8.3 Continuous Money Flow

If the *rate of money flow (change in money per unit time)*, $f(t)$, over some time period $[0, T]$, with interest rate r, are known, the *present value of money flow* is,

$$P = \int_0^T f(t)e^{-rt}\,dt,$$

and also the *accumulated amount of money flow at time T* is,

$$A = e^{rT} \int_0^T f(t)e^{-rt}\,dt.$$

Exercise 8.3 (Continuous Money Flow)

1. *Money versus money flow*

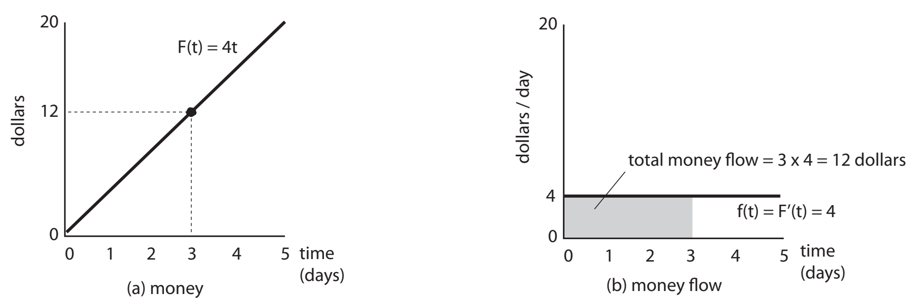

Figure 8.10 (Money versus money flow)

$Y_1 = 4x$, $Y_2 = 4$ WINDOW 0 5 1 0 20 5 1

(a) *Money, figure (a).* After 3 days,

$$F(3) = 4(3) =$$

(i) **12** (ii) **15** (iii) **20** dollars

(b) *Money flow, figure (b).* Change in money per day is

$$f(t) = F'(t) = \frac{d}{dt}(4t) = 4 \cdot t^{1-1} =$$

(i) **3** (ii) **4** (iii) **5** dollars per day

(c) *Total money flow, figure (b).* Total money flow after 3 days is

$$\int_0^3 4\,dt = \int_0^3 4t^0\,dt = \left[4 \cdot \frac{1}{0+1} t^{0+1}\right]_0^3 = 4(3) - 4(0) =$$

(i) **12** (ii) **15** (iii) **20** dollars

MATH fnInt($Y_2, X, 0, 3$)

also, notice integral of money flow (total amount of money flow over time) equals money

(d) *Total money flow, in general.* Total money flow over $[0, T]$ time is

$$\int_0^T f(t)\,dt$$

(i) **True** (ii) **False**

this does *not* account for any interest earned on money

2. *More total money flow.*

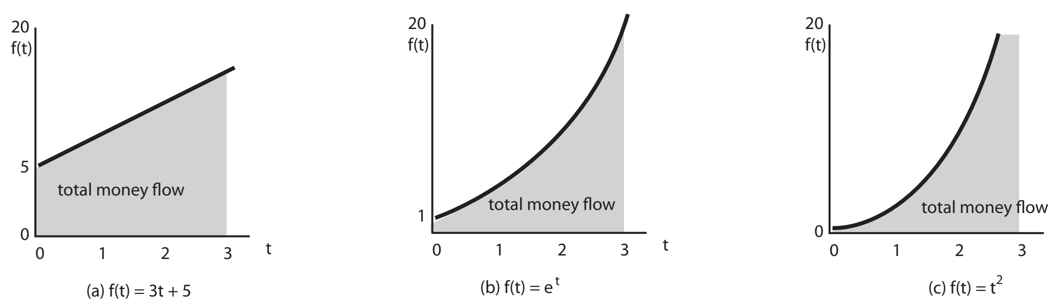

Figure 8.11 (More total money flow)

$Y_1 = 3x + 5$, $Y_2 = e^x$, $Y_3 = x^3$, WINDOW 0 3 1 0 20 5 1

(a) $f(t) = 3t + 5, [0, 3]$, *figure (a)*

$$\int_0^3 f(t)\,dt = \int_0^3 (3t+5)\,dt = \left[3 \cdot \frac{1}{1+1} t^{1+1} + 5 \cdot \frac{1}{0+1} t^{0+1}\right]_0^3 = \left[3 \cdot \frac{3}{2} t^2 + 5t\right]_{t=0}^{t=3} =$$

(i) **28** (ii) **28.5** (iii) **29** dollars

MATH fnInt($Y_1, X, 0, 3$)

(b) $f(t) = e^t, [0, 3]$, *figure (b)*

$$\int_0^3 f(t)\,dt = \int_0^3 e^t\,dt = \left[e^t\right]_{t=0}^{t=3} \approx$$

(i) **17.09** (ii) **18.09** (iii) **19.09** dollars

MATH fnInt($Y_2, X, 0, 3$)

(c) $f(t) = t^3, [0, 3]$, *figure (c)*

$$\int_0^3 f(t)\,dt = \int_0^3 t^3\,dt = \left[\frac{1}{3+1}t^{3+1}\right]_{t=0}^{t=3} = \left[\frac{1}{4}t^4\right]_{t=0}^{t=3} \approx$$

(i) **20.25** (ii) **21.25** (iii) **22.25** dollars

MATH fnInt($Y_3, X, 0, 3$)

3. *Present value of money flow (with continuously compounding interest).*

(a) *Find P if $f(t) = 4, r = 0.07, [0, 3]$.* If *rate* of annual income (in 100,000s) is $f(t) = 4$, what is the *present* value of this income over a 3 year period, assuming an annual interest rate of 7% compounded continuously?

$$P = \int_0^T f(t)e^{-rt}\,dt = \int_0^3 4e^{-0.07t}\,dt = \left[-\frac{4}{0.07}e^{-0.07t}\right]_{t=0}^{t=3} = \left[-\frac{4}{0.07}e^{-0.07(3)}\right] - \left[-\frac{4}{0.07}e^{-0.07(0)}\right] \approx$$

(i) **10.82** (ii) **11.25** (iii) **13.35** 100,000 dollars

$Y_1 = 4e^{-0.07x}$, MATH fnInt($Y_1, X, 0, 3$)

(b) *Find P if $f(t) = e^t, r = 0.07, [0, 3]$.* If rate of annual income (in 100,000s) is $f(t) = e^t$, what is the present value of this income over a 3 year period, assuming an annual interest rate of 7% compounded continuously?

$$\int_0^T f(t)e^{-rt}\,dt = \int_0^3 e^t e^{-0.07t}\,dt = \int_0^3 e^{0.93t}\,dt = \left[\frac{1}{0.93}e^{0.93t}\right]_{t=0}^{t=3} = \left[\frac{1}{0.93}e^{0.93(3)}\right] - \left[\frac{1}{0.93}e^{0.93(0)}\right] \approx$$

(i) **16.43** (ii) **17.25** (iii) **20.27** 100,000 dollars

$Y_2 = e^{0.93x}$, MATH fnInt($Y_2, X, 0, 3$)

(c) *Find P if $f(t) = 3t + 5, r = 0.07, [0, 3]$.*

$$P = \int_0^T f(t)e^{-rt}\,dt = \int_0^3 (3t+5)e^{-0.07t}\,dt$$

where we guess $u = 3t + 5$ and $dv = e^{-0.07t}\,dt$,

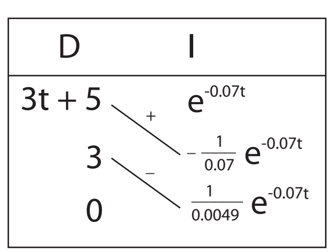

Figure 8.12 (Column integration, $\int(3t+5)e^{-0.07t}\,dx$)

in column D, successive differentiation of guess $u = f(t) = t$ gives
guess $f(t) = 3t + 5$,
then $\frac{d}{dt}(3t+5) = 3 \cdot t^{1-1} + 0 = 3$
and finally $\frac{d}{dt}1 = 0$

in column I, successive integration of guess $dv = f(t) = e^{-0.07t}$
guess $f(t) = e^{-0.07t}$,
then $\int e^{-0.07t}\, dt = -\frac{1}{0.07}e^{-0.07t}$
and finally $-\frac{1}{0.07}\int e^{-0.07t}\, dt = \frac{1}{-0.07} \cdot \frac{1}{-0.07} \cdot e^{-0.07t} = \frac{1}{0.0049}e^{-0.07t}$

then, multiplying items in columns as indicated (with indicated signs)

$$\int (3t+5)e^{-0.07t}\, dt = (3t+5) \cdot \left(-\frac{1}{0.07}e^{-0.07t}\right) - 3 \cdot \left(\frac{1}{0.0049}e^{-0.07t}\right) + C =$$

(i) $\boldsymbol{\left(-\frac{3t+5}{0.07} - \frac{3}{0.0049}\right) e^{-0.07t} + C}$
(ii) $\boldsymbol{\left(\frac{3t+5}{0.07} + \frac{3}{0.0049}\right) e^{-0.07t} + C}$
(iii) $\boldsymbol{\left(\frac{3t+5}{0.07} - \frac{3}{0.0049}\right) e^{-0.07t} + C}$

so

$$\begin{aligned} P &= \int_0^3 (3t+5)e^{-0.07t}\, dt \\ &= \left[\left(-\frac{3t+5}{0.07} - \frac{3}{0.0049}\right) e^{-0.07t}\right]_{t=0}^{t=3} \\ &= \left[\left(-\frac{3(3)+5}{0.07} - \frac{3}{0.0049}\right) e^{-0.07(3)}\right] - \left[\left(-\frac{3(0)+5}{0.07} - \frac{3}{0.0049}\right) e^{-0.07(0)}\right] \approx \end{aligned}$$

(i) **25.28** (ii) **26.25** (iii) **31.19** 100,000 dollars
$\text{Y}_3 = (3x+5)e^{-0.07x}$, MATH fnInt($\text{Y}_3, X, 0, 3$)

4. *Accumulated amount of money flow at time T (with continuously compounding interest).*

   (a) *Find A if $f(t) = 4, r = 0.07, [0, 3]$.* If rate of annual income (in 100,000s) is $f(t) = 4$, what is the *accumulated (future)* value of this income over a 3 year period, assuming an annual interest rate of 7% compounded continuously?

$$\begin{aligned} A &= e^{rT}\int_0^T f(t)e^{-rt}\, dt = e^{0.07(3)}\int_0^3 4e^{-0.07t}\, dt \\ &= e^{0.07(3)} \cdot \left[-\frac{4}{0.07}e^{-0.07t}\right]_{t=0}^{t=3} = e^{0.07(3)} \cdot \left\{\left[-\frac{4}{0.07}e^{-0.07(3)}\right] - \left[-\frac{4}{0.07}e^{-0.07(0)}\right]\right\} \approx \end{aligned}$$

(i) **10.82** (ii) **11.25** (iii) **13.35** 100,000 dollars
multiply $e^{0.07(3)}$ times MATH fnInt$(Y_1, X, 0, 3)$

recall, from above, total money flow over 3 years is

$$\int_0^T f(t)\,dt = \int_0^3 4\,dt = \int_0^3 4t^0\,dt = \left[4 \cdot \frac{1}{0+1} t^{0+1}\right]_0^3 = 4(3) - 4(0) = 12$$

so, total *interest* earned over 3 years is

$$A - \text{ total money flow} = e^{rT}\int_0^T f(t)e^{-rt}\,dt - \int_0^T f(t)\,dt \approx 13.35 - 12 =$$

(i) **1.35** (ii) **1.65** (iii) **1.85** 100,000 dollars
accumulated money flow, A, includes interest, whereas total money flow does not

(b) *Find A if $f(t) = e^t, r = 0.07, [0, 3]$.*

$$\begin{aligned} A &= e^{rT}\int_0^T f(t)e^{-rt}\,dt = e^{0.07(3)}\int_0^3 e^t e^{-0.07t}\,dt \\ &= e^{0.07(3)}\int_0^3 e^{0.93t}\,dt = e^{0.07(3)}\left[\frac{1}{0.93}e^{0.93t}\right]_{t=0}^{t=3} = e^{0.07(3)}\left\{\left[\frac{1}{0.93}e^{0.93(3)}\right] - \left[\frac{1}{0.93}e^{0.93(0)}\right]\right\} \approx \end{aligned}$$

(i) **16.43** (ii) **17.25** (iii) **20.27** 100,000 dollars
multiply $e^{0.07(3)}$ times MATH fnInt$(Y_2, X, 0, 3)$

recall, from above, total money flow over 3 years is

$$\int_0^T f(t)\,dt = \int_0^3 e^t\,dt = \left[e^t\right]_{t=0}^{t=3} \approx 19.09$$

so, total interest earned over 3 years is

$$A - \text{ total money flow} = e^{rT}\int_0^T f(t)e^{-rt}\,dt - \int_0^T f(t)\,dt \approx 20.27 - 19.09 =$$

(i) **1.18** (ii) **1.25** (iii) **1.35** 100,000 dollars

(c) *Find A if $f(t) = 3t + 5, r = 0.07, [0, 3]$.*

$$\begin{aligned} A &= e^{rT}\int_0^T f(t)e^{-rt}\,dt \\ &= e^{0.07(3)}\int_0^3 (3t+5)e^{-0.07t}\,dt \\ &= e^{0.07(3)}\left[\left(-\frac{3t+5}{0.07} - \frac{3}{0.0049}\right)e^{-0.07t}\right]_{t=0}^{t=3} \\ &= e^{0.07(3)}\left\{\left[\left(-\frac{3(3)+5}{0.07} - \frac{3}{0.0049}\right)e^{-0.07(3)}\right] - \left[\left(-\frac{3(0)+5}{0.07} - \frac{3}{0.0049}\right)e^{-0.07(0)}\right]\right\} \approx \end{aligned}$$

(i) **25.28** (ii) **26.25** (iii) **31.19** 100,000 dollars
multiply $e^{0.07(3)}$ times MATH fnInt($Y_3, X, 0, 3$)

recall, from above, total money flow over 3 years is

$$\int_0^3 f(t)\,dt = \int_0^3 (3t+5)\,dt = \left[3 \cdot \frac{1}{1+1} t^{1+1} + 5 \cdot \frac{1}{0+1} t^{0+1}\right]_0^3 = \left[3 \cdot \frac{3}{2} t^2 + 5t\right]_{t=0}^{t=3} = 28.5$$

so, total interest earned over 3 years is

$$A - \text{ total money flow} = e^{rT} \int_0^T f(t)e^{-rt}\,dt - \int_0^T f(t)\,dt \approx 31.19 - 28.50 =$$

(i) **2.69** (ii) **3.25** (iii) **3.35** 100,000 dollars

8.4 Improper Integrals

We will look at some examples of how to integrate *improper* integrals (integrals where one or the other of the limits of integration are $\pm\infty$); in particular,

- $\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$
- $\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx$
- $\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$

Exercise 8.4 (Improper Integrals)

1. *Power Functions and Improper Integration*

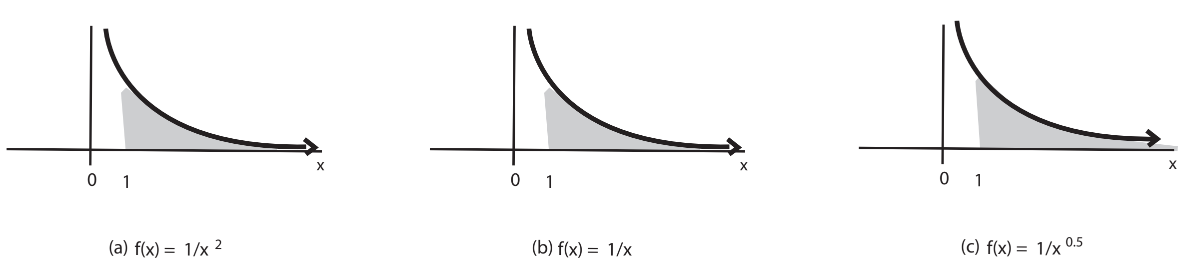

Figure 8.13 (Power Functions)

(a) *Find* $\int_1^\infty x^{-2}\,dx$*, figure (a).*

$$\int_1^\infty x^{-2}\,dx = \lim_{b\to\infty} \left[\frac{1}{-2+1} x^{-2+1}\right]_1^b =$$

(choose one or more)
(i) $\mathbf{\lim_{b\to\infty}\left[-x^{-1}\right]_1^b}$
(ii) $\mathbf{\lim_{b\to\infty}\left[-b^{-1}-(-(1)^{-1})\right]}$
(iii) $\mathbf{\lim_{b\to\infty}\left[-\frac{1}{b}-(-1)\right]}$

so

$$\int_1^\infty x^{-2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{b}-(-1)\right] =$$

(i) $\mathbf{-\frac{1}{3}}$ (ii) **1** (iii) **diverges**
because $\lim_{b\to\infty} -\frac{1}{b} = 0$

(b) *Find* $\int_1^\infty x^{-1}\,dx = \int_1^\infty \frac{1}{x}\,dx$*, figure (b).*

$$\int_1^\infty \frac{1}{x}\,dx =$$

(choose one or more)
(i) $\mathbf{\lim_{b\to\infty}\left[\ln x\right]_1^b}$
(ii) $\mathbf{\lim_{b\to\infty}\left[\ln b-\ln 1\right]}$
(iii) $\mathbf{\lim_{b\to\infty}\left[\ln b-0\right]}$

so

$$\int_1^\infty x^{-1}\,dx = \lim_{b\to\infty}\left[\ln b-0\right] =$$

(i) $\mathbf{-\frac{1}{3}}$ (ii) $\mathbf{\frac{1}{3}}$ (iii) **diverges**
because $\lim_{b\to\infty}\ln b = \infty$;
replace b with increasingly large numbers in your calculator to convince yourself

(c) *Find* $\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \int_1^\infty x^{-0.5}\,dx$*, figure (c).*

$$\int_1^\infty x^{-0.5}\,dx = \lim_{b\to\infty}\left[\frac{1}{-0.5+1}x^{-0.5+1}\right]_1^b =$$

(choose one or more)
(i) $\mathbf{\lim_{b\to\infty}\left[2x^{0.5}\right]_1^b}$
(ii) $\mathbf{\lim_{b\to\infty}\left[2(b)^{0.5}-0.5(1)^{0.5}\right]}$
(iii) $\mathbf{\lim_{b\to\infty}\left[2\sqrt{b}-0.5\right]}$

so

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{b\to\infty}\left[2\sqrt{b}-0.5\right] =$$

(i) $\mathbf{-\frac{1}{3}}$ (ii) $\mathbf{\frac{1}{3}}$ (iii) **diverges**
because $\lim_{b\to\infty} 2\sqrt{b} = \infty$

(d) *Find $\int_1^\infty \frac{1}{x^{-2}}\,dx = \int_1^\infty x^2\,dx$, figure (c).*

$$\int_1^\infty x^2\,dx = \lim_{b\to\infty}\left[\frac{1}{2+1}x^{2+1}\right]_1^b =$$

(choose one or more)
(i) $\mathbf{\lim_{b\to\infty}\left[\frac{1}{3}x^3\right]_1^b}$
(ii) $\mathbf{\lim_{b\to\infty}\left[\frac{1}{3}(b)^3 - \frac{1}{3}(1)^3\right]}$
(iii) $\mathbf{\lim_{b\to\infty}\left[\frac{1}{3}(b)^3 - \frac{1}{3}\right]}$

so

$$\int_1^\infty \frac{1}{x^{-2}}\,dx = \lim_{b\to\infty}\left[\frac{1}{3}(b)^3 - \frac{1}{3}\right] =$$

(i) $\mathbf{-\frac{1}{3}}$ (ii) $\mathbf{\frac{1}{3}}$ (iii) **diverges**
because $\lim_{b\to\infty}\frac{1}{3}(b)^3 = \infty$

(e) (i) **True** (ii) **False**

$$\int_1^\infty \frac{1}{x^\alpha}\,dx$$

converges if $\alpha > 1$ and diverges if $\alpha \le 1$ (such as $\alpha = 0.5$ or $\alpha = -2$, say)

2. *Power Function $f(x) = \frac{1}{x^4}$ and Improper Integration*

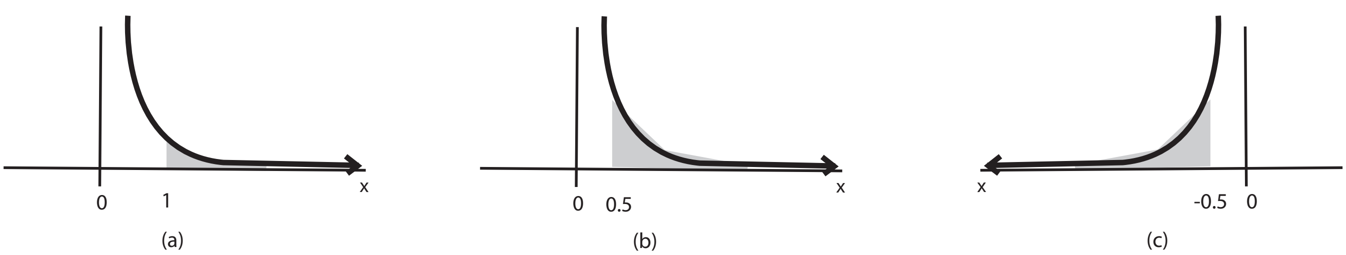

Figure 8.14 ($f(x) = \frac{1}{x^4}$)

(a) *Find $\int_1^\infty x^{-4}\,dx$, figure (a).*

$$\int_1^\infty x^{-4}\,dx = \lim_{b\to\infty}\left[\frac{1}{-4+1}x^{-4+1}\right]_1^b =$$

(choose one or more)
(i) $\mathbf{\lim_{b\to\infty}\left[-\frac{1}{3}x^{-3}\right]_1^b}$
(ii) $\mathbf{\lim_{b\to\infty}\left[-\frac{1}{3}b^{-3} - \left(-\frac{1}{3}(1)^{-3}\right)\right]}$
(iii) $\mathbf{\lim_{b\to\infty}\left[-\frac{1}{3}\frac{1}{b^3} - \left(-\frac{1}{3}\right)\right]}$

so

$$\int_1^\infty x^{-4}\,dx = \lim_{b\to\infty}\left[-\frac{1}{3}\frac{1}{b^3} - \left(-\frac{1}{3}\right)\right] =$$

(i) $-\frac{1}{3}$ (ii) $\frac{1}{3}$ (iii) $\frac{2}{3}$

in other words, $\int_{0.5}^\infty x^{-4}\,dx$ is (i) **convergent** (ii) **divergent**
because it exists; it is $\frac{1}{3}$.

(b) *Find* $\int_{0.5}^\infty x^{-4}\,dx$*, figure (b).*

$$\int_{0.5}^\infty x^{-4}\,dx = \lim_{b\to\infty}\left[\frac{1}{-4+1}x^{-4+1}\right]_{0.5}^b =$$

(choose one or more)
(i) $\lim_{b\to\infty}\left[-\frac{1}{3}x^{-3}\right]_{0.5}^b$
(ii) $\lim_{b\to\infty}\left[-\frac{1}{3}b^{-3} - \left(-\frac{1}{3}(0.5)^{-3}\right)\right]$

so

$$\int_{0.5}^\infty x^{-4}\,dx = \lim_{b\to\infty}\left[-\frac{1}{3b^3} + \frac{1}{3}2^3\right] =$$

(i) **0** (ii) $-\frac{8}{3}$ (iii) $\frac{8}{3}$

in other words, $\int_{0.5}^\infty x^{-4}\,dx$ is (i) **convergent** (ii) **divergent**
because it exists; it is $\frac{8}{3}$.

(c) *Find* $\int_{-\infty}^{-0.5} x^{-4}\,dx$*, figure (c).*

$$\int_{-\infty}^{-0.5} x^{-4}\,dx = \lim_{a\to-\infty}\left[\frac{1}{-4+1}x^{-4+1}\right]_a^{-0.5} =$$

(choose one or more)
(i) $\lim_{a\to-\infty}\left[-\frac{1}{3}x^{-3}\right]_a^{-0.5}$
(ii) $\lim_{a\to-\infty}\left[-\frac{1}{3}(-0.5)^{-3} - \left(-\frac{1}{3}(a)^{-3}\right)\right]$

so

$$\int_{-\infty}^{-0.5} x^{-4}\,dx = \lim_{a\to-\infty}\left[\frac{1}{3}2^3 + \frac{1}{3a^3}\right] =$$

(i) **0** (ii) $-\frac{8}{3}$ (iii) $\frac{8}{3}$

in other words, $\int_{-\infty}^{0.5} x^{-4}\,dx$ is (i) **convergent** (ii) **divergent**
because it exists; it is $\frac{8}{3}$.

(d) (i) **True** (ii) **False**

$$\int_0^\infty x^{-4}\,dx, \quad \int_{-\infty}^0 x^{-4}\,dx \quad \text{and} \quad \int_{-\infty}^\infty x^{-4}\,dx$$

do not exist (are divergent) because $x^{-4} = \frac{1}{x^4}$ does not exist at $x = 0$

3. *More Improper Integrals*

(a) *Find* $\int_0^\infty e^{3x}\,dx$

$$\int_0^\infty e^{3x}\,dx =$$

(choose one or more)
(i) $\boldsymbol{\lim_{b\to\infty}\left[\frac{1}{3}e^{3x}\right]_0^b}$
(ii) $\boldsymbol{\lim_{b\to\infty}\left[\frac{1}{3}e^{3b} - \frac{1}{3}e^{3(0)}\right]}$
(iii) $\boldsymbol{\lim_{b\to\infty}\left[\frac{1}{3}e^{3b} - \frac{1}{3}\right]}$

so

$$\int_0^\infty e^{3x}\,dx = \lim_{b\to\infty}\left[\frac{1}{3}e^{3b} - \frac{1}{3}\right] =$$

(i) $\boldsymbol{\frac{1}{3}}$ (ii) $\boldsymbol{-\frac{1}{3}}$ (iii) **divergent**
because $\lim_{b\to\infty}\frac{1}{3}e^{3b} = \infty$

(b) *Find* $\int_{-\infty}^0 e^{3x}\,dx$

$$\int_{-\infty}^0 e^{3x}\,dx =$$

(choose one or more)
(i) $\boldsymbol{\lim_{a\to-\infty}\left[\frac{1}{3}e^{3x}\right]_a^0}$
(ii) $\boldsymbol{\lim_{a\to-\infty}\left[\frac{1}{3}e^{3(0)} - \frac{1}{3}e^{3a}\right]}$
(iii) $\boldsymbol{\lim_{a\to-\infty}\left[\frac{1}{3} - \frac{1}{3}e^{3a}\right]}$

so

$$\int_{-\infty}^0 e^{3x}\,dx = \lim_{a\to-\infty}\left[\frac{1}{3} - \frac{1}{3}e^{3a}\right] =$$

(i) $\boldsymbol{\frac{1}{3}}$ (ii) $\boldsymbol{-\frac{1}{3}}$ (iii) **divergent**
because $\lim_{a\to-\infty}\frac{1}{3}e^{3a} = 0$

(c) *Find* $\int_{-\infty}^\infty e^{3x}\,dx$

$$\int_{-\infty}^\infty e^{3x}\,dx = \int_{-\infty}^0 e^{3x}\,dx + \int_0^\infty e^{3x}\,dx =$$

(choose one or more)
(i) $\mathbf{\lim_{b\to\infty}\left[\frac{1}{3}e^{3x}\right]_0^b + \lim_{a\to-\infty}\left[\frac{1}{3}e^{3x}\right]_a^0}$
(ii) $\mathbf{\lim_{b\to\infty}\left[\frac{1}{3}e^{3b} - \frac{1}{3}e^{3(0)}\right] + \lim_{a\to-\infty}\left[\frac{1}{3}e^{3(0)} - \frac{1}{3}e^{3a}\right]}$
(iii) $\mathbf{\lim_{b\to\infty}\left[\frac{1}{3}e^{3b} - \frac{1}{3}\right] + \lim_{a\to-\infty}\left[\frac{1}{3} - \frac{1}{3}e^{3a}\right]}$

so

$$\int_{-\infty}^{\infty} e^{3x}\,dx = \lim_{b\to\infty}\left[\frac{1}{3}e^{3b} - \frac{1}{3}\right] + \lim_{a\to-\infty}\left[\frac{1}{3} - \frac{1}{3}e^{3a}\right] =$$

(i) $\mathbf{\frac{1}{3}}$ (ii) $\mathbf{-\frac{1}{3}}$ (iii) **divergent**
because $\lim_{b\to\infty}\frac{1}{3}e^{3b} = \infty$

(d) (i) **True** (ii) **False**

$$\begin{aligned}\int_{-\infty}^{\infty} e^{3x}\,dx &= \int_{-\infty}^{0} e^{3x}\,dx + \int_{0}^{\infty} e^{3x}\,dx \\ &= \int_{-\infty}^{1} e^{3x}\,dx + \int_{1}^{\infty} e^{3x}\,dx \\ &= \int_{-\infty}^{-1} e^{3x}\,dx + \int_{-1}^{\infty} e^{3x}\,dx \\ &= \int_{-\infty}^{c} e^{3x}\,dx + \int_{c}^{\infty} e^{3x}\,dx\end{aligned}$$

where c can be *any* value.

Typically, though, c is something "easy" to work with, such as $c = 0$ in this case.

(e) *Find* $\int_{-\infty}^{-1}(3x^{-2} - 1)\,dx$

$$\int_{-\infty}^{-1}\left(3x^{-2} - 1\right)\,dx = \lim_{a\to-\infty}\left[\frac{1}{-2+1}x^{-2+1} - \frac{1}{0+1}x^{0+1}\right]_a^{-1} =$$

(choose one or more)
(i) $\mathbf{\lim_{a\to-\infty}\left[-3x^{-1} - x\right]_a^{-1}}$
(ii) $\mathbf{\lim_{a\to-\infty}\left[(-3(-1)^{-1} - (-1)) - (-3(a)^{-1} - a)\right]}$
(iii) $\mathbf{\lim_{a\to-\infty}\left[4 - (-\frac{3}{a} - a)\right]}$

so

$$\int_{-\infty}^{-1}\left(3x^{-2} - 1\right)\,dx = \lim_{a\to-\infty}\left[4 + \frac{3}{a} + a\right] =$$

(i) $\mathbf{\frac{1}{3}}$ (ii) $\mathbf{-\frac{1}{3}}$ (iii) **divergent**
because, although $\lim_{a\to-\infty}\frac{3}{a} = 0$, $\lim_{a\to-\infty} a = -\infty$

(f) *Find* $\int_0^\infty \left[\left(-\frac{7}{2}x^{-1/2}\right)e^{\left(-7\sqrt{x}\right)}\right]\,dx$

guess $u =$ (choose *two*!) (i) $\boldsymbol{-\frac{7}{2}x^{-1/2}}$ (ii) $\boldsymbol{-7\sqrt{x}}$ (iii) $\boldsymbol{-7x^{1/2}}$,

then $du =$ (i) $\boldsymbol{-\frac{7}{2}x^{-1/2}\,dx}$ (ii) $\boldsymbol{\sqrt{x}\,dx}$ (i) $\boldsymbol{-7\sqrt{x}\,dx}$

substituting $u = -7\sqrt{x}$ and $du = -\frac{7}{2}x^{-1/2}\,dx$ into $\int f(x)\,dx$,

$$\int \left[\left(-\frac{7}{2}x^{-1/2}\right)e^{\left(-7\sqrt{x}\right)}\right]\,dx =$$

(i) $\boldsymbol{\int e^u\,du}$ (ii) $\boldsymbol{\int \left[\frac{7}{2}e^u\right]\,du}$ (iii) $\boldsymbol{\int \left[\frac{2}{7}e^{3u}\right]\,du}$

but $u = -7\sqrt{x}$ and so

$$\int \left[\left(-\frac{7}{2}x^{-1/2}\right)e^{\left(-7\sqrt{x}\right)}\right]\,dx = \int e^u\,du = e^u + C =$$

(i) $\boldsymbol{\left(-2x^4 + 7x\right)^7 + C}$ (ii) $\boldsymbol{\frac{10}{21}\left(7 + 4x^2\right)^7 + C}$ (iii) $\boldsymbol{e^{-7\sqrt{x}} + C}$

and so

$$\int_0^{\infty} \left[\left(-\frac{7}{2}x^{-1/2}\right)e^{\left(-7\sqrt{x}\right)}\right]\,dx =$$

(choose one or more)
(i) $\boldsymbol{\lim_{b\to\infty}\left[e^{-7\sqrt{x}}\right]_0^b}$
(ii) $\boldsymbol{\lim_{b\to\infty}\left[e^{-7\sqrt{b}} - \left(e^{-7\sqrt{0}}\right)\right]}$
(iii) $\boldsymbol{\lim_{b\to\infty}\left[e^{-7\sqrt{b}} - 1\right]}$

so

$$\int_0^{\infty} \left[\left(-\frac{7}{2}x^{-1/2}\right)e^{\left(-7\sqrt{x}\right)}\right]\,dx = \lim_{b\to\infty}\left[e^{-7\sqrt{b}} - 1\right] =$$

(i) $\boldsymbol{-1}$ (ii) $\boldsymbol{1}$ (iii) **divergent**

because $\lim_{b\to\infty} e^{-7\sqrt{b}} = 0$