3.7 The Hypergeometric Probability Distribution

The *hypergeometric* distribution, the probability of y successes when sampling with*out*[15] replacement n items from a population with r successes and $N-r$ failures, is

$$p(y) = P(Y = y) = \frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}}, \quad 0 \le y \le r, \quad 0 \le n-y \le N-r,$$

and its expected value (mean), variance and standard deviation are,

$$\mu = E(Y) = \frac{nr}{N}, \quad \sigma^2 = V(Y) = n\left(\frac{r}{N}\right)\left(\frac{N-r}{N}\right)\left(\frac{N-n}{N-1}\right), \quad \sigma = \sqrt{V(Y)}.$$

Exercise 3.7 (The Hypergeometric Probability Distribution)

1. *Hypergeometric: televisions.* Seven television ($n = 7$) tubes are chosen at random from a shipment of $N = 240$ television tubes of which $r = 15$ are defective.

   (a) The probability that $y = 4$ of the chosen televisions are defective is

   $$p(4) = \frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}} = \text{(choose one)}$$

   (i) $\frac{\binom{\mathbf{15}}{\mathbf{4}}\times\binom{\mathbf{225}}{\mathbf{3}}}{\binom{\mathbf{240}}{\mathbf{4}}}$ (ii) $\frac{\binom{\mathbf{15}}{\mathbf{3}}\times\binom{\mathbf{225}}{\mathbf{3}}}{\binom{\mathbf{240}}{\mathbf{7}}}$ (iii) $\frac{\binom{\mathbf{15}}{\mathbf{4}}\times\binom{\mathbf{225}}{\mathbf{3}}}{\binom{\mathbf{240}}{\mathbf{7}}}$

   (b) The probability $y = 4$ of the chosen televisions are defective is $p(4) =$ (choose one)
   (i) **0.0003069** (ii) **0.0005069** (iii) **0.0006069** (iv) **0.0007069**.
   PRGM HPMF ENTER ENTER (again!) 240 ENTER 7 ENTER 15 ENTER 4 ENTER.

   (c) The probability $y = 5$ of the chosen televisions are defective is $p(5) =$ (choose one)
   (i) **0.000007069** (ii) **0.000009084** (iii) **0.00010069** (iv) **0.00013059**.
   PRGM HPMF ENTER ENTER (again!) 240 ENTER 7 ENTER 15 ENTER 5 ENTER.

[15] The hypergeometric assumption of sampling with*out* replacement is more realistic than the binomial assumption of sampling with replacement.

(d) The probability *at most* $y = 5$ of the chosen televisions are defective is $P(Y \leq 5) =$ (choose one)
(i) **0.900** (ii) **0.925** (iii) **0.950** (iv) **0.999**.
PRGM HCMF ENTER ENTER (again!) 240 ENTER 7 ENTER 15 ENTER 5 ENTER.

(e) The probability *at least* $y = 1$ of the chosen televisions are defective is $P(Y \geq 1) = 1 - P(Y < 1) = 1 - P(Y = 0) =$ (choose one)
(i) **0.367** (ii) **0.435** (iii) **0.545** (iv) **0.633**.
PRGM HCMF ENTER ENTER (again!) 240 ENTER 7 ENTER 15 ENTER 0 ENTER, then subtract from 1.

(f) *Expectation.* The expected number of defective TVs chosen is $\mu = E(Y) = \frac{nr}{N} = \frac{(7)(15)}{240} \approx$ (circle one)
(i) **0.4375** (ii) **0.7375** (iii) **0.9375** (iv) **1.2375**.

(g) *Expected cost.* If it costs \$75 to repair a defective television, the expected total repair cost is
$E(C) = E(75Y) = 75E(Y) = 75(0.4375) \approx$ (circle one)
(i) **\$23.75** (ii) **\$28.75** (iii) **\$32.82** (iv) **\$36.37**.

(h) *Variance.* The variance in the number of defective TVs chosen
$\sigma^2 = V(Y) = 7\left(\frac{15}{240}\right)\left(\frac{240-15}{240}\right)\left(\frac{240-7}{240-1}\right) = n\left(\frac{r}{N}\right)\left(\frac{N-r}{N}\right)\left(\frac{N-n}{N-1}\right) =$
(i) **0.29986** (ii) **0.39986** (iii) **0.49986** (iv) **0.69986**.

(i) *Standard deviation in costs.* If it costs \$75 to repair a defective television, the standard deviation in total repair costs is
$\sigma = \sqrt{V(C)} = \sqrt{V(75Y)} = \sqrt{75^2 V(Y)} = 75\sqrt{V(Y)} \approx 75\sqrt{0.39986} \approx$
(i) **39.46** (ii) **44.86** (iii) **45.98** (iv) **47.43**.

2. *Binomial approximation to the hypergeometric: televisions.* Seven television ($n = 7$) tubes are chosen at random from a shipment of $N = 240$ television tubes of which $r = 15$ are defective.

(a) We sample with*out* replacement; that is, every time a TV is chosen, we do *not* replace it to be potentially chosen again. In other words, the chance of choosing a defective TV, every time a TV is chosen, *changes* or *depends* on the number of defective TVs that were chosen before it.
(i) **True** (ii) **False**

(b) If the sample size, n, is small relative to the number of televisions, N, $\frac{n}{N} < 0.05$, say, the hypergeometric can be approximated by a binomial. The chance, $p = \frac{r}{N}$, of choosing a defective TV, every time a TV is chosen, does not change "that much" when $\frac{n}{N} < 0.05$. Since $\frac{n}{N} = \frac{15}{240} = 0.0625 > 0.05$, the binomial will probably approximate the hypergeometric (choose one)
(i) **very closely.** (ii) **somewhat closely.** (iii) **not closely at all.**

(c) Since $p = \frac{r}{N} = \frac{15}{240} = 0.0625$ and so a binomial approximation to:
$p(5) =$ (i) **0.00157** (ii) **0.00908** (iii) **0.00106** (iv) **0.00139**,
2nd DISTR binompdf 240, 0.0625, 5 ENTER
as compared to 0.000009084 for the hypergeometric,
$E(Y) = np = (7)(0.0625) =$
(i) **0.3998** (ii) **0.4375** (iii) **0.5345** (iv) **0.8345**,
as compared to 0.4375 for the hypergeometric,
$V(Y) = npq = (7)(0.0625)(1 - 0.0625) \approx$
(i) **0.3998** (ii) **0.4102** (iii) **0.7345** (iv) **0.8345**,
as compared to 0.3999 for the hypergeometric.

(d) In general,

$$\lim_{N \to \infty} \frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}} = \binom{n}{y} p^y q^{n-y}$$

where $p = \frac{r}{N}$.
(i) **True** (ii) **False**

3. *Hypergeometric: capture–recapture.* To determine number of perch, N, in Lake Fishalot, $r = 45$ are captured at random from the lake, tagged and let go back into the lake. A short while later, another $n = 32$ perch are captured, of which $y = 2$ are found to be tagged. Approximately how many perch are in Lake Fishalot?

(a) The chance two of the second group of captured fish are tagged is

$$p(2) = \frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}} = \text{(choose one)}$$

(i) $\frac{\binom{\mathbf{32}}{\mathbf{2}} \times \binom{\boldsymbol{N}}{\mathbf{43}}}{\binom{\boldsymbol{N}}{\mathbf{3200}}}$ (ii) $\frac{\binom{\mathbf{45}}{\mathbf{2}} \times \binom{\boldsymbol{N} - \mathbf{45}}{\mathbf{32} - \mathbf{2}}}{\binom{\boldsymbol{N}}{\mathbf{32}}}$ (iii) $\frac{\binom{\mathbf{45}}{\mathbf{43}} \times \binom{\boldsymbol{N}}{\mathbf{2}}}{\binom{\boldsymbol{N}}{\mathbf{3200}}}$,

(b) *Guess* $N = 500$. In this case, chance two of 32 fish chosen are tagged is

$$p(2) = \frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}} = \frac{\binom{45}{2} \times \binom{500-45}{32-2}}{\binom{500}{32}} = \text{(choose one)}$$

(i) **0.24** (ii) **0.26** (iii) **0.29** (iv) **0.32**.
PRGM HPMF ENTER ENTER (again!) 500 ENTER 32 ENTER 45 ENTER 2 ENTER.

(c) *Guess* $N = 750$. In this case, chance two of 32 fish chosen are tagged is

$$p(2) = \frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}} = \frac{\binom{45}{2} \times \binom{750-45}{32-2}}{\binom{750}{32}} = \text{(choose one)}$$

(i) **0.24** (ii) **0.26** (iii) **0.29** (iv) **0.32**.

PRGM HPMF ENTER ENTER (again!) 750 ENTER 32 ENTER 45 ENTER 2 ENTER.

(d) *Guess* $N = 1000$. In this case, chance two of 32 fish chosen are tagged is $p(2) =$ (choose one) (i) **0.24** (ii) **0.26** (iii) **0.29** (iv) **0.32**.

PRGM HPMF ENTER ENTER (again!) 1000 ENTER 32 ENTER 45 ENTER 2 ENTER.

(e) A summary of results are given in the following table.

N	500	750	1000
$p(2)$	0.24	0.29	0.26

Since the *largest* chance that 2 of 32 fish chosen are tagged is 0.29, then it seems from the choices given, the number of fish in Lake Fishalot is $N =$ (choose one) (i) **500** (ii) **750** (iii) **1000** (iv) **1250**.

(f) The approximation to N would improve if we had more than three $p(2)$ to choose from; however, more effort would be required in calculating the extra $p(2)$. Differentiating

$$\frac{\binom{45}{2} \times \binom{N-45}{32-2}}{\binom{N}{32}}$$

with respect to N and then setting to zero, to locate the maximum N is also possible, but difficult to do.
(i) **True** (ii) **False**

3.8 The Poisson Probability Distribution

The function of the *Poisson* distribution is

$$p(y) = P(Y = y) = \frac{\lambda^y}{y!}e^{-\lambda}, \quad y = 0, 1, \ldots, \lambda > 0,$$

where $e = 2.71828\ldots$ and expected value, variance and standard deviation are,

$$\mu = E(Y) = \lambda, \quad \sigma^2 = V(X) = \lambda, \quad \sigma = \sqrt{\lambda}.$$

Exercise 3.8 (The Poisson Probability Distribution)

1. *Poisson: accidents.* There are an average of $\lambda = 3$ accidents per year along the I–95 stretch of highway between Michigan City, Indiana, and St. Joseph, Michigan.

   (a) The chance there are $y = 4$ accidents is
   $p(4) = \frac{\lambda^y}{y!}e^{-\lambda} =$ (choose one)
   (i) $\boldsymbol{\frac{3^4}{4!}e^{-3}}$ (ii) $\boldsymbol{\frac{3^4}{4!}e^{-4}}$ (iii) $\boldsymbol{\frac{4^3}{4!}e^{-4}}$ (iv) $\boldsymbol{\frac{4^4}{3!}e^{-3}}$

   (b) The chance there are $y = 4$ accidents is
   $p(4) \approx$ (choose one) (i) **0.13** (ii) **0.17** (iii) **0.18** (iv) **0.21**.
   2nd DISTR poissonpdf(ENTER 3, 4) ENTER.

   (c) The chance there are $y = 3$ accidents is
   $p(3) \approx$ (choose one) (i) **0.17** (ii) **0.18** (iii) **0.22** (iv) **0.24**.
   2nd DISTR poissonpdf(ENTER 3, 3) ENTER.

   (d) The probability there is *at least* $y = 2$ accidents is
   $P(Y \geq 2) = 1 - P(Y < 2) = 1 - P(Y \leq 1) \approx$ (choose one)
   $p(4) \approx$ (choose one) (i) **0.78** (ii) **0.79** (iii) **0.80** (iv) **0.81**.
   Subtract 2nd DISTR poissoncdf(ENTER 3, 1) ENTER from one; notice: poissoncdf, not poissonpdf!

   (e) *Expectation.* The expected number of accidents is
   $\mu = E(Y) = \lambda =$ (circle one)
   (i) **1** (ii) **2** (iii) **3** (iv) **4**.

   (f) *Expected cost.* If it costs \$500,000 per accident, the expected yearly cost is
   $E(C) = E(500000Y) = 500000E(Y) = 500000(3) =$ (circle one)
   (i) **\$500,000** (ii) **\$1,000,000** (iii) **\$1,500,000** (iv) **\$2,000,000**.

   (g) *Variance.* The variance in the number of accidents per year is
   $\sigma^2 = V(Y) = \lambda =$ (circle one)
   (i) **1** (ii) **2** (iii) **3** (iv) **4**.

   (h) *Standard deviation.* Standard deviation in number of accidents per year
   $\sigma = \sqrt{\lambda} = \sqrt{3} \approx$ (circle one)
   (i) **1.01** (ii) **1.34** (iii) **1.73** (iv) **1.96**.

2. *Poisson: photons.* A piece of iron is bombarded with electrons and, as a consequence, releases a number of photons. A number, y, of the photon particles released hit a magnetic detection field that surround the piece of iron being tested. It is found that an *average* of $\lambda = 5$ particles hit the magnetic detection field *per microsecond.*

   (a) The chance $y = 2$ particles hit the field per microsecond is
   $p(2) = \frac{5^2}{2!}e^{-5} \approx$ (choose one) (i) **0.06** (ii) **0.07** (iii) **0.08** (iv) **0.09**.
   2nd DISTR poissonpdf(ENTER 5, 2) ENTER

(b) The chance $y = 0$ particles hit the field per microsecond is $p(0) \approx$ (choose one) (i) **0.007** (ii) **0.008** (iii) **0.08** (iv) **0.009**.
2nd DISTR poissonpdf(ENTER 5, 0) ENTER

(c) If an average of $\lambda = 5$ particles hit the field every one microsecond time interval, then, in a *two* microsecond time interval, $\lambda = 2 \times 5 =$ (choose one) (i) **10** (ii) **15** (iii) **20** (iv) **25**.

(d) The chance $y = 3$ particles hit the field in two microseconds is $p(3) = \frac{10^3}{3!}e^{-10} \approx$ (choose one)
(i) **0.007** (ii) **0.008** (iii) **0.009** (iv) **0.010**.
2nd DISTR poissonpdf(ENTER 10, 3) ENTER

(e) The chance $y = 21$ particles hit the field in four microseconds (so $\lambda = 4 \times 5 = 20$) is $p(21) = \frac{20^{21}}{21!}e^{-20} \approx$ (choose one)
(i) **0.073** (ii) **0.085** (iii) **0.093** (iv) **0.101**.
2nd DISTR poissonpdf(ENTER 20, 21) ENTER

(f) *Standard deviation.* Standard deviation in number of field hits per microsecond is $\sigma = \sqrt{\lambda} = \sqrt{5} \approx$ (circle one)
(i) **1.51** (ii) **1.74** (iii) **2.13** (iv) **2.24**.

3. *Poisson approximation of the binomial: photons.* The Poisson distribution can be used to approximate the binomial distribution by letting $\lambda = np$. This is a fairly good approximation if $np < 7$.

(a) If $n = 2000$ particles are released by iron per microsecond, and there is a chance $p = 0.005$ that a particle hits the surrounding field per microsecond, then $\lambda = np = 2000(0.005) =$ (circle one)
(i) **5** (ii) **10** (iii) **15** (iv) **20**.
The chance $y = 15$ particles hit the field in a one microsecond period, is either *(approximate) Poisson*[16]
$p(15) = \frac{10^{15}}{15!}e^{-10} \approx$ (circle one)
(i) **0.00234** (ii) **0.0347** (iii) **0.0445** (iv) **0.0645**
2nd DISTR poissonpdf(ENTER 10, 15) ENTER
or *(exact) binomial*
$p(15) = \frac{2000!}{15!(2000-15)!} \times (0.005)^{15} \times (0.995)^{1985} \approx$ (circle one)
(i) **0.00234** (ii) **0.0346** (iii) **0.0445** (iv) **0.0645**.
2nd DISTR binompdf(ENTER 2000, 0.005, 15) ENTER

(b) If $n = 1000$ and $p = 0.01$, then $\lambda = np = 1000(0.01) =$ (circle one)
(i) **5** (ii) **10** (iii) **15** (iv) **20**.
Either *(approximate) Poisson*
$p(12) = \frac{10^{12}}{12!}e^{-10} \approx$ (circle one)

[16] It is a little surprising Poisson approximation to binomial is close because $\lambda = 2000(0.005) = 10 > 7$.

(i) **0.0948** (ii) **0.1247** (iii) **0.1345** (iv) **0.1445**
2nd DISTR poissonpdf(ENTER 10, 12) ENTER
or *(exact) binomial*
$p(12) = \frac{2000!}{12!(2000-12)!} \times (0.005)^{12} \times (0.995)^{1988} \approx$ (circle one)
(i) **0.0952** (ii) **0.1046** (iii) **0.1245** (iv) **0.1345**.
2nd DISTR binompdf(ENTER 1000, 0.01, 12) ENTER

4. *Poisson process: photons.* A random variable has a Poisson distribution if the Poisson process conditions are satisfied. For the photons example, this would mean the following assumptions are satisfied.

   - P(no "hit" occurs in time subinterval)$= 1 - p$
   - P(one "hit" occurs in time subinterval)$= p$
   - P(two or more "hits" occurs in time subinterval)$= 0$
   - Occurrence of hit in each time subinterval is independent of occurrence of event in other nonoverlapping subintervals.

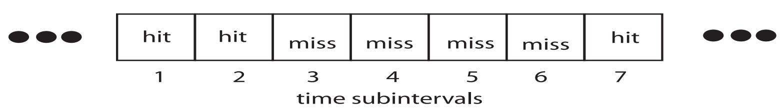

Figure 3.5: Poisson process: sequence of photon hits

   (a) The Poisson process assumes all time subintervals are created so small *one and only one* photon *could* hit the magnetic detection field during each subinterval. During seven time subintervals shown in Figure 3.5, three hits occur in time subintervals 1, 2 and 7 and four misses occur in time subintervals 3, 4, 5 and 6.
   (i) **True** (ii) **False**

   (b) The Poisson process assumes the chance a photon hits the magnetic detection field during any one time subinterval is p and the chance it misses is $q = 1 - p$. It is impossible to have more than one photon hit the magnetic detection field during any time subinterval. Consequently, the time subintervals are *infinitesimally small.*
   (i) **True** (ii) **False**

   (c) The Poisson process assumes a hit in each time subinterval is independent of any other hit in other nonoverlapping subintervals.
   (i) **True** (ii) **False**

(d) If there was a 20% chance, $p = 0.2$, a photon hit the magnetic detection field during any time subinterval then the chance of $y = 3$ hits in the $n = 7$ time subintervals is given by the *binomial*,

$$p(3) = \binom{n}{y} p^y q^{n-y} = \binom{7}{3} 0.2^3 0.8^{7-3} \approx$$

(i) **0.089** (ii) **0.115** (iii) **0.124** (iv) **0.134**.
2nd DISTR binompdf(ENTER 7, 0.2, 3) ENTER

(e) If a large number of time subintervals are considered; in other words, as, n gets bigger, in fact, as $n \to \infty$, it becomes increasingly difficult to calculate the probability of a number of hits using the binomial. In this case, the poisson is used instead.
(i) **True** (ii) **False**

(f) In general,

$$\lim_{n \to \infty} \binom{n}{y} p^y q^{n-y} = \frac{\lambda^y}{y!} e^{-\lambda}$$

where $\lambda = np$.
(i) **True** (ii) **False**

3.9 Moments and Moment–Generating Functions

Moment–generating functions, $m(t)$, are useful in calculating the *moments* of the distribution of any[17] random variable Y. Furthermore, $m(t)$ uniquely identifies any probability distribution. Consequently, it is possible to use either the probability distribution or its associated (and possibly easier to mathematically manipulate) moment–generating function when working with probability distributions.

- The *moment of random variable Y taken about the origin* is defined by,

$$\mu'_k = E\left(Y^k\right).$$

- The *moment of random variable Y taken about its mean* (or *kth central moment of Y*) is defined by,

$$\mu_k = E((Y - \mu)^k).$$

[17]This is true as long as $m(t)$ exists. The function $m(t)$ exists as long as there is a constant b such that $m(t) < \infty$ for $|t| < b$.

- In the discrete case, the moment–generating function of Y is defined by,

$$\begin{aligned} m(t) = E\left(e^{tY}\right) &= \sum_y e^{ty} p(y) \\ &= \sum_y \left[1 + ty + \frac{(ty)^2}{2!} + \cdots\right] p(y) \\ &= \sum_y p(y) + t\sum_y yp(y) + \frac{t^2}{2!}\sum_y y^2 p(y) + \cdots \\ &= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots \end{aligned}$$

- Function $m(t)$ "generates" moments of a distribution by successively differentiating $m(t)$ and evaluating the results at $t = 0$,

$$\left.\frac{d^k m(t)}{dt^k}\right]_{t=0} = m^{(k)}(0) = \mu_k', \quad k = 1, 2, \ldots$$

Exercise 3.9 (Moments and Moment–Generating Functions)

1. *Poisson. Assume* moment–generating function for poisson is

$$m(t) = e^{\lambda(e^t - 1)}.$$

   (a) *Determine $E(Y)$; that is, show $E(Y) = \lambda$ using $m(t)$.* First notice

$$E(Y) = E\left(Y^1\right) = \mu_1'.$$

   So

$$\mu_1' = m^{(1)}(0) = \left.\frac{d^1 m(t)}{dt^1}\right]_{t=0} = \left.\frac{de^{\lambda(e^t-1)}}{dt}\right]_{t=0} = \left[e^{\lambda(e^t-1)} \cdot \lambda e^t\right]_{t=0} = e^{\lambda(e^0-1)} \cdot \lambda e^0 =$$

   (i) $\boldsymbol{\lambda}$ (ii) $\mathbf{2}\boldsymbol{\lambda}$ (iii) $\mathbf{3}\boldsymbol{\lambda}$ (iv) $\boldsymbol{\lambda} + \mathbf{1}$.

   (b) *Determine $E\left(Y^2\right)$.* First notice

$$E\left(Y^2\right) = \mu_2'.$$

   So

$$\mu_2' = m^{(2)}(0) = \left.\frac{d^2 m(t)}{dt^2}\right]_{t=0} = \left.\frac{d^2 e^{\lambda(e^t-1)}}{dt^2}\right]_{t=0} = \left[e^{\lambda(e^t-1)} \cdot \left(\lambda e^t\right)^2 + e^{\lambda(e^t-1)} \cdot \lambda e^t\right]_{t=0}$$

   which equals $e^{\lambda(e^0-1)} \cdot \left(\lambda e^0\right)^2 + e^{\lambda(e^0-1)} \cdot \lambda e^0 =$ (choose one)
   (i) $\boldsymbol{\lambda}$ (ii) $\boldsymbol{\lambda}^\mathbf{2} + \boldsymbol{\lambda}$ (iii) $\boldsymbol{\lambda} + \mathbf{3}\boldsymbol{\lambda}$ (iv) $\boldsymbol{\lambda}^\mathbf{3} + \boldsymbol{\lambda}$.

(c) *Determine $V(Y)$; that is, show $V(Y) = \lambda$.*

$$V(Y) = E\left(Y^2\right) - E(Y)^2 = \mu_2' - (\mu_1')^2 = (\lambda^2 + \lambda) - \lambda^2 \ =$$

(i) $\boldsymbol{\lambda}$ (ii) $\boldsymbol{\lambda^2 + \lambda}$ (iii) $\boldsymbol{\lambda + 3\lambda}$ (iv) $\boldsymbol{\lambda^3 + \lambda}$.

2. *Binomial. Assume* moment–generating function for binomial is

$$m(t) = \left(pe^t + q\right)^n.$$

(a) *Determine $E(Y)$; that is, show $E(Y) = np$ using $m(t)$.*

$$\mu_1' = m^{(1)}(0) = \left.\frac{d\left(pe^t + q\right)^n}{dt}\right]_{t=0} = \left[n\left(pe^t + q\right)^{n-1} pe^t\right]_{t=0}$$

which equals $n\left(pe^0 + q\right)^{n-1} pe^0 =$ (choose one)
(i) $\boldsymbol{\lambda}$ (ii) $\boldsymbol{np}$ (iii) $\boldsymbol{2np}$ (iv) $\boldsymbol{npq}$.

(b) *Determine $E\left(Y^2\right)$.*

$$\mu_2' = m^{(2)}(0) = \left.\frac{d^2\left(pe^t + q\right)^n}{dt^2}\right]_{t=0} = \left[n(n-1)\left(pe^t + q\right)^{n-1}\left(pe^t\right)^2 + n\left(pe^t + q\right)^{n-1} pe^t\right]_{t=0}$$

which is $n(n-1)\left(pe^0 + q\right)^{n-1}\left(pe^0\right)^2 + n\left(pe^0 + q\right)^{n-1} pe^0 =$ (choose one)
(i) $\boldsymbol{np(n-1)}$ (ii) $\boldsymbol{np^2(n-1)^2 + np}$ (iii) $\boldsymbol{np^2(n-1) + np}$.

(c) *Determine $V(Y)$; that is, show $V(Y) = npq$.*

$$V(Y) = E\left(Y^2\right) - E(Y)^2 = \mu_2' - (\mu_1')^2 = (np^2(n-1) + np) - (np)^2 = np(1-p) \ =$$

(i) $\boldsymbol{n}$ (ii) $\boldsymbol{np}$ (iii) $\boldsymbol{2np}$ (iv) $\boldsymbol{npq}$.

3. *Geometric. Assume* moment–generating function for geometric is

$$m(t) = \frac{pe^t}{1 - qe^t}.$$

(a) *Determine $E(Y)$; that is, show $E(Y) = \frac{1}{p}$ using $m(t)$.*

$$\mu_1' = m^{(1)}(0) = \left.\frac{d}{dt}\left(\frac{pe^t}{1 - qe^t}\right)\right]_{t=0} = \left[\frac{pe^t}{\left(1 - qe^t\right)^2}\right]_{t=0} =$$

(i) $\boldsymbol{\frac{1}{p}}$ (ii) $\boldsymbol{\frac{p}{p^3}}$ (iii) $\boldsymbol{\frac{1}{p}}$ (iv) $\boldsymbol{\frac{q}{p}}$.
Hint: Since $p + q = 1, p = 1 - q$.

(b) *Determine $E\left(Y^2\right)$.*

$$\mu_2' = m^{(2)}(0) = \left.\frac{d^2}{d^2t}\left(\frac{pe^t}{1 - qe^t}\right)\right]_{t=0} = \left[\frac{\left(1 - qe^t\right)^2 pe^t - 2pe^t\left(1 - qe^t\right)\left(-qe^t\right)}{\left(1 - qe^t\right)^4}\right]_{t=0} =$$

(i) $\boldsymbol{\frac{1}{p}}$ (ii) $\boldsymbol{\frac{p}{p^3}}$ (iii) $\boldsymbol{\frac{1+q}{p^2}}$ (iv) $\boldsymbol{\frac{q-1}{p}}$.

(c) *Determine $V(Y)$.*

$$V(Y) = E\left(Y^2\right) - E(Y)^2 = \mu_2' - (\mu_1')^2 = \left(\frac{1+q}{p^2}\right) - \left(\frac{1}{p}\right)^2 =$$

(i) $\boldsymbol{\frac{q}{p^2}}$ (ii) $\boldsymbol{\frac{p}{p^3}}$ (iii) $\boldsymbol{\frac{1+q}{p^2}}$ (iv) $\boldsymbol{\frac{q-1}{p}}$.

4. *Uniqueness of moment–generating functions.*

(a) Moment–generating function $m(t) = e^{1(e^t-1)}$ corresponds to (choose one)
(i) **poisson** (ii) **binomial** (iii) **geometric** (iv) **negative binomial**.
Consequently, $\mu = E(Y) = \lambda =$ (circle one)
(i) **1** (ii) **2** (iii) **3** (iv) **4**,
and $\sigma = \sqrt{\lambda} \approx$ (circle one)
(i) **1.00** (ii) **2.41** (iii) **3.41** (iv) **4.41**,
and so $P(0 \leq Y \leq 3) = P(Y \leq 3) \approx$
(circle one) (i) **0.41** (ii) **0.61** (iii) **0.91** (iv) **0.98**.
2nd DISTR poissoncdf(ENTER 1, 3) ENTER
and so the probability Y is within two standard deviations of the mean is,

$$\begin{aligned} P(|Y-\mu| \leq 2\sigma) &= P(\mu - 2\sigma \leq Y \leq \mu + 2\sigma) \\ &\approx P(1-2(1) \leq Y \leq 1+2(1)) \\ &= P(-2 \leq Y \leq 3) \\ &= P(0 \leq Y \leq 3) \ = \end{aligned}$$

(circle one) (i) **0.41** (ii) **0.61** (iii) **0.91** (iv) **0.98**,
2nd DISTR poissoncdf(ENTER 1, 3) ENTER

(b) Moment–generating function $m(t) = (0.2e^t + 0.8)^5$ corresponds to
(i) **poisson** (ii) **binomial** (iii) **geometric** (iv) **negative binomial**.
Consequently, $\mu = E(Y) = np =$ (circle one)
(i) **1** (ii) **2** (iii) **3** (iv) **4**,
and $\sigma = \sqrt{npq} \approx$ (circle one)
(i) **0.41** (ii) **0.51** (iii) **0.89** (iv) **1.21**,
and so the probability Y is within two standard deviations of the mean is,

$$\begin{aligned} P(|Y-\mu| \leq 2\sigma) &= P(\mu - 2\sigma \leq Y \leq \mu + 2\sigma) \\ &\approx P(1-2(0.89) \leq Y \leq 1+2(0.89)) \\ &= P(-0.78 \leq Y \leq 2.78) \\ &= P(0 \leq Y \leq 2) \ = \end{aligned}$$

(circle one) (i) **0.64** (ii) **0.78** (iii) **0.88** (iv) **0.94**.
2nd DISTR binomcdf(ENTER 5, 0.2, 2) ENTER

(c) Moment–generating function $m(t) = \frac{0.4e^t}{1-0.6e^t}$ corresponds to
(i) **poisson** (ii) **binomial** (iii) **geometric** (iv) **negative binomial**.
Consequently, $\mu = E(Y) = \frac{1}{p} =$ (circle one)
(i) **1.5** (ii) **2.5** (iii) **3.5** (iv) **4.5**,
and $\sigma = \sqrt{\frac{q}{p^2}} \approx$ (circle one)
(i) **1.41** (ii) **1.51** (iii) **1.39** (iv) **1.94**,
and so the probability Y is within two standard deviations of the mean is,

$$\begin{aligned} P(|Y-\mu| \leq 2\sigma) &= P(\mu - 2\sigma \leq Y \leq \mu + 2\sigma) \\ &\approx P(2.5 - 2(1.94) \leq Y \leq 2.5 + 2(1.94)) \\ &= P(-1.38 \leq Y \leq 6.38) \\ &= P(1 \leq Y \leq 6) = \end{aligned}$$

(circle one) (i) **0.64** (ii) **0.78** (iii) **0.87** (iv) **0.95**.
2nd DISTR geometcdf(ENTER 0.4, 6) ENTER

5. *Functions of random variables and associated moment–generating function.*

(a) *Moment–generating function for $m(t) = m(0)$.*
For poisson moment–generating function, $m(t) = e^{\lambda(e^t-1)}$,

$$m(0) = e^{\lambda(e^0-1)} =$$

(choose one) (i) **1** (ii) **2** (iii) **3** (iv) **4**.
For geometric moment–generating function, $m(t) = \frac{pe^t}{1-qe^t}$,

$$m(0) = \frac{pe^0}{1-qe^0} =$$

(choose one) (i) **1** (ii) **2** (iii) **3** (iv) **4**.
For negative binomial moment–generating function, $m(t) = \left(\frac{pe^t}{1-qe^t}\right)^r$,

$$m(0) = \left(\frac{pe^0}{1-qe^0}\right)^r =$$

(choose one) (i) **1** (ii) **2** (iii) **3** (iv) **4**.
In general, since $m(t) = E\left(e^{tY}\right)$,

$$m(0) = E\left(e^{(0)Y}\right) = E(1) =$$

(choose one) (i) **1** (ii) **2** (iii) **3** (iv) **4**.

(b) *Moment–generating function for $W = 5Y$.*
In general, since $m_Y(t) = E\left(e^{tY}\right)$,

$$m_W(t) = E\left(e^{tW}\right) = E\left(e^{t(5Y)}\right) = E\left(e^{(5t)Y}\right) =$$

(choose one) (i) $\boldsymbol{5m_Y(t)}$ (ii) $\boldsymbol{m_Y(5t)}$ (iii) $\boldsymbol{m_W(5t)}$ (iv) $\boldsymbol{m_Y(5t^2)}$.
For poisson moment–generating function, $m_Y(t) = e^{\lambda(e^t-1)}$,

$$m_W(t) = m_Y(5t) =$$

(choose one) (i) $\boldsymbol{e^{\lambda(5e^t-1)}}$ (ii) $\boldsymbol{e^{\lambda(e^{5t}-5)}}$ (iii) $\boldsymbol{e^{\lambda(e^{5t^2}-1)}}$ (iv) $\boldsymbol{e^{\lambda(e^{5t}-1)}}$.
For negative moment–generating function, $m_Y(t) = \left(\frac{pe^t}{1-qe^t}\right)^r$,

$$m_W(t) = m_Y(5t) =$$

(i) $\boldsymbol{\left(\frac{pe^t}{1-qe^t}\right)^r}$ (ii) $\boldsymbol{\left(\frac{5pe^t}{1-qe^{5t}}\right)^r}$ (iii) $\boldsymbol{\left(\frac{pe^{5t}}{1-qe^{5t}}\right)^r}$ (iv) $\boldsymbol{\left(\frac{5pe^{5t}}{1-5qe^{5t}}\right)^r}$.

(c) *Moment–generating function for $W = 5Y + 3$.*
In general, since $m_Y(t) = E\left(e^{tY}\right)$,

$$m_W(t) = E\left(e^{tW}\right) = E\left(e^{t(5Y+3)}\right) = E\left(e^{(5t)Y+3t}\right) = e^{3t}E\left(e^{(5t)Y}\right) =$$

(i) $\boldsymbol{5e^{3t}m_Y(t)}$ (ii) $\boldsymbol{e^{3t}m_Y(5t)}$ (iii) $\boldsymbol{e^{3t}m_W(5t)}$ (iv) $\boldsymbol{e^{3t}m_Y(5t^2)}$.
For geometric moment–generating function, $m_Y(t) = \frac{pe^t}{1-qe^t}$,

$$m_W(t) = e^{3t}m_Y(5t) =$$

(i) $\boldsymbol{e^{3t}\frac{pe^{3t}}{1-qe^t}}$ (ii) $\boldsymbol{e^{5t}\frac{pe^{5t}}{1-qe^t}}$ (iii) $\boldsymbol{e^{3t}\frac{pe^t}{1-5qe^t}}$ (iv) $\boldsymbol{e^{3t}\frac{pe^{5t}}{1-qe^{5t}}}$.
For negative moment–generating function, $m_Y(t) = \left(\frac{pe^t}{1-qe^t}\right)^r$,

$$m_W(t) = e^{3t}m_Y(5t) =$$

(i) $\boldsymbol{\left(\frac{pe^t}{1-qe^t}\right)^r}$ (ii) $\boldsymbol{e^{5t}\left(\frac{5pe^t}{1-qe^{5t}}\right)^r}$ (iii) $\boldsymbol{e^{3t}\left(\frac{pe^{5t}}{1-qe^{5t}}\right)^r}$ (iv) $\boldsymbol{e^{3t}\left(\frac{5pe^{5t}}{1-5qe^{5t}}\right)^r}$.

6. *More on moment–generating functions.*

(a) *Deriving distribution from moment–generating function.* What is the distribution of Y if

$$m(t) = \frac{1}{2}e^t + \frac{1}{3}e^{2t} + \frac{1}{6}e^{2t}?$$

Since

$$m(t) = E\left(e^{tY}\right) = \sum_y e^{ty}p(y) = \frac{1}{2}e^t + \frac{1}{3}e^{2t} + \frac{1}{6}e^{3t}$$

this implies when
(i) $\boldsymbol{Y = 1, p(1) = \frac{1}{2}; Y = 2, p(2) = \frac{1}{3}}$ and when $\boldsymbol{Y = 3, p(3) = \frac{1}{6}}$
(ii) $\boldsymbol{Y = 1, p(1) = \frac{1}{6}; Y = 2, p(2) = \frac{1}{3}}$ and when $\boldsymbol{Y = 3, p(3) = \frac{1}{2}}$
(iii) $\boldsymbol{Y = 1, p(1) = \frac{1}{3}; Y = 2, p(2) = \frac{1}{3}}$ and when $\boldsymbol{Y = 3, p(3) = \frac{1}{3}}$

(b) *Deriving binomial moment–generating function.*

$$\begin{aligned} m(t) = E\left(e^{tY}\right) &= \sum_{0}^{n} e^{ty} p(y) \\ &= \sum_{0}^{n} e^{ty} \binom{n}{y} p^y q^{n-y} \\ &= \sum_{0}^{n} \binom{n}{y} (pe^t)^y q^{n-y} \\ &= \left(pe^t + q\right)^n \end{aligned}$$

(c) *Summary of moment–generating functions.*

DISCRETE	$p(y)$	$m(t)$	μ	σ^2
Binomial	$\binom{n}{y} p^y q^{n-y}$	$(pe^t + q)^n$	np	npq
Poisson	$e^{-\lambda}\frac{\lambda^y}{y!}$	$e^{\lambda(e^t-1)}$	λ	λ
Geometric	$q^{y-1}p$	$\frac{pe^t}{1-qe^t}$	$1/p$	q/p^2
Negative Binomial	$\binom{y-1}{r-1} p^r q^{y-r}$	$\left(\frac{pe^t}{1-qe^t}\right)^r$	r/p	rq/p^2

3.10 Probability–Generating Functions

Not covered.

3.11 Tchebysheff's Theorem

Tchebysheff's Theorem states, for random variable Y with finite μ and σ^2 and for $k > 0$,

$$P(|Y - \mu| < k\sigma) \geq 1 - \frac{1}{k^2} \quad \text{or} \quad P(|Y - \mu| \geq k\sigma) \leq \frac{1}{k^2}.$$

These two (equivalent) inequalities allow us to specify (very loose) lower bounds on probabilities when the distribution is not known.

Exercise 3.11 (Tchebysheff's Theorem)

1. *Tchebysheff's theorem and binomial: lawyer.* A lawyer estimates she wins 40% ($p = 0.4$) of her cases. Assume each trial is independent of one another and, in general, this problem obeys the conditions of a binomial experiment. The lawyer presently represents 10 ($n = 10$) defendants.

(a) The lawyer's probability of winning is given by the binomial distribution

$$p(y) = \binom{n}{y} p^y q^{n-y}, \quad n = 10, p = 0.4, r = 0, 1, \ldots, 10,$$

with corresponding table of

y	0	1	2	3	4	5	6	7	8	9	10
$p(y)$	0.006	0.040	0.121	0.215	0.251	0.201	0.111	0.043	0.011	0.002	0.000

For example, the chance of her winning 4 of 10 cases is (choose one)
(i) **0.121** (ii) **0.215** (iii) **0.251** (iv) **0.351**.

(b) $\mu = np = (10)(0.4) =$ (choose one) (i) **3** (ii) **4** (iii) **5** (iv) **6**.

(c) $\sigma = \sqrt{npq} = \sqrt{(10)(0.4)(0.6)} \approx$ (choose one)
(i) **1.32** (ii) **1.55** (iii) **1.67** (iv) **2.03**.

(d) According to Tchebysheff's theorem, the probability the number of wins is *within* $k = 2$ standard deviations of the mean number of wins *is at least*

$$P(|Y - \mu| < k\sigma) \geq 1 - \frac{1}{k^2} = 1 - \frac{1}{2^2} =$$

(i) **0.75** (ii) **0.85** (iii) **0.95** (iv) **0.98**.
In actual fact, since $\mu = 4$ and $\sigma \approx 1.55$,

$$\begin{aligned} P(|Y - \mu| < k\sigma) &= P(\mu - k\sigma < Y < \mu + k\sigma) \\ &\approx P(4 - 2(1.55) < Y < 4 + 2(1.5)) \\ &= P(0.9 < Y < 7.1) \\ &= P(1 \leq Y \leq 7) \ = \end{aligned}$$

(i) **0.56** (ii) **0.76** (iii) **0.88** (iv) **0.98**.
2nd DISTR binomcdf(ENTER 10, 0.4, 7) ENTER
subtract 2nd DISTR binomcdf(ENTER 10, 0.4,0) ENTER
Tchebysheff's approximation, 0.75, is a (very) low bound on the actual probability, 0.98.

(e) According to Tchebysheff's theorem, the probability the number of wins is *within* $k = 2.5$ standard deviations of the mean number of wins *is at least*

$$P(|Y - \mu| < k\sigma) \geq 1 - \frac{1}{k^2} = 1 - \frac{1}{2.5^2} =$$

(i) **0.75** (ii) **0.84** (iii) **0.95** (iv) **0.98**.
In actual fact, since $\mu = 4$ and $\sigma \approx 1.55$,

$$\begin{aligned} P(|Y - \mu| < k\sigma) &= P(\mu - k\sigma < Y < \mu + k\sigma) \\ &\approx P(4 - 2.5(1.55) < Y < 4 + 2.5(1.5)) \\ &= P(0.125 < Y < 7.875) \\ &= P(1 \leq Y \leq 7) \ = \end{aligned}$$

(i) **0.56** (ii) **0.76** (iii) **0.88** (iv) **0.98**.
Tchebysheff's approximation, 0.84, is a (very) low bound on the actual probability, 0.98.

(f) According to Tchebysheff's theorem, the probability the number of wins is *beyond* $k = 2.5$ standard deviations from the mean number of wins *is at most*

$$P(|Y - \mu| \geq k\sigma) \leq \frac{1}{k^2} = \frac{1}{2.5^2} =$$

(i) **0.12** (ii) **0.16** (iii) **0.25** (iv) **0.34**.
In actual fact, since $\mu = 4$ and $\sigma \approx 1.55$,

$$\begin{aligned} P(|Y - \mu| \geq k\sigma) &= P(Y \leq \mu - k\sigma) + P(Y \geq \mu + k\sigma) \\ &\approx P(Y \leq 4 - 2.5(1.55)) + P(Y \geq 4 + 2.5(1.5)) \\ &= P(Y \leq 0.125) + P(Y \geq 7.875) \\ &= P(Y = 0) + P(8 \leq Y \leq 10) \\ &= 1 - P(1 \leq Y \leq 7) \ = \end{aligned}$$

(i) **0.01** (ii) **0.02** (iii) **0.03** (iv) **0.04**.
Tchebysheff's approximation, 0.16, is a (very) high bound on the actual probability, 0.02.

2. *Tchebysheff's theorem: Ph levels in soil.* Assume the Ph levels in soil samples taken at Sand Dunes Park, Indiana, have a mean and standard deviation of $\mu = 10$ and $\sigma = 3$ respectively.

(a) According to Tchebysheff's theorem, the probability the Ph level is *within* $k = 2$ standard deviations of the mean Ph level *is at least*

$$P(|Y - \mu| < k\sigma) \geq 1 - \frac{1}{k^2} = 1 - \frac{1}{2^2} =$$

(i) **0.75** (ii) **0.85** (iii) **0.95** (iv) **0.98**.
The actual probability cannot be calculated here because the probability distribution of the Ph levels is unknown in this case. Tchebysheff's approximation, 0.75, will be a low bound on whatever is the actual probability.

(b) If 400 soil samples are taken, what number will be *at least within* $k = 2$ standard deviations of the mean Ph level? Since $P(|Y - \mu| < 2\sigma) \geq 0.75$, at least $0.75(400) =$ (choose one) (i) **200** (ii) **300** (iii) **400** (iv) **500**.

(c) According to Tchebysheff's theorem, the probability the Ph level is *beyond* $k = 2.5$ standard deviations from the mean Ph level *is at most*

$$P(|Y - \mu| \geq k\sigma) \leq \frac{1}{k^2} = \frac{1}{2.5^2} =$$

(i) **0.12** (ii) **0.16** (iii) **0.25** (iv) **0.34**.
Tchebysheff's approximation, 0.16, will be a (very) high bound on whatever is the actual probability.

(d) According to Tchebysheff's theorem,

$$\begin{aligned} P(5 < Y < 15) &= P(\mu - k\sigma < Y < \mu + k\sigma) \\ &= P\left(10 - \frac{5}{3}(3) < Y < 10 + \frac{5}{3}(3)\right) \\ &\geq 1 - \frac{1}{k^2} = 1 - \frac{1}{\left(\frac{5}{3}\right)^2} = \end{aligned}$$

(circle one) (i) **0.54** (ii) **0.64** (iii) **0.75** (iv) **0.84**.

(e) If $P(|Y - \mu| \geq k\sigma) \leq 0.35$, then $\frac{1}{k^2} = 0.35$ or $k = \sqrt{\frac{1}{0.35}} \approx$ (choose one)
(i) **1.12** (ii) **1.16** (iii) **1.25** (iv) **1.69**.

3.12 Summary