# Chapter 23

# Multifactor Studies

We look at three–factor studies. We find out they are a natural generalization of two–factor studies.

## 23.1 ANOVA Model I for Three–Factor Studies

SAS program: att5-23-1-roc-threeANOVA,means

**Exercise 23.1 (Three–Factor Study Notation and Model: ROC, Temperature, Noise and Gender)** Consider the effect of air temperature, noise and gender on the ROC of deer mice. Data for the Male mice level of Factor A, (when $i = 1$) is given by

| | Factor C, noise → | $k = 1$, low | $k = 2$, medium | $k = 3$, high | column ave |
|---|---|---|---|---|---|
| Factor B, | $j = 1$, $0^o$ F | 10.3, 7.2 | 9.1, 5.4 | 6.1, 2.1 | $\overline{Y}_{11\cdot\cdot} = 6.7$ |
| temperature | $j = 2$, $10^o$ F | 1.8, 9.8 | 12.1, 4.2 | 5.1, 6.2 | $\overline{Y}_{12\cdot\cdot} = 6.5$ |
| | $j = 3$, $20^o$ F | 1.2, 8.1 | 6.5, 4.1 | 1.2, 2.1 | $\overline{Y}_{13\cdot\cdot} = 3.9$ |
| | $j = 4$, $30^o$ F | 12.4, 15.1 | 16.1, 17.2 | 18.1, 19.1 | $\overline{Y}_{14\cdot\cdot} = 16.3$ |
| | column ave | $\overline{Y}_{1\cdot1\cdot} = 8.2$ | $\overline{Y}_{1\cdot2\cdot} = 9.3$ | $\overline{Y}_{1\cdot3\cdot} = 7.5$ | $\overline{Y}_{1\cdot\cdot\cdot} = 8.4$ |

and data for the Female mice level of Factor A, ($i = 2$) is given by,

| | Factor C, noise → | $k = 1$, low | $k = 2$, medium | $k = 3$, high | column ave |
|---|---|---|---|---|---|
| Factor B, | $j = 1$, $0^o$ F | 9.3, 6.2 | 8.1, 5.4 | 8.1, 7.1 | $\overline{Y}_{21\cdot\cdot} = 7.4$ |
| temperature | $j = 2$, $10^o$ F | 2.8, 9.8 | 10.1, 4.2 | 5.1, 6.2 | $\overline{Y}_{22\cdot\cdot} = 6.4$ |
| | $j = 3$, $20^o$ F | 4.2, 5.1 | 7.5, 4.1 | 6.2, 7.1 | $\overline{Y}_{23\cdot\cdot} = 5.7$ |
| | $j = 4$, $30^o$ F | 5.4, 5.1 | 11.1, 12.2 | 8.1, 9.1 | $\overline{Y}_{24\cdot\cdot} = 8.5$ |
| | column ave | $\overline{Y}_{2\cdot1\cdot} = 6.0$ | $\overline{Y}_{2\cdot2\cdot} = 7.8$ | $\overline{Y}_{2\cdot3\cdot} = 7.1$ | $\overline{Y}_{2\cdot\cdot\cdot} = 7.0$ |

1. *Notation.*
   From the tables above and SAS,

   (a) *Observations*

   i. $Y_{ijkm} = Y_{1111} =$ (circle one) **10.3** / **7.2** / **1.8**
   ii. $Y_{1121} =$ (circle one) **10.3** / **7.2** / **9.1**
   iii. $Y_{2121} =$ (circle one) **5.4** / **7.2** / **8.1**
   iv. $Y_{1311} =$ (circle one) **1.2** / **19.1** / **17.2**

   (b) *Cell means*

   i. cell mean,
   $\overline{Y}_{111\cdot} = \frac{1}{2}(17.5) =$ (circle one) **6.5** / **7.5** / **8.8**
   ii. cell mean,
   $\overline{Y}_{211\cdot} = \frac{1}{2}(15.5) =$ (circle one) **5.8** / **6.6** / **7.8**
   iii. cell mean,
   $\overline{Y}_{143\cdot} = \frac{1}{2}(37.2) =$ (circle one) **18.6** / **19.1** / **22.2**

   (c) *Main factor means*

   i. factor A, level 1, mean,
   $\overline{Y}_{1\cdots} = \frac{1}{bcn}\sum_{j=1}^{b}\sum_{k=1}^{c}\sum_{m=1}^{n} Y_{1jkm} = \frac{1}{bcn}Y_{1\cdots} = \frac{1}{4(3)(2)}(200.6) =$
   (circle one) **6.5** / **6.7** / **8.4**
   ii. factor B, level 2, mean,
   $\overline{Y}_{\cdot 2\cdot\cdot} =$ (circle one) **6.5** / **9.1** / **9.4**
   iii. factor C, level 3, mean,
   $\overline{Y}_{\cdot\cdot 3\cdot} =$ (circle one) **7.3** / **9.1** / **10.0**
   iv. overall (grand) mean,
   $\overline{Y}_{\cdots\cdot} = \frac{1}{2(4)(3)(2)}(368.2) =$ (circle one) **7.7** / **8.4** / **9.6**

   (d) *Fitted (predicted) values*

   i. fitted value, $\hat{Y}_{1111} = \overline{Y}_{111\cdot} =$ (circle one) **6.5** / **7.5** / **8.75**
   ii. fitted value, $\hat{Y}_{1112} = \overline{Y}_{111\cdot} =$ (circle one) **3.45** / **5.5** / **8.75**
   iii. fitted value, $\hat{Y}_{2312} = \overline{Y}_{231\cdot} =$ (circle one) **4.65** / **7.5** / **8.75**

   (e) *Residuals*

   i. cell residual,
   $e_{1111} = Y_{1111} - \hat{Y}_{1111} = Y_{1111} - \overline{Y}_{111\cdot} = 10.3 - 8.75 =$
   (circle one) **1.55** / **1.7** / **1.9**
   ii. cell residual,
   $e_{2111} = Y_{2111} - \hat{Y}_{2111} = 9.3 - 7.75 =$ (circle one) **1.55** / **1.7** / **1.9**

   (f) *Factor effects*

   i. factor A effect estimator,
   $\hat{\alpha}_1 = \overline{Y}_{1\cdots} - \overline{Y}_{\cdots\cdot} = 8.4 - 7.7 =$ (circle one) **0.7** / **0.9** / **1.1**

ii. factor B effect estimator,
$\hat{\beta}_1 = \overline{Y}_{\cdot 1 \cdot\cdot} - \overline{Y}_{\cdot\cdot\cdot\cdot} = 6.45 - 7.7 =$ (circle one) **−1.2** / **−0.2** / **0.2**

iii. factor C effect estimator,
$\hat{\gamma}_2 = \overline{Y}_{\cdot\cdot 2 \cdot} - \overline{Y}_{\cdot\cdot\cdot\cdot} = 8.6 - 7.7 =$ (circle one) **0.1** / **0.2** / **0.9**

iv. interaction AB effect estimator,
$(\hat{\alpha\beta})_{12} = \overline{Y}_{12\cdot\cdot} - \overline{Y}_{1\cdot\cdot\cdot} - \overline{Y}_{\cdot 2\cdot\cdot} + \overline{Y}_{\cdot\cdot\cdot\cdot} = 6.533 - 8.358 - 6.45 + 7.671 =$
(circle one) **−0.344** / **−0.604** / **−0.788**

v. interaction AC effect estimator,
$(\hat{\alpha\gamma})_{13} = \overline{Y}_{1\cdot 3\cdot} - \overline{Y}_{1\cdot\cdot\cdot} - \overline{Y}_{\cdot\cdot 3\cdot} + \overline{Y}_{\cdot\cdot\cdot\cdot} = 7.500 - 8.358 - 7.313 + 7.671 =$
(circle one) **−0.1** / **−0.2** / **−0.5**

vi. interaction ABC effect estimator,
$(\hat{\alpha\beta\gamma})_{111} =$
$= \overline{Y}_{111\cdot} - \overline{Y}_{11\cdot\cdot} - \overline{Y}_{1\cdot 1\cdot} - \overline{Y}_{\cdot 11\cdot} + \overline{Y}_{1\cdot\cdot\cdot} + \overline{Y}_{\cdot 1\cdot\cdot} + \overline{Y}_{\cdot\cdot 1\cdot} - \overline{Y}_{\cdot\cdot\cdot\cdot} =$
$= 8.750 - 6.700 - 8.238 - 8.250 + 8.358 + 7.033 + 7.113 - 7.671 =$
(circle one) **0.1** / **0.395** / **0.415**

2. *Treatment Mean Plots.*
The treatment mean plots are more complicated for a three–factor ANOVA than for a two–factor ANOVA[1]. Looking at the SAS output plots.

(a) *Factor A (gender) significant?*
Since the mean roc levels are *different* for the two gender plots (0–20, gender=1 and 4–12, gender=2), it appears as though gender (choose one) **is** / **is not** significant.

(b) *Factor B (temperature) significant?*
Since the mean roc levels are *different* for different temperatures in both gender plots (the different colored plus–signs are at different mean levels in both plots), it appears as though temperature (choose one) **is** / **is not** significant.

(c) *Factor C (noise) significant?*
Since the mean roc levels are more or less the *same* for different noise levels in both gender plots (each color of plus–signs are more or less at the same mean level for each color in each plot), it appears as though noise (choose one) **is** / **is not** significant.

(d) *Interaction BC (temperature–noise) significant?*
Since the different colored graphs are more or less *parallel* to each other in both gender plots, it appears as though temperature–noise interaction (choose one) **is** / **is not** significant.

[1]We are *not* going to worry about the *detailed* interpretation of treatment plots for three–factor data sets.

(e) *Other interactions significant?*
It (choose one) **is / is not** possible to tell from the given treatment mean plots whether any of the other interactions (AC, AB or ABC) are significant or not[2]. Other treatment mean plots would need to be created to look for these other interactions.

3. *Model.* Match the columns.

| |
|---|
| Model A<br>$(Y_{ijkm} = \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk} + \varepsilon_{ijkm}$ |
| Model B<br>$Y_{ijkm} = \mu_{ijkm} + \varepsilon_{ijkm}$ |

| Description | Model |
|---|---|
| Cell means model | |
| Factor effects model | |

## 23.2 Analysis of Variance

SAS program: att5-23-2-roc-threeANOVA

The three–factor analysis of variance (ANOVA) procedure, has the following ANOVA table, where the seven (7) tests are conducted using various $F^*$ statistics, where all seven statistics are calculated by dividing the mean squares of the main factor or interaction by the mean squares of the error.

| Source | Degrees of Freedom, *df* | Sum Of Squares, *SS* | Mean Squares, *MS* |
|---|---|---|---|
| Factor A | $a-1$ | $SSA = nbc\sum(\overline{Y}_{i\ldots} - \overline{Y}_{\ldots.})^2$ | $MSTA = \frac{SSA}{a-1}$ |
| Factor B | $b-1$ | $SSB = nac\sum(\overline{Y}_{\cdot j\cdot\cdot} - \overline{Y}_{\ldots.})^2$ | $MSTB = \frac{SSB}{b-1}$ |
| Factor B | $c-1$ | $SSC = nab\sum(\overline{Y}_{\cdot\cdot k\cdot} - \overline{Y}_{\ldots.})^2$ | $MSTB = \frac{SSC}{c-1}$ |
| Interaction AB | $(a-1)(b-1)$ | $SSAB = nc\sum\sum(\overline{Y}_{ij\cdot\cdot} - \overline{Y}_{i\ldots} - \overline{Y}_{\cdot j\cdot\cdot} - \overline{Y}_{\ldots.})^2$ | $MSTAB = \frac{SSAB}{(a-1)(b-1)}$ |
| Interaction AC | $(a-1)(c-1)$ | $SSAC = nb\sum\sum(\overline{Y}_{i\cdot k\cdot} - \overline{Y}_{i\ldots} - \overline{Y}_{\cdot\cdot k\cdot} - \overline{Y}_{\ldots.})^2$ | $MSTAB = \frac{SSAC}{(a-1)(c-1)}$ |
| Interaction BC | $(b-1)(b-1)$ | $SSBC = na\sum\sum(\overline{Y}_{\cdot jk\cdot} - \overline{Y}_{\cdot j\cdot\cdot} - \overline{Y}_{\cdot\cdot k\cdot} - \overline{Y}_{\ldots.})^2$ | $MSTAB = \frac{SSBC}{(b-1)(c-1)}$ |
| Interaction ABC | $(a-1)(b-1)(c-1)$ | $SSABC = n\sum\sum(\overline{Y}_{ijk\cdot} - \overline{Y}_{ij\cdot\cdot} - \overline{Y}_{i\cdot k\cdot} - \overline{Y}_{\cdot jk\cdot} - \overline{Y}_{\ldots})^2$ | $MSTAB = \frac{SSABC}{(a-1)(b-1)(c-1)}$ |
| | | $+\overline{Y}_{i\ldots} + \overline{Y}_{\cdot j\cdot\cdot} + \overline{Y}_{\cdot\cdot k\cdot} - \overline{Y}_{\ldots.})^2$ | |
| Error | $abc(n-1)$ | $SSE = \sum\sum\sum(Y_{ijkm} - \overline{Y}_{ijk\cdot})^2$ | $MSE = \frac{SSE}{abc(n-1)}$ |
| Total | $nabc-1$ | $SSTO = \sum\sum(Y_{ijkm} - \overline{Y}_{\ldots.})^2$ | |

**Exercise 23.2 (Three–Factor Study: Rate of oxygen (roc) of mice)**
Data for the Male mice level of Factor A, (when $i = 1$) is given by

[2]It turns out that parallel plots often, although not always, imply no interaction. However, more detailed discussion is given in the Neter et al. text.

| | Factor C, noise → | $k = 1$, low | $k = 2$, medium | $k = 3$, high | column ave |
|---|---|---|---|---|---|
| Factor B, | $j = 1$, $0^o$ F | 10.3, 7.2 | 9.1, 5.4 | 6.1, 2.1 | $\overline{Y}_{11\cdot\cdot} = 6.7$ |
| temperature | $j = 2$, $10^o$ F | 1.8, 9.8 | 12.1, 4.2 | 5.1, 6.2 | $\overline{Y}_{12\cdot\cdot} = 6.5$ |
| | $j = 3$, $20^o$ F | 1.2, 8.1 | 6.5, 4.1 | 1.2, 2.1 | $\overline{Y}_{13\cdot\cdot} = 3.9$ |
| | $j = 4$, $30^o$ F | 12.4, 15.1 | 16.1, 17.2 | 18.1, 19.1 | $\overline{Y}_{14\cdot\cdot} = 16.3$ |
| | column ave | $\overline{Y}_{1\cdot1\cdot} = 8.2$ | $\overline{Y}_{1\cdot2\cdot} = 9.3$ | $\overline{Y}_{1\cdot3\cdot} = 7.5$ | $\overline{Y}_{1\cdot\cdot\cdot} = 8.4$ |

and data for the Female mice level of Factor A, $(i = 2)$ is given by,

| | Factor C, noise → | $k = 1$, low | $k = 2$, medium | $k = 3$, high | column ave |
|---|---|---|---|---|---|
| Factor B, | $j = 1$, $0^o$ F | 9.3, 6.2 | 8.1, 5.4 | 8.1, 7.1 | $\overline{Y}_{21\cdot\cdot} = 7.4$ |
| temperature | $j = 2$, $10^o$ F | 2.8, 9.8 | 10.1, 4.2 | 5.1, 6.2 | $\overline{Y}_{22\cdot\cdot} = 6.4$ |
| | $j = 3$, $20^o$ F | 4.2, 5.1 | 7.5, 4.1 | 6.2, 7.1 | $\overline{Y}_{23\cdot\cdot} = 5.7$ |
| | $j = 4$, $30^o$ F | 5.4, 5.1 | 11.1, 12.2 | 8.1, 9.1 | $\overline{Y}_{24\cdot\cdot} = 8.5$ |
| | column ave | $\overline{Y}_{2\cdot1\cdot} = 6.0$ | $\overline{Y}_{2\cdot2\cdot} = 7.8$ | $\overline{Y}_{2\cdot3\cdot} = 7.1$ | $\overline{Y}_{2\cdot\cdot\cdot} = 7.0$ |

| Source | *df* | *SS* | *MS* |
|---|---|---|---|
| Factor A (Gender) | 1 | 22.6875 | 22.6875 |
| Factor B (Temperature) | 3 | 393.0891667 | 131.0297222 |
| Factor C (Noise) | 2 | 20.48666667 | 10.243333 |
| Interaction AB | 3 | 172.8958333 | 57.631944 |
| Interaction AC | 2 | 7.125 | 3.5625 |
| Interaction BC | 6 | 55.67333333 | 9.2788889 |
| Interaction ABC | 6 | 34.54166667 | 5.7569444 |
| Error | 24 | 174.5 | 7.270833 |
| Total | 47 | 880.9991667 | |

After checking to see that none of the interactions are significant, test for main effects at $\alpha = 0.05$.

1. *Test Interaction (Gender × Temperature, AB)*

   (a) *Statement*
   The statement of the test is (check none, one or more):

   i. $H_0:$ all $(\alpha\beta)_{ij} = 0$ versus $H_a:$ not all $(\alpha\beta)_{ij} = 0$.

   ii. $H_0: \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ versus $H_a:$ $\alpha_1 \neq \alpha_3, \alpha_1 \neq \alpha_2$.

   iii. $H_0: \alpha_1 = \alpha_2 = 0$ versus $H_a:$ at least one $\alpha_i \neq 0$, $i = 1, 2, 3$.

   iv. $H_0: \mu_{1\cdot\cdot} = \mu_{2\cdot\cdot}$ versus $H_a:$ at least one $\mu_{i\cdot} \neq \mu_{j\cdot}$, $i, j = 1, 2$.

   (b) *Test*
   From SAS, the test statistic is

$F^* =$ (circle one) **3.12** / **7.93** / **3576.98**
the p–value, with 3 and 24 degrees of freedom, is

$$\text{p–value} = P(F \geq 7.93)$$

which equals (circle one) **0.0008** / **0.09** / **0.43**.
The level of significance is 0.05.

(c) *Conclusion*
Since the p–value, 0.0008, is smaller than the level of significance, 0.05, we (circle one) **accept** / **reject** the null hypothesis that the AB interaction effect is zero.

(d) In fact, since the Gender $\times$ Temperature interaction is not zero, we (choose one) **can** / **cannot** test the Gender main effect or Temperature main effect. However, we *are* still able to test the Noise main effect.

2. *Test Interaction (Temperature $\times$ Noise, BC)*

(a) *Statement*
The statement of the test is (check none, one or more):

i. $H_0:$ all $(\beta\gamma)_{ij} = 0$ versus $H_a:$ not all $(\beta\gamma)_{ij} = 0$.
ii. $H_0: \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ versus $H_a: \alpha_1 \neq \alpha_3, \alpha_1 \neq \alpha_2$.
iii. $H_0: \alpha_1 = \alpha_2 = 0$ versus
$H_a:$ at least one $\alpha_i \neq 0,\ i = 1, 2, 3, 4$.
iv. $H_0: \mu_{1\cdot\cdot} = \mu_{2\cdot\cdot}$ versus
$H_a:$ at least one $\mu_{i\cdot} \neq \mu_{j\cdot},\ i, j = 1, 2$.

(b) *Test*
From SAS,
$F^* =$ (circle one) **1.28** / **7.93** / **3576.98**
the p–value, with 6 and 24 degrees of freedom, is

$$\text{p–value} = P(F \geq 1.28)$$

which equals (circle one) **0.0008** / **0.09** / **0.31**.
The level of significance is 0.05.

(c) *Conclusion*
Since the p–value, 0.31, is larger than the level of significance, 0.05, we (circle one) **accept** / **reject** the null hypothesis that the BC (Temperature $\times$ Noise) interaction effect is zero.

3. *Test Factor (Noise, C)*

(a) *Statement*
The statement of the test is (check none, one or more):

i. $H_0 : \beta_1 = \beta_2 = \beta_3 = 0$ versus $H_a :\ \beta_i \neq \beta_j,\ \ i \neq j,\ \ i, j = 1, 2, 3.$
ii. $H_0 : \beta_1 = \beta_2 = \beta_3 = 0$ versus $H_a :\ \beta_1 \neq \beta_3, \beta_1 \neq \beta_2.$
iii. $H_0 : \gamma_1 = \gamma_2 = \gamma_3 = 0$ versus
$H_a :$ at least one $\gamma_i \neq 0,\ i = 1, 2, 3, 4.$
iv. $H_0 : \mu_{\cdot 1 \cdot} = \mu_{\cdot 2 \cdot} = \mu_{\cdot 3 \cdot} = \mu_{\cdot 4 \cdot}$ versus
$H_a :$ at least one $\mu_{i\cdot} \neq \mu_{j\cdot},\ i, j = 1, 2, 3, 4.$

(b) *Test*
From SAS,
$F^* =$ (circle one) **18.02** / **357.69** / **3576.98**
the p–value, with 3 and 24 degrees of freedom, is given by

$$\text{p–value} = P(F \geq 1.41)$$

which equals (circle one) **0.26** / **0.34** / **0.43**.
The level of significance is 0.05.

(c) *Conclusion*
Since the p–value, 0.26, is larger than the level of significance, 0.05, we (circle one) **accept** / **reject** the null hypothesis that the main Noise effect is zero.

## 23.3 Evaluation of Appropriateness of ANOVA Model

SAS program: att5-23-3-roc-threeANOVA,residuals

As discussed previously, residuals plots are useful in assessing the validity of the fit of the model to the data.

**Exercise 23.3 (Residual analysis: Rate of oxygen (roc) of mice)**
Data for the Male mice level of Factor A, (when $i = 1$) is given by

| | Factor C, noise → | $k = 1$, low | $k = 2$, medium | $k = 3$, high | column ave |
|---|---|---|---|---|---|
| Factor B, | $j = 1$, $0^o$ F | 10.3, 7.2 | 9.1, 5.4 | 6.1, 2.1 | $\overline{Y}_{11\cdot\cdot} = 6.7$ |
| temperature | $j = 2$, $10^o$ F | 1.8, 9.8 | 12.1, 4.2 | 5.1, 6.2 | $\overline{Y}_{12\cdot\cdot} = 6.5$ |
| | $j = 3$, $20^o$ F | 1.2, 8.1 | 6.5, 4.1 | 1.2, 2.1 | $\overline{Y}_{13\cdot\cdot} = 3.9$ |
| | $j = 4$, $30^o$ F | 12.4, 15.1 | 16.1, 17.2 | 18.1, 19.1 | $\overline{Y}_{14\cdot\cdot} = 16.3$ |
| | column ave | $\overline{Y}_{1\cdot 1\cdot} = 8.2$ | $\overline{Y}_{1\cdot 2\cdot} = 9.3$ | $\overline{Y}_{1\cdot 3\cdot} = 7.5$ | $\overline{Y}_{1\cdot\cdot\cdot} = 8.4$ |

and data for the Female mice level of Factor A, ($i = 2$) is given by,

| | Factor C, noise → | $k = 1$, low | $k = 2$, medium | $k = 3$, high | column ave |
|---|---|---|---|---|---|
| Factor B, | $j = 1$, $0^o$ F | 9.3, 6.2 | 8.1, 5.4 | 8.1, 7.1 | $\overline{Y}_{21\cdot\cdot} = 7.4$ |
| temperature | $j = 2$, $10^o$ F | 2.8, 9.8 | 10.1, 4.2 | 5.1, 6.2 | $\overline{Y}_{22\cdot\cdot} = 6.4$ |
| | $j = 3$, $20^o$ F | 4.2, 5.1 | 7.5, 4.1 | 6.2, 7.1 | $\overline{Y}_{23\cdot\cdot} = 5.7$ |
| | $j = 4$, $30^o$ F | 5.4, 5.1 | 11.1, 12.2 | 8.1, 9.1 | $\overline{Y}_{24\cdot\cdot} = 8.5$ |
| | column ave | $\overline{Y}_{2\cdot 1\cdot} = 6.0$ | $\overline{Y}_{2\cdot 2\cdot} = 7.8$ | $\overline{Y}_{2\cdot 3\cdot} = 7.1$ | $\overline{Y}_{2\cdot\cdot\cdot} = 7.0$ |

Provide various residual plots to check the appropriateness of using three–factor ANOVA model for this data.

1. *Residual plots*

   (a) *Residual vs gender*
   From SAS, this residual plot appears to indicate (choose best one)
      i. nothing–it is random–all is fine.
      ii. non–constant variance

   (b) *Residual vs temperature*
   This residual plot appears to indicate (choose best one)
      i. nothing–it is random–all is fine.
      ii. non–constant variance

   (c) *Residual vs noise*
   This residual plot appears to indicate (choose best one)
      i. nothing–it is random–all is fine.
      ii. non–constant variance

   (d) *Residual vs predicted*
   This residual plot appears to indicate (choose best one)
      i. nothing–it is random–all is fine.
      ii. non–constant variance

2. *Normal probability plot*
   **True** / **False** The normal probability plot is linear.

## 23.4 Analysis of Factor Effects

SAS program: att5-23-4-roc-threeANOVA,analysis

The main factor level means can be analyzed in detail when there are no interactions, whereas the treatment means must be analyzed in detail when there are interactions.

**Exercise 23.4 (Analysis of factor effects)**
Data for the Male mice level of Factor A, (when $i = 1$) is given by

| | Factor C, noise → | $k = 1$, low | $k = 2$, medium | $k = 3$, high | column ave |
|---|---|---|---|---|---|
| Factor B, | $j = 1$, $0^o$ F | 10.3, 7.2 | 9.1, 5.4 | 6.1, 2.1 | $\overline{Y}_{11\cdot\cdot} = 6.7$ |
| temperature | $j = 2$, $10^o$ F | 1.8, 9.8 | 12.1, 4.2 | 5.1, 6.2 | $\overline{Y}_{12\cdot\cdot} = 6.5$ |
| | $j = 3$, $20^o$ F | 1.2, 8.1 | 6.5, 4.1 | 1.2, 2.1 | $\overline{Y}_{13\cdot\cdot} = 3.9$ |
| | $j = 4$, $30^o$ F | 12.4, 15.1 | 16.1, 17.2 | 18.1, 19.1 | $\overline{Y}_{14\cdot\cdot} = 16.3$ |
| | column ave | $\overline{Y}_{1\cdot 1\cdot} = 8.2$ | $\overline{Y}_{1\cdot 2\cdot} = 9.3$ | $\overline{Y}_{1\cdot 3\cdot} = 7.5$ | $\overline{Y}_{1\cdot\cdot\cdot} = 8.4$ |

and data for the Female mice level of Factor A, ($i = 2$) is given by,

| | Factor C, noise → | $k = 1$, low | $k = 2$, medium | $k = 3$, high | column ave |
|---|---|---|---|---|---|
| Factor B, | $j = 1$, $0^o$ F | 9.3, 6.2 | 8.1, 5.4 | 8.1, 7.1 | $\overline{Y}_{21\cdot\cdot} = 7.4$ |
| temperature | $j = 2$, $10^o$ F | 2.8, 9.8 | 10.1, 4.2 | 5.1, 6.2 | $\overline{Y}_{22\cdot\cdot} = 6.4$ |
| | $j = 3$, $20^o$ F | 4.2, 5.1 | 7.5, 4.1 | 6.2, 7.1 | $\overline{Y}_{23\cdot\cdot} = 5.7$ |
| | $j = 4$, $30^o$ F | 5.4, 5.1 | 11.1, 12.2 | 8.1, 9.1 | $\overline{Y}_{24\cdot\cdot} = 8.5$ |
| | column ave | $\overline{Y}_{2\cdot 1\cdot} = 6.0$ | $\overline{Y}_{2\cdot 2\cdot} = 7.8$ | $\overline{Y}_{2\cdot 3\cdot} = 7.1$ | $\overline{Y}_{2\cdot\cdot\cdot} = 7.0$ |

1. *Preliminary comments*
   **True / False**
   In this case, the noise main effect can be analyzed in detail because, from above, none of the interactions involving Noise are significant. However, the gender main effect and the temperature main effect cannot be analyzed in detail separately because the interaction between these two effects is significant. Consequently, the treatment means for the gender and temperature interaction are analyzed in detail.

2. *Confidence Interval, Multiple Pairwise, Noise Factor C, Bonferroni.*
   Since there are $c = 3$ levels of Factor C (Noise), there are a total of $\frac{c(c-1)}{2} = \frac{3(2)}{2} = 3$ possible pairwise comparisons. This is not a large number of pairwise comparisons, but determine a Bonferroni multiple pairwise confidence interval

anyway. In particular, determine a 95% confidence interval for $\mu_{\cdot\cdot 1} - \mu_{\cdot\cdot 2}$. Since

$$\begin{aligned}
\hat{D} &= \hat{\mu}_{\cdot\cdot 1} - \hat{\mu}_{\cdot\cdot 2} \\
&= 7.11 - 8.59 \\
&= -1.48 \\
s\left\{\hat{D}\right\} &= \sqrt{\frac{2MSE}{abn}} \\
&= \sqrt{\frac{2(7.27)}{(2)(4)(2)}} \\
&= 0.953 \\
t(1-\alpha/2g; abc(n-1)) &= t(1-0.05/(2(3)); (2)(4)(3)(1)) \\
&= t(0.992; 24) \\
&= 2.592
\end{aligned}$$

the confidence interval for $\mu_{\cdot\cdot 1} - \mu_{\cdot\cdot 2}$ is given by

$$\hat{D} \pm t(1-\alpha/2g; abc(n-1))s\left\{\hat{D}\right\} = -1.48 \pm (2.592)(0.953) =$$

(choose one) $\mathbf{(-3.95, 1.00)}$ / $\mathbf{(-5.98, -2.98)}$ / $\mathbf{(-7.381, -3.409)}$

3. *Test, Treatment Mean, Gender and Temperature,* $\mu_{23\cdot}$.
   Test if the ROC of mice subjected to Factor A, level 2 and Factor B, level 3, is greater than 5.5 units at $\alpha = 0.05$.

   (a) *Statement*
   The statement of the test is (check none, one or more):
   i. $H_0: \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ versus $H_a: \alpha_1 \neq \alpha_2, \alpha_1 = \alpha_3$.
   ii. $H_0: \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ versus $H_a: \alpha_1 \neq \alpha_3, \alpha_1 \neq \alpha_2$.
   iii. $H_0: \mu_{23\cdot} = 5.5$ versus
   $H_a: \mu_{23\cdot} > 5.5$.
   iv. $H_0: \mu_{1\cdot} = \mu_{2\cdot} = \mu_{3\cdot} = \mu_{4\cdot}$ versus
   $H_a:$ at least one $\mu_{i\cdot} \neq \mu_{j\cdot}$, $i, j = 1, 2, 3, 4$.

   (b) *Test*
   Since (from SAS output)

$$\begin{aligned}
\hat{\mu}_{23\cdot} &= \overline{Y}_{23\cdot} \\
&= 5.7 \\
s\left\{\hat{\mu}_{23\cdot}\right\} &= \sqrt{\frac{MSE}{cn}} \\
&= \sqrt{\frac{7.27}{3(2)}} \\
&= 1.101
\end{aligned}$$

the test statistic is

$$t^* = [\hat{\mu}_{23\cdot} - c]/s\left\{\hat{\mu}_{23\cdot}\right\} = [5.7 - 5.5]/1.101 = 0.182$$

the p–value, with $abc(n-1) = (2)(4)(3)(2-1) = 24$ degrees of freedom, is given by

$$\text{p–value} = P(t \geq 0.182)$$

which equals (circle one) **0.00** / **0.37** / **0.43**.
The level of significance is 0.05.

(c) *Conclusion*
Since the p–value, 0.43, is larger than the level of significance, 0.05, we (circle one) **accept** / **reject** the null hypothesis that the ROC of mice, subjected to factor A, level 2 and Factor B, level 3, is equal to 5.5 units.

4. *Confidence Interval, Multiple Pairwise of Treatment Means, $\mu_{11\cdot} - \mu_{23\cdot}$, Tukey.* Since there are $a = 2$ levels of factor A (gender) and $b = 4$ levels of factor B (temperature), there are a total of $\frac{ab(ab-1)}{2} = \frac{8(7)}{2} = 28$ possible pairwise comparisons. Determine a Tukey 95% confidence interval for $\mu_{11\cdot} - \mu_{23\cdot}$. Since

$$\begin{aligned}
\hat{D} &= \overline{Y}_{11\cdot\cdot} - \overline{Y}_{23\cdot\cdot} \\
&= 6.7 - 5.7 \\
&= 1.0 \\
s\left\{\hat{D}\right\} &= \sqrt{\frac{MSE}{cn}\sum c_i^2} \\
&= \sqrt{\frac{7.27}{6}\left((1)^2 + (-1)^2\right)} \\
&= 1.557 \\
\frac{1}{\sqrt{2}}q(1-\alpha; ab, abc(n-1)) &= \frac{1}{\sqrt{2}}q(1-0.05; (2)(4), 24) \\
&= \frac{1}{\sqrt{2}}q(0.95; 8, 24) \\
&= \frac{1}{\sqrt{2}}(4.68) \\
&= 3.31
\end{aligned}$$

the confidence interval for $\mu_{11\cdot} - \mu_{23\cdot}$ is given by

$$\hat{D} \pm \frac{1}{\sqrt{2}}q(1-\alpha; ab, abc(n-1))s\left\{\hat{D}\right\} = 1.0 \pm (1.557)(3.31) =$$

(choose one) $\mathbf{(-4.15, 6.15)}$ / $\mathbf{(-5.98, -2.98)}$ / $\mathbf{(-5.433, -3.527)}$

## 23.5 Example of Three–Factor Study

This a very good example to look over.

## 23.6 Unequal Sample Sizes in Multifactor Studies

Inference in the three–factor ANOVA when there are unequal sample sizes is essentially the same as for this case in the previously discussed two–factor ANOVA.