^2}$.

(e) $\lim_{h\to 0} 3 =$ (i) **0** (ii) **3** (iii) **5**.

(f) $\lim_{h\to 0} \frac{(x+h)^2-x^2}{h} =$ (i) $\boldsymbol{2x}$ (ii) $\boldsymbol{2h}$ (iii) **0**.
Hint: $\frac{(x+h)^2-x^2}{h} = \frac{x^2+2xh+h^2-x^2}{h} = \frac{2xh+h^2}{h} = 2x+h$

12. *Application: lawyer fees.* Lawyer fees per hour, $f(t)$, versus number of years of experience, $t$, are given in graph below.

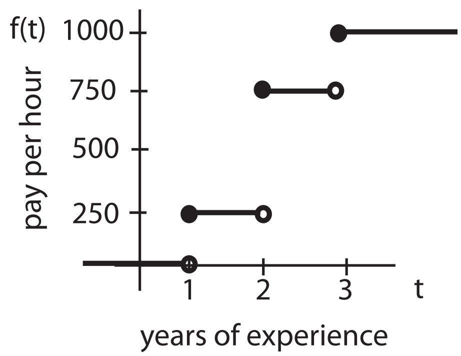



Figure 3.8 (Step function: lawyer fees)

(a) $\lim_{t\to 2^-} f(t) =$ (i) **does not exist** (ii) **\$250** (iii) **\$750** (iv) **\$1000**

(b) $\lim_{t\to 2^+} f(t) =$ (i) **does not exist** (ii) **\$250** (iii) **\$750** (iv) **\$1000**

(c) $\lim_{t\to 2} f(t) =$ (i) **does not exist** (ii) **\$250** (iii) **\$750** (iv) **\$1000**

(d) $\lim_{t\to \infty} f(t) =$ (i) **does not exist** (ii) **\$250** (iii) **\$750** (iv) **\$1000**

13. *Another application: average cost.* Monthly fixed costs of using machine I are \$15,000 and marginal costs of manufacturing one widget using machine I is \$20. Consequently, *average* costs are

$$\overline{C}(x) = \frac{20x + 15000}{x}, \quad x > 0$$

(Set WINDOW to 0 700 1 1 0 1000 1 1 before graphing function.)

(a) $\lim_{x\to 0^-} \overline{C}(x) =$ (i) **does not exist** (ii) **0** (iii) **20** (iv) $\boldsymbol{\infty}$

(b) $\lim_{x\to 0^+} \overline{C}(x) =$ (i) **does not exist** (ii) **0** (iii) **20** (iv) $\boldsymbol{\infty}$

(c) $\lim_{x\to \infty} \overline{C}(x) =$ (i) **does not exist** (ii) **0** (iii) **20** (iv) $\boldsymbol{\infty}$
Hint: $\overline{C}(x) = \frac{20x+15000}{x} = \frac{20x}{x} + \frac{15000}{x} = 20 + 15000\left(\frac{1}{x}\right)$

## 3.2 Continuity

Roughly speaking, a function is continuous if its graph can be drawn without lifting the pencil from the paper. A function $f(x)$ is continuous at $x = c$ if

1. $f(c)$ is defined,
2. $\lim_{x \to c} f(x)$ exists,
3. $\lim_{x \to c} f(x) = f(c)$.

If $f(x)$ is not continuous, it is *discontinuous*. A function is continuous on an *open* interval, $(a, b)$, if it is continuous on every $x$ in the interval; a function is continuous on a *closed* interval, $[a, b]$, if it is continuous

- on the open interval $(a, b)$,
- from the right at $x = a$,
- from the left at $x = b$.

Polynomial and exponential functions are continuous for all $x$; rational functions, $f(x) = \frac{p(x)}{q(x)}$, are continuous for all $x$ where $q(x) > 0$; logarithmic functions, $f(x) = \log_a x$, $a > 0$, $a \neq 1$, are continuous for all $x > 0$; root functions, $f(x) = \sqrt{ax + b}$, $ax + b \geq 0$, are continuous for all $x$ where $ax + b \geq 0$.

**Exercise 3.2 (Continuity)** Consider the following graphs which demonstrate various different types of functions with *discontinuities*.

1. *Checking for continuity: some first examples.*

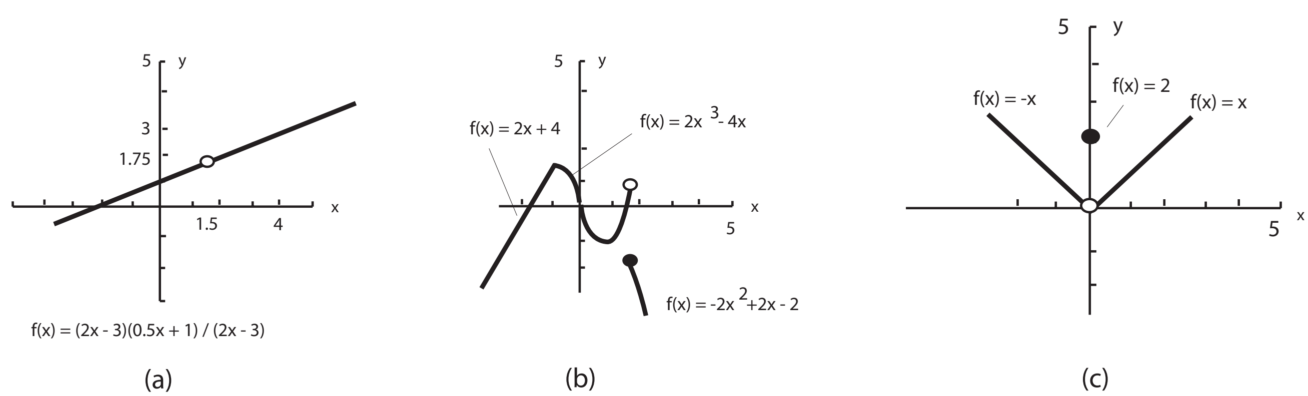



Figure 3.9 (Three types of discontinuity)

(use WINDOW -5 5 1 -5 5 1, use dotted line, use 2nd TEST for in/equalities,
For (a), type $y = \frac{(2x-3)(0.5x+1)}{2x-3}$ into $\text{Y}_1 =$,
For (b), type $y = (2x+4)(X \leq -1) + (2x^3 - 4x)(x > -1)(x < 1.5) + (-2x^2 + 2x - 2)(x \geq 1.5)$ into $\text{Y}_2 =$
For (c), type $y = (-x)(X < 0) + (2)(x = 0) + (x)(x > 0)$ into $\text{Y}_3 =$

(a) *Figure (a),* $f(x) = \frac{(2x-3)(0.5x+1)}{2x-3}$. *Continuous at* $x = 1.5$*?*

i. Since $f(1.5) = \frac{(2(1.5)-3)(0.5(1.5)+1)}{2(1.5)-3} =$ (i) **3** (ii) **4** (iii) **not defined**, at $x = 1.5$, function $f(x)$ is
(i) **continuous**
(ii) **discontinuous**
(iii) **cannot tell if dis/continuous using only this information**

ii. (i) **True** (ii) **False** Since the first of three conditions for continuity is violated; namely, $f(1.5)$ is not defined (is a removable discontinuity), $f(x)$ is discontinuous at $x = 1.5$. It is not necessary to check the other two conditions for continuity since it is now known $f(x)$ is discontinuous.

(b) *Figure (b). Continuous at $x = 1.5$?*

$$f(x) = \begin{cases} 2x+4 & \text{if } x \leq -1 \\ 2x^3 - 4x & \text{if } -1 < x < 1.5 \\ -2x^2 + 2x - 2 & \text{if } 1.5 \leq x, \end{cases}$$

i. Since $f(1.5) = -2(1.5)^2 + 2(1.5) - 2 =$
(i) **−3.5** (ii) **−4** (iii) **not defined**,
so, at $x = 1.5$, function $f(x)$ is
(i) **continuous**
(ii) **discontinuous**
(iii) **cannot tell if continuous using only this information**

ii. since $\lim_{x\to 1.5^-} f(x) =$ (i) **does not exist** (ii) **−3.5** (iii) **0.75**
and $\lim_{x\to 1.5^+} f(x) =$ (i) **does not exist** (ii) **−3.5** (iii) **0.75**
so $\lim_{x\to 1.5} f(x) =$ (i) **does not exist** (ii) **−3.5** (iii) **0.75**
so, at $x = 1.5$, function $f(x)$ is
(i) **continuous**
(ii) **discontinuous**
(iii) **cannot tell if continuous using only this information**

iii. (i) **True** (ii) **False** Since the second of three conditions for continuity is violated; namely, $\lim_{x\to 1.5} f(x)$ does not exist, $f(x)$ is discontinuous at $x = 1.5$. It is not necessary to check the third condition for continuity since it is now known $f(x)$ is discontinuous.

(c) *Figure (c). Continuous at $x = 0$?*

$$f(x) = \begin{cases} x & \text{if } x > 0 \\ 2 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

i. Since $f(0) =$
(i) **0** (ii) **2** (iii) **not defined**,
so, at $x = 0$, function $f(x)$ is

(i) **continuous**
(ii) **discontinuous**
(iii) **cannot tell if continuous using only this information**

ii. since $\lim_{x \to 0^-} f(x) =$ (i) **does not exist** (ii) **0** (iii) **2**
and $\lim_{x \to 0^+} f(x) =$ (i) **does not exist** (ii) **0** (iii) **2**
so $\lim_{x \to 0} f(x) =$ (i) **does not exist** (ii) **0** (iii) **2**
so, at $x = 0$, function $f(x)$ is
(i) **continuous**
(ii) **discontinuous**
(iii) **cannot tell if continuous using only information so far**

iii. since $\lim_{x \to 0} f(x) = 0$ (i) **equals** (ii) **does not equal** $f(0) = 2$
so, at $x = 0$, function $f(x)$ is
(i) **continuous**
(ii) **discontinuous**
(iii) **cannot tell if continuous using only information so far**

iv. (i) **True** (ii) **False** Since the third of three conditions for continuity is violated; namely, $\lim_{x \to 0} f(x) \neq f(0)$, $f(x)$ is discontinuous at $x = 0$. It was necessary to check all three conditions for continuity in this case.

2. *More checking for continuity,* $f(x) = \frac{x^2+x-6}{x^2-4}$.

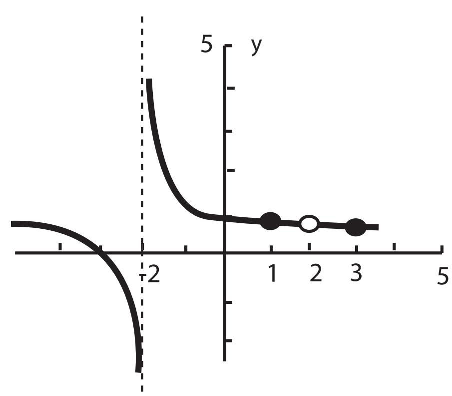



Figure 3.10 ($f(x) = \frac{x^2+x-6}{x^2-4}$)

(a) *Continuous at* $x = 2$*?*

i. Since $f(2) = \frac{(2)^2+2-6}{(2)^2-4} =$ (i) **3** (ii) **4** (iii) **not defined**,
at $x = 2$, function $f(x)$ is
(i) **continuous**
(ii) **discontinuous**
(iii) **cannot tell if dis/continuous using only this information**

ii. (i) **True** (ii) **False** Since the first of three conditions for continuity is violated; namely, $f(2)$ is not defined (is a removable discontinuity),

$f(x)$ is discontinuous at $x = 2$. It is not necessary to check the other two conditions for continuity since it is now known $f(x)$ is discontinuous.

(b) *Continuous at $x = -2$?*

i. Since $f(-2) = \frac{(-2)^2+(-2)-6}{(-2)^2-4} =$ (i) **3** (ii) **4** (iii) **not defined**,
at $x = -2$, function $f(x)$ is
(i) **continuous**
(ii) **discontinuous**
(iii) **cannot tell if dis/continuous using only this information**

ii. (i) **True** (ii) **False** Since the first of three conditions for continuity is violated; namely, $f(-2)$ is not defined (is a removable discontinuity), $f(x)$ is discontinuous at $x = -2$. It is not necessary to check the other two conditions for continuity since it is now known $f(x)$ is discontinuous.

(c) *Continuous at $x = 1$?*

i. Since $f(1) = \frac{(1)^2+(1)-6}{(1)^2-4} =$
(i) $\mathbf{\frac{4}{3}}$ (ii) $\mathbf{\frac{6}{5}}$ (iii) **not defined**,
so, at $x = 1$, function $f(x)$ is
(i) **continuous**
(ii) **discontinuous**
(iii) **cannot tell if continuous using only this information**

ii. since $\lim_{x\to 1^-} f(x) =$ (i) **does not exist** (ii) $\mathbf{\frac{4}{3}}$ (iii) $\mathbf{\frac{6}{5}}$
and $\lim_{x\to 1^+} f(x) =$ (i) **does not exist** (ii) $\mathbf{\frac{4}{3}}$ (iii) $\mathbf{\frac{6}{5}}$
so $\lim_{x\to 1} f(x) =$ (i) **does not exist** (ii) $\mathbf{\frac{4}{3}}$ (iii) $\mathbf{\frac{6}{5}}$
so, at $x = 1$, function $f(x)$ is
(i) **continuous**
(ii) **discontinuous**
(iii) **cannot tell if continuous using only information so far**

iii. since $\lim_{x\to 1} f(x) = \frac{4}{3}$ (i) **equals** (ii) **does not equal** $f(1) = \frac{4}{3}$
so, at $x = 1$, function $f(x)$ is
(i) **continuous**
(ii) **discontinuous**
(iii) **cannot tell if continuous using only information so far**

iv. (i) **True** (ii) **False** Since all three conditions for continuity are satisfied, function is continuous at $x = 1$.

(d) *Continuous at $x = 3$?*

i. Since $f(3) = \frac{(3)^2+(3)-6}{(3)^2-4} =$
(i) $\mathbf{\frac{4}{3}}$ (ii) $\mathbf{\frac{6}{5}}$ (iii) **not defined**,
so, at $x = 3$, function $f(x)$ is

(i) **continuous**
(ii) **discontinuous**
(iii) **cannot tell if continuous using only this information**

ii. since $\lim_{x \to 3^-} f(x) =$ (i) **does not exist** (ii) $\mathbf{\frac{4}{3}}$ (iii) $\mathbf{\frac{6}{5}}$
and $\lim_{x \to 3^+} f(x) =$ (i) **does not exist** (ii) $\mathbf{\frac{4}{3}}$ (iii) $\mathbf{\frac{6}{5}}$
so $\lim_{x \to 3} f(x) =$ (i) **does not exist** (ii) $\mathbf{\frac{4}{3}}$ (iii) $\mathbf{\frac{6}{5}}$
so, at $x = 3$, function $f(x)$ is
(i) **continuous**
(ii) **discontinuous**
(iii) **cannot tell if continuous using only information so far**

iii. since $\lim_{x \to 3} f(x) = \frac{4}{3}$ (i) **equals** (ii) **does not equal** $f(1) = \frac{4}{3}$
so, at $x = 3$, function $f(x)$ is
(i) **continuous**
(ii) **discontinuous**
(iii) **cannot tell if continuous using only information so far**

iv. (i) **True** (ii) **False** Since all three conditions for continuity are satisfied, function is continuous at $x = 3$.

3. *And more checking for continuity.* Identify all $x$ where $f(x)$ is discontinuous.

(a) Function $f(x) = \frac{(x-2)(x+2)(x-5)}{(x-2)(x+2)(x-1)}$
is discontinuous at (circle none, one or more)
(i) $\boldsymbol{x = -2}$ (ii) $\boldsymbol{x = 1}$ (iii) $\boldsymbol{x = 2}$ (iv) **nowhere**

(b) Function $f(x) = \frac{(x-2)(x+3)(x-5)}{(x-2)(x+2)(x-1)}$
is discontinuous at (circle none, one or more)
(i) $\boldsymbol{x = -2}$ (ii) $\boldsymbol{x = 1}$ (iii) $\boldsymbol{x = 2}$ (iv) **nowhere**

(c) Function $f(x) = \frac{2x^2+2}{x^2+5}$
is discontinuous at (circle none, one or more)
(i) $\boldsymbol{x = -2}$ (ii) $\boldsymbol{x = 1}$ (iii) $\boldsymbol{x = 2}$ (iv) **nowhere**
(Hint: Is $x^2 + 5$ ever equal to zero?)

(d) Function $f(x) = 16$
is discontinuous at (circle none, one or more)
(i) $\boldsymbol{x = -2}$ (ii) $\boldsymbol{x = 1}$ (iii) $\boldsymbol{x = 2}$ (iv) **nowhere**

(e) Function $f(x) = 3x^2 + 2x - 7$
is discontinuous at (circle none, one or more)
(i) $\boldsymbol{x = -2}$ (ii) $\boldsymbol{x = 1}$ (iii) $\boldsymbol{x = 2}$ (iv) **nowhere**

(f) Function $f(x) = 7^x$
is discontinuous at (circle none, one or more)
(i) $\boldsymbol{x = -2}$ (ii) $\boldsymbol{x = 1}$ (iii) $\boldsymbol{x = 2}$ (iv) **nowhere**

(g) Function $f(x) = \ln|x|$
is discontinuous at (circle none, one or more)
(i) $\boldsymbol{x = 0}$ (ii) $\boldsymbol{x = 1}$ (iii) $\boldsymbol{x = 2}$ (iv) **nowhere**

4. *Identify k which makes function continuous (not discontinuous!)*

(a) Function

$$f(x) = \begin{cases} kx^2 & \text{if } x \leq 3 \\ x + 3k & \text{if } x > 3 \end{cases}$$

is continuous at (i) $\boldsymbol{k = \frac{1}{2}}$ (ii) $\boldsymbol{k = \frac{1}{3}}$ (iii) $\boldsymbol{k = \frac{1}{4}}$ (iv) **nowhere**
($kx^2 = x + 3k$, or $kx^2 - 3k = x$ or $k(x^2 - 3) = x$ or $k = \frac{x}{x^2-3} = \frac{3}{3^2-3} = \frac{1}{2}$)

(b) Function

$$f(x) = \begin{cases} kx^3 & \text{if } x \leq 4 \\ x + 4k & \text{if } x > 4 \end{cases}$$

is continuous at (i) $\boldsymbol{k = \frac{1}{2}}$ (ii) $\boldsymbol{k = \frac{1}{3}}$ (iii) $\boldsymbol{k = \frac{1}{4}}$ (iv) **nowhere**
($kx^3 = x + 4k$, or $kx^3 - 4k = x$ or $k(x^3 - 4) = x$ or $k = \frac{x}{x^3-4} = \frac{4}{4^2-4} = \frac{1}{3}$)

5. *Application: lawyer fees.* Lawyer fees per hour, $f(t)$, versus number of years of experience, $t$, are given in graph below.

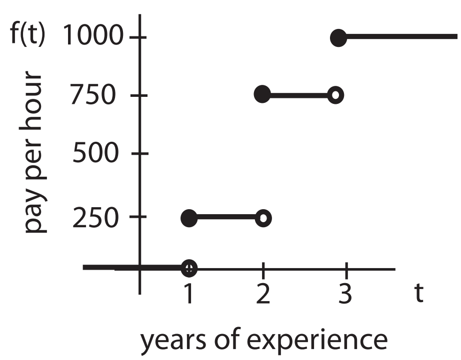


Figure 3.11 (Step function: lawyer fees)

Lawyer fees $f(t)$ are discontinuous at (circle none, one or more)
(i) $\boldsymbol{t = 1}$ (ii) $\boldsymbol{t = 2}$ (iii) $\boldsymbol{t = 3}$ (iv) **nowhere**

6. *Application: intermediate value theorem.*
(i) **True** (ii) **False** If a function $f(x)$ is continuous on closed interval $[a, b]$, then $f(x)$ takes on every value between $f(a)$ and $f(b)$. For example, on closed interval $[0, 1]$ where $f(0) = -2$ and $f(1) = 3$, continuous function $f(x)$ must take on every value between $-2$ and 3; for instance the value 0.