3.3 Rates of Change

The *average rate of change* of $f(x)$ with respect to x as x changes from a to b is

$$\frac{f(b) - f(a)}{b - a}.$$

The *instantaneous rate of change* of $f(x)$ at $x = a$ is

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{b \to a} \frac{f(b) - f(a)}{b - a},$$

assuming limit exists and where $\frac{f(a+h)-f(a)}{h}$ and $\frac{f(b)-f(a)}{b-a}$ are different versions of the *difference quotient*. These formulas serve as an intermediate step towards understanding the derivative.

Exercise 3.3 (Rates of Change)

1. *Applications: average rate of change.*

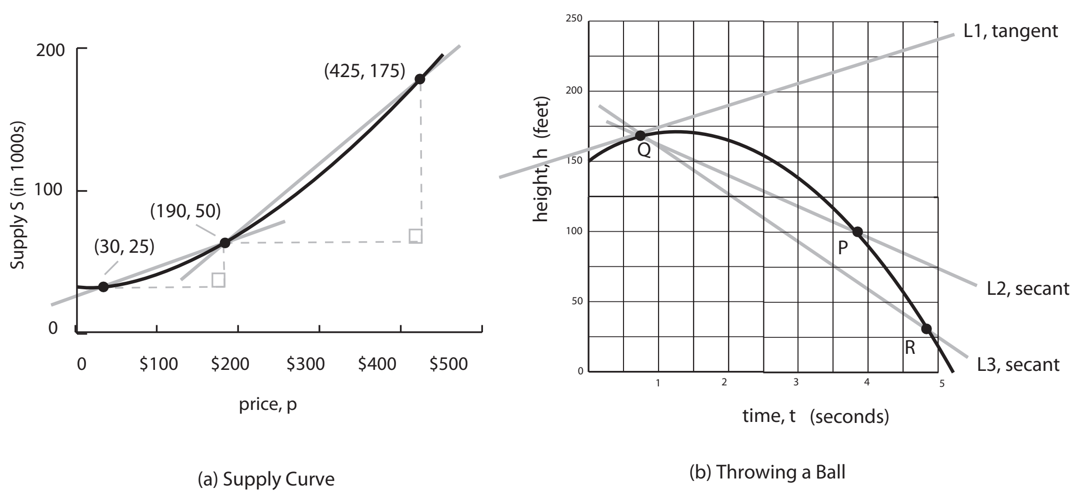

Figure 3.12 (Examples of average rates of change)

(a) *Figure (a). Supply curve and average rates of change.*

i. Table version of points in figure (a) is

A. *table A*

price p	30	190	425
supply 1000s	25,000	50,000	175,000

B. *table B*

price p	25	50	175
supply 1000s	30,000	190,000	425,000

ii. Average change in supply with respect to price from $(a, f(a)) = (\$30, 25,000)$ to $(b, f(b)) = (\$190, 50,000)$ is $\frac{f(b)-f(a)}{b-a} = \frac{50{,}000-25{,}000}{190-30} =$
(i) **0.15625** (ii) **1.5625** (iii) **156.25** units per dollar.

iii. Average change in supply with respect to price from $a = \$190$ to $b = \$425$ is $\frac{f(b)-f(a)}{b-a} = \frac{175{,}000-50{,}000}{425-190} =$
(i) **432.91** (ii) **531.91** (iii) **656.25** units per dollar.

iv. Average change in supply with respect to price from $a = \$30$ to $b = \$425$ is $\frac{f(b)-f(a)}{b-a} = \frac{175{,}000-25{,}000}{425-30} =$
(i) **232.91** (ii) **331.91** (iii) **379.75** units per dollar.

v. The line that passes through, for example, the points ($30, 25,000) and ($190, 50,000) is called a (i) **secant** (ii) **tangent** line.

vi. (i) **True** (ii) **False** In general, slope of secant between two points is called average rate of change.

vii. Average rate of change (i) **changes** (ii) **remains the same** for different points.

(b) *Figure (b). Throwing a ball and average rates of change.*
Average change in height per second (average *velocity*) of a ball thrown ...

i. ...from Q at $(a, f(a)) = (0.75, 165)$ to P at $(b, f(b)) = (3.75, 100)$ is $\frac{f(b)-f(a)}{b-a} = \frac{100-165}{3.75-0.75} \approx$
(i) **−21.67** (ii) **−11.23** (iii) **21.67** feet per second.

ii. ...Q at $(a, f(a)) = (0.75, 165)$ to R at $(b, f(b)) = (4.75, 30)$ is $\frac{f(b)-f(a)}{b-a} = \frac{30-165}{4.75-0.75} \approx$
(i) **−51.67** (ii) **−41.23** (iii) **−33.75** feet per second.

iii. Time $b - a = 3.75 - 0.75 = 3$ seconds associated with Q to P is
(i) **smaller** (ii) **larger**
than $b - a = 4.75 - 0.75 = 4$ seconds associated with Q to R.

iv. As b approaches a closely, average velocity, $\frac{f(b)-f(a)}{b-a}$, approaches slope of *tangent* line (i) **L1** (ii) **L2** (iii) **L3**

(c) *More figure (b). Throwing a ball and average rates of change.*
Suppose graph of function given in Figure (b) is

$$h = 150 + 32t - 12t^2$$

where h is height (in feet) and t is time (in seconds).

i. (i) **True** (ii) **False** A few (time, height) points associated with h are:

time t	1	2	3	4	5
height h	170	166	138	86	10

(Type $150 + 32x - 12x^2$ into $Y_1 =$, then 2nd TBLSET 0 1 Ask Auto, then 2nd TABLE and type 1 2 3 4 5 into X.)

ii. Average velocity, $a = 1$ to $b = 2$ seconds is $\frac{f(b)-f(a)}{b-a} = \frac{166-170}{2-1} =$
(i) **−4** (ii) **−16** (iii) **−32** feet per second.

iii. Average velocity, $a = 1$ to $b = 3$ seconds is $\frac{f(b)-f(a)}{b-a} = \frac{138-170}{3-1} =$
(i) **−4** (ii) **−16** (iii) **−32** feet per second.

iv. If $b = a + h$, then difference quotient becomes

$$\frac{f(b)-f(a)}{b-a} = \frac{f(a+h)-f(a)}{(a+h)-a} = \frac{f(a+h)-f(a)}{h}$$

So, if $h = 1$ and $a = 1$, then $b = a + h =$ (i) **2** (ii) **1.5** (iii) **1.25**
or if $h = \frac{1}{2}$ and $a = 1$, then $b = a + h =$ (i) **2** (ii) **1.5** (iii) **1.25**
or if $h = \frac{1}{4}$ and $a = 1$, then $b = a + h =$ (i) **2** (ii) **1.5** (iii) **1.25**
so, as h approaches 0, b approaches a

(d) *Interest when compounding k Times Per Year:* $A = P\left(1+\frac{r}{k}\right)^{kt}$
Value of \$500 invested with 3% annual interest compounded quarterly is

$$A = P\left(1+\frac{r}{k}\right)^{kt} = 500\left(1+\frac{0.03}{4}\right)^{4t} = 500(1.0075)^{4t}.$$

i. (i) **True** (ii) **False** A few (time, dollar) points associated with A are:

time t	1	2	3	4	5
future value A	515.17	530.80	546.90	563.50	580.59

(Type $500(1.0075)^{4x}$ into Y_1 =, then 2nd TBLSET 0 1 Ask Auto,
then 2nd TABLE and type 1 2 3 4 5 into X.)

ii. Average change in dollars per year, $a = 1$ year to $b = 2$ years is
$\frac{f(b)-f(a)}{b-a} = \frac{f(2)-f(1)}{2-1} = \frac{f(2)-f(1)}{1} = \frac{530.80-515.17}{1} =$
(i) **15.63** (ii) **16.11** (iii) **19.57** dollars per year.

iii. *Again, but using the other difference quotient.*
Average change in dollars per year, $a = 1$ year to $h = 1$ year later is
$\frac{f(a+h)-f(a)}{h} = \frac{f(1+1)-f(1)}{1} = \frac{f(2)-f(1)}{1} = \frac{530.80-515.17}{1} =$
(i) **15.63** (ii) **16.11** (iii) **19.57** dollars per year.

iv. *Using the other difference quotient again.*
Average change in dollars per year, $a = 1$ year to $h = 3$ years later is
$\frac{f(a+h)-f(a)}{h} = \frac{f(1+3)-f(1)}{3} = \frac{f(4)-f(1)}{3} = \frac{563.50-515.17}{3} =$
(i) **15.63** (ii) **16.11** (iii) **19.57** dollars per year.

2. *Applications: instantaneous rate of change.*

(a) *Throwing a ball.* Suppose graph of function given in figure is

$$y = 150 + 32x - 12x^2$$

where y is height (in feet) and x is time (in seconds).

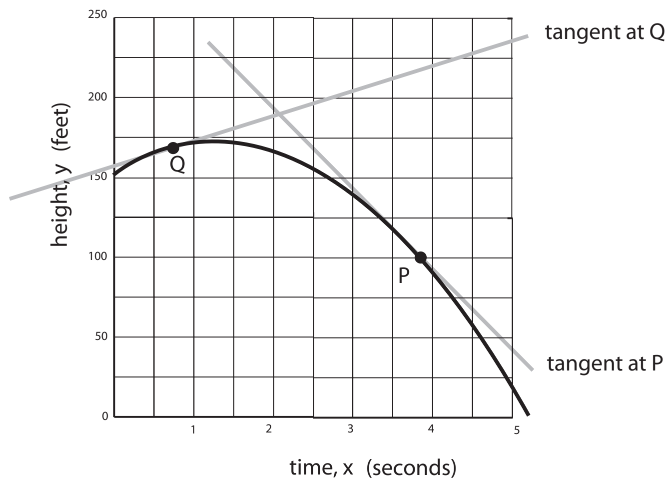

Figure 3.13 (Throwing a ball and instantaneous rates of change)

i. *Instantaneous change in feet per second* (instantaneous velocity) at Q at $(a, f(a)) = (0.75, 165)$, since $a = 0.75$, is

$$\begin{aligned} f(a+h) &= f(0.75+h) \\ &= 150 + 32(0.75+h) - 12(0.75+h)^2 \\ &= 150 + 24 + 32h - 12(0.5625 + 1.5h + h^2) \\ &= 150 + 24 + 32h - 6.75 - 18h - 12h^2 \\ &= 167.25 + 14h - 12h^2 \end{aligned}$$

and

$$f(a) = f(0.75) = 150 + 32(0.75) - 12(0.75)^2 =$$

(i) **167.25** (ii) **190.75** (iii) **200.75** feet
so

$$\begin{aligned} \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h\to 0} \frac{f(0.75+h) - f(0.75)}{h} \\ &= \lim_{h\to 0} \frac{(167.25 + 14h - 12h^2) - 167.25}{h} \\ &= \lim_{h\to 0} \frac{14h - 12h^2}{h} \\ &= \lim_{h\to 0} (14 - 12h) = \end{aligned}$$

(i) **14** (ii) **15** (iii) **16** feet per second.

ii. *Instantaneous velocity* at P, where $a = 3.75$, is

$$\begin{aligned} f(a+h) &= f(3.75+h) \\ &= 150 + 32(3.75+h) - 12(3.75+h)^2 \\ &= 150 + 120 + 32h - 12(14.0625 + 7.5h + h^2) \end{aligned}$$

$$\begin{aligned} &= 150 + 120 + 32h - 168.75 - 90h - 12h^2 \\ &= 101.25 - 58h - 12h^2 \end{aligned}$$

and

$$f(a) = f(3.75) = 150 + 32(3.75) - 12(3.75)^2 =$$

(i) **121.25** (ii) **131.25** (iii) **101.25**
so

$$\begin{aligned} \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h\to 0} \frac{f(3.75+h) - f(3.75)}{h} \\ &= \lim_{h\to 0} \frac{(101.25 - 58h - 12h^2) - 101.25}{h} \\ &= \lim_{h\to 0} \frac{-58h - 12h^2}{h} \\ &= \lim_{h\to 0} (-58 - 12h) = \end{aligned}$$

(i) **−60** (ii) **−62** (iii) **−58** feet per second.

iii. *A second method: instantaneous velocity* at P, where $a = 3.75$, is

$$f(b) = 150 + 32b - 12b^2$$

and

$$f(a) = f(3.75) = 150 + 32(3.75) - 12(3.75)^2 =$$

(i) **121.25** (ii) **101.25** (iii) **131.25**
so

$$\begin{aligned} \lim_{b\to a} \frac{f(b) - f(a)}{b - a} &= \lim_{b\to 3.75} \frac{f(b) - f(3.75)}{b - 3.75} \\ &= \lim_{b\to 3.75} \frac{(150 + 32b - 12b^2) - 101.25}{b - 3.75} \\ &= \lim_{b\to 3.75} \frac{48.75 + 32b - 12b^2}{b - 3.75} \\ &= \lim_{b\to 3.75} \frac{(-12b - 13)(b - 3.75)}{b - 3.75} \quad \text{(tricky factorization)} \\ &= \lim_{b\to 3.75} (-12b - 13) = \end{aligned}$$

(i) **−58** (ii) **−60** (iii) **−62** feet per second.

iv. *A third method: compounding interest.* Value of \$500 invested with 3% annual interest compounded quarterly is

$$y = 500(1.0075)^{4x}.$$

So, at year $a = 3$

$$f(a+h) = f(3+h) = 500(1.0075)^{4(3+h)} = 500(1.0075)^{12+4h}$$

and

$$f(a) = f(3) = 500(1.0075)^{4(3)} = 500(1.0075)^{12}$$

so

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{f(3+h)-f(3)}{h} = \lim_{h\to 0} \frac{500(1.0075)^{12+4h} - 500(1.0075)^{12}}{h}$$

so, because it is difficult to compute limit directly, use table method:

h	0.1	0.01	0.001	0.0001	0.00001
$\frac{500(1.0075)^{12+4h} - 500(1.0075)^{12}}{h}$	16.37	16.348	16.34613	16.34591	16.34589

(Type $\frac{500(1.0075)^{12+4X} - 500(1.0075)^{12}}{X}$ into Y_1 =, then 2nd TBLSET 0 1 Ask Auto, then 2nd TABLE and type 0.1 0.01 0.001 0.0001 0.00001 into X.)

so instantaneous rate is approximately
(i) **16.35** (ii) **16.11** (iii) **15.57** dollars per year.

3. *Last example of average and instantaneous rates of change for* $y = x^2$.

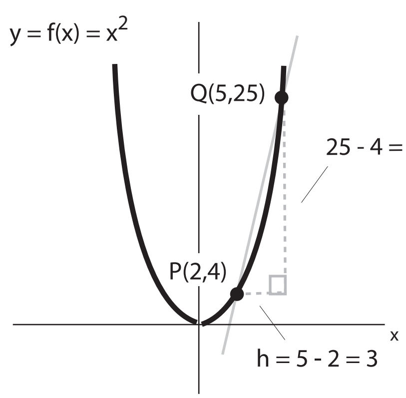

Figure 3.14 (Average and instantaneous rate of change, $y = x^2$.)

(a) *Average* rate of change from $a = 2$ to $b = 5$ is
$\frac{f(b)-f(a)}{b-a} = \frac{f(5)-f(2)}{5-2} = \frac{25-4}{5-2} =$ (i) **3** (ii) **4** (iii) **5**

(b) *Average* rate of change where $a = 2$ and $h = 3$ is
$\frac{f(a+h)-f(a)}{h} = \frac{f(2+3)-f(2)}{3} = \frac{25-4}{3} =$ (i) **5** (ii) **6** (iii) **7**.

(c) *Instantaneous* rate of change for $y = x^2$ at P, where $a = 2$, is
since $f(a+h) = f(2+h) = (2+h)^2 = 4 + 2h + h^2$

and $f(a) = f(2) = (2)^2 =$ (i) **3** (ii) **4** (iii) **5**
so

$$\begin{aligned}\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} &= \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} \\ &= \lim_{h\to 0}\frac{(4+2h+h^2)-4}{h} \\ &= \lim_{h\to 0}\frac{2h-h^2}{h} \\ &= \lim_{h\to 0}(2-h) =\end{aligned}$$

(i) **2** (ii) **3** (iii) **4**

(d) *Instantaneous* rate of change for $y = x^2$ at Q, where $a = 5$, is since $f(a+h) = f(5+h) = (5+h)^2 = 25 + 10h + h^2$
and $f(a) = f(5) = (5)^2 =$ (i) **23** (ii) **24** (iii) **25**
so

$$\begin{aligned}\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} &= \lim_{h\to 0}\frac{f(5+h)-f(5)}{h} \\ &= \lim_{h\to 0}\frac{(25+10h+h^2)-25}{h} \\ &= \lim_{h\to 0}\frac{10h-h^2}{h} \\ &= \lim_{h\to 0}(10-h) =\end{aligned}$$

(i) **8** (ii) **9** (iii) **10**

3.4 Definition of the Derivative

Two equivalent definitions of the *derivative* of $f(x)$ at x are

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{b\to x}\frac{f(b)-f(x)}{b-x}$$

if the limit exists, and where *function* $f'(x)$ is read "f-prime of x". Function $f'(x)$ is both the instantaneous rate of change of $y = f(x)$ at x and also the *slope* of the *tangent line* at x. The tangent line to graph of $y = f(x)$ at point $(x_1, f(x_1))$ is

$$y - f(x_1) = f'(x_1)(x - x_1).$$

If $f'(x)$ exists, $f(x)$ is *differentiable* and the steps which produce $f'(x)$ is called *differentiation*. A function f is differentiable if *all* of the following conditions are satisfied,

- f is *continuous*,
- f is *smooth*,
- f does *not* have a vertical tangent line,

and *non*differentiable is any *one* of the following conditions are satisfied,

- f is *dis*continuous (there are "jumps", "holes", asymptotes in the function) because a slope cannot be where there is nothing (points b, c and e in Figure);
- f is *not* smooth (there is "sharp corner" in the function) because there are different conflicting slopes (but not *one* slope) at this point (point d in Figure);
- f has a vertical tangent line because the "run" is zero in the rise/run formula for the slope which would make the slope undefined at this point (point a).

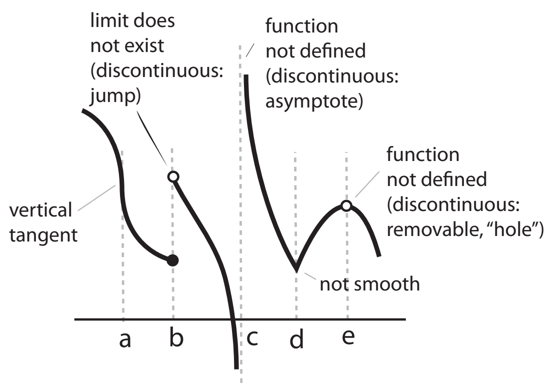

Figure 3.15 (Different types of nondifferentiability)

Exercise 1.4 (Definition of the Derivative)

1. *Derivatives of* $y = x^2$ *at* $x = -1, 0, 3$.
   since

$$f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$$

and

$$f(x+h) - f(x) = \left(x^2 + 2xh + h^2\right) - x^2 =$$

(i) $\boldsymbol{2x + h^2}$ (ii) $\boldsymbol{2xh + h}$ (iii) $\boldsymbol{2xh + h^2}$
and

$$\frac{f(x+h) - f(x)}{h} = \frac{2xh + h^2}{h} =$$

(i) $\boldsymbol{2x + h}$ (ii) $\boldsymbol{2xh^2 + h}$ (iii) $\boldsymbol{2xh + h}$
and

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (2x + h) =$$

(i) **2** (ii) $\boldsymbol{2x}$ (iii) $\boldsymbol{h}$

then derivative $f'(-1) = 2(-1) =$ (i) $\boldsymbol{-2}$ (ii) $\boldsymbol{-3}$ (iii) $\boldsymbol{-4}$
and $f'(0) = 2(0) =$ (i) **0** (ii) **1** (iii) **2**
and $f'(3) = 2(3) =$ (i) **7** (ii) **8** (iii) **6**

2. *Derivatives of $y = x^3$ at $x = -1, 0, 3$.*
since

$$f(x+h) = (x+h)^3 = (x+h)(x+h)(x+h) = x^3 + 3x^2h + 3xh^2 + h^3$$

and

$$f(x+h) - f(x) = \left(x^3 + 3x^2h + 3xh^2 + h^3\right) - x^3 =$$

(i) $\boldsymbol{3x^2 + 3xh^2 + h^3}$ (ii) $\boldsymbol{3x^2h + 3xh^2 + h^3}$ (iii) $\boldsymbol{3xh + 3xh^2 + h^3}$
and

$$\frac{f(x+h) - f(x)}{h} = \frac{3x^2h + 3xh^2 + h^3}{h} =$$

(i) $\boldsymbol{3x + 3x + h^2}$ (ii) $\boldsymbol{3x^2 + 3xh + h^2}$ (iii) $\boldsymbol{3x^2 + 3xh + h}$
and

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \left(3x^2 + 3xh + h^2\right) =$$

(i) $\boldsymbol{3x}$ (ii) $\boldsymbol{3x^2}$ (iii) $\boldsymbol{3xh}$

then derivative $f'(-1) = 3(-1)^2 =$ (i) **3** (ii) **4** (iii) $\boldsymbol{-3}$
and $f'(0) = 3(0)^2 =$ (i) **0** (ii) **1** (iii) **2**
and $f'(3) = 3(3)^2 =$ (i) **27** (ii) **28** (iii) **29**

3. *Derivatives, tangent lines of $y = -3x^2$ at $x = -1, 0, 3$.*
since

$$f(x+h) = -3(x+h)^2 = -3(x^2 + 2xh + h^2) = -3x^2 - 6xh - 3h^2$$

and

$$f(x+h) - f(x) = \left(-3x^2 - 6xh - 3h^2\right) - \left(-3x^2\right) =$$

(i) $\boldsymbol{6xh - 3h^2}$ (ii) $\boldsymbol{-6xh + 3h^2}$ (iii) $\boldsymbol{-6xh - 3h^2}$
and

$$\frac{f(x+h) - f(x)}{h} = \frac{-6xh - 3h^2}{h} =$$

(i) $\boldsymbol{-6xh - 3h}$ (ii) $\boldsymbol{-6x + 3h}$ (iii) $\boldsymbol{-6x - 3h}$
and

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (-6x - 3h) =$$

(i) $\boldsymbol{6x}$ (ii) $\boldsymbol{-3h}$ (iii) $\boldsymbol{-6x}$

then derivative $f'(-1) = -6(-1) =$ (i) **7** (ii) **6** (iii) **8**
and $y = f(x) = f(-1) = -3(-1)^2 =$ (i) $\boldsymbol{-3}$ (ii) $\boldsymbol{-4}$ (iii) $\boldsymbol{-5}$
and so tangent line at $x_1 = -1$ is

$$\begin{aligned} y - f(x_1) &= f'(x_1)(x - x_1) \\ y - f(-1) &= f'(-1)(x - (-1)) \\ y - (-3) &= 6(x - (-1)) \end{aligned}$$

or (i) $\boldsymbol{y + 3 = 6(x + 1)}$ (ii) $\boldsymbol{y = 6x + 3}$ (iii) $\boldsymbol{y = -6x - 3}$

and $f'(0) = -6(0) =$ (i) **1** (ii) **2** (iii) **0**
and $y = f(x_1) = f(-1) = -3(0)^2 =$ (i) $\boldsymbol{-3}$ (ii) $\boldsymbol{-2}$ (iii) **0**
and so tangent line at $x_1 = 0$ is

$$\begin{aligned} y - f(x_1) &= f'(x_1)(x - x_1) \\ y - f(0) &= f'(0)(x - (-1)) \\ y - (0) &= 0(x - (0)) \end{aligned}$$

or (i) $\boldsymbol{y = 0}$ (ii) $\boldsymbol{y = 6x + 3}$ (iii) $\boldsymbol{y = -6x - 3}$

and $f'(3) = -6(3) =$ (i) $\boldsymbol{-19}$ (ii) $\boldsymbol{-18}$ (iii) $\boldsymbol{-20}$
and $y = f(x_1) = f(-1) = -3(3)^2 =$ (i) $\boldsymbol{-28}$ (ii) $\boldsymbol{-29}$ (iii) $\boldsymbol{-27}$
and so tangent line at $x_1 = 3$ is

$$\begin{aligned} y - f(x_1) &= f'(x_1)(x - x_1) \\ y - f(3) &= f'(3)(x - (3)) \\ y - (-27) &= -18(x - 3) \end{aligned}$$

or (i) $\boldsymbol{y + 27 = -18(x - 3)}$ (ii) $\boldsymbol{y = -18x + 27}$ (iii) $\boldsymbol{y = -6x - 3}$

4. *Derivatives, tangent lines of* $y = \frac{4}{x}$ *at* $x = -1, 0, 3$.
   since
$$f(x+h) = \frac{4}{x+h}$$
and
$$f(x+h) - f(x) = \frac{4}{x+h} - \frac{4}{x} = \frac{4x}{(x+h)x} - \frac{4(x+h)}{x(x+h)} = \frac{4x - 4(x+h)}{(x+h)x} =$$
(i) $\boldsymbol{\frac{-4}{x^2+xh}}$ (ii) $\boldsymbol{\frac{-4h}{x^2+h}}$ (iii) $\boldsymbol{\frac{-4h}{x^2+xh}}$

and
$$\frac{f(x+h)-f(x)}{h} = \frac{\frac{-4h}{x^2+xh}}{h} \times \frac{\frac{1}{h}}{\frac{1}{h}} = \frac{-4h}{(x^2+xh)h} =$$
(i) $\boldsymbol{\frac{-4}{x+xh}}$ (ii) $\boldsymbol{\frac{-4}{x^2+xh}}$ (iii) $\boldsymbol{\frac{-4}{x^2+h}}$

and
$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0}\left(\frac{-4}{x^2+xh}\right) =$$
(i) $\boldsymbol{\frac{-4}{x}}$ (ii) $\boldsymbol{\frac{-4}{x^2}}$ (iii) $\boldsymbol{\frac{-1}{4x^2}}$

then derivative $f'(-1) = \frac{-4}{(-1)^2} =$ (i) $\mathbf{-4}$ (ii) $\mathbf{-5}$ (iii) $\mathbf{-6}$
and $y = f(x_1) = f(-1) = \frac{4}{-1} =$ (i) $\mathbf{-5}$ (ii) $\mathbf{-4}$ (iii) $\mathbf{-6}$
and so tangent line at $x_1 = -1$ is
$$\begin{aligned} y - f(x_1) &= f'(x_1)(x - x_1) \\ y - f(-1) &= f'(-1)(x-(-1)) \\ y - (-4) &= -4(x - (-1)) \end{aligned}$$
or (i) $\boldsymbol{y + 4 = -4(x+1)}$ (ii) $\boldsymbol{y = -4x - 8}$ (iii) $\boldsymbol{y = -6x - 3}$

and $f'(0) = \frac{-4}{(0)^2} =$ (i) **does not exist** (ii) $\mathbf{-5}$ (iii) $\mathbf{-6}$
and $y = f(x_1) = f(0) = \frac{4}{0} =$ (i) **does not exist** (ii) **5** (iii) **6**
and so tangent line at $x_1 = 0$ (i) $\boldsymbol{y = -4x}$ (ii) **does not exist**

and $f'(3) = \frac{-4}{(3)^2} =$ (i) $\mathbf{-\frac{4}{9}}$ (ii) $\mathbf{-4}$ (iii) $\mathbf{-9}$
and $y = f(x_1) = f(3) = \frac{4}{3} =$ (i) **4** (ii) **3** (iii) $\mathbf{\frac{4}{3}}$
and so tangent line at $x_1 = 3$ is
$$\begin{aligned} y - f(x_1) &= f'(x_1)(x - x_1) \\ y - f(3) &= f'(3)(x - (3)) \\ y - \left(\frac{4}{3}\right) &= -\frac{4}{9}(x-3) \end{aligned}$$

or (i) $\boldsymbol{y + 27 = -18(x - 3)}$ (ii) $\boldsymbol{y = -6x - 3}$ (iii) $\boldsymbol{y = -\frac{4}{9}x + \frac{8}{3}}$

5. *Checking for differentiability: discontinuity (hole).*

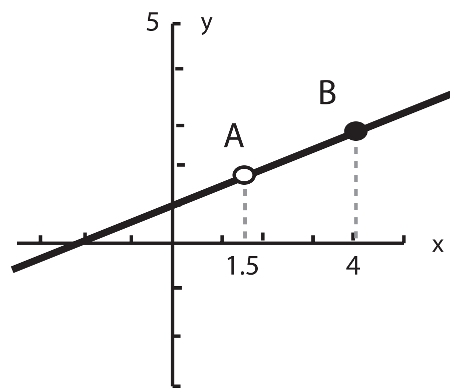

Figure 3.16 (Differentiability of $f(x) = \frac{(2x-3)(0.5x+1)}{2x-3}$)

(Type $\frac{(2x-3)(0.5x+1)}{2x-3}$ into Y_1 =, WINDOW -5 5 1 -5 5 1, then GRAPH.)

(a) Function f at point A (i) **is** (ii) **is not** differentiable because
   i. **all conditions are satisfied for differentiability here.**
   ii. **it is discontinuous (jump, hole, asymptote) here.**
   iii. **it is not smooth (has a sharp corner) here,**
   iv. **it has a vertical ("infinite slope") tangent line here.**

(b) Function f at point B (i) **is** (ii) **is not** differentiable because
   i. **all conditions are satisfied for differentiability here,**
   ii. **it is discontinuous (jump, hole, asymptote) here.**
   iii. **it is not smooth (has a sharp corner) here.**
   iv. **it has a vertical ("infinite slope") tangent line here.**

   and $f'(4) =$ (i) **0.5** (ii) **1.0** (iii) **1.5**
   (2nd CALC dy/dx, 4 ENTER, gives $dy/dx = 0.5$.)

6. *Checking for differentiability: not smooth (sharp corner).*

$$f(x) = |x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x \le 0 \end{cases}$$

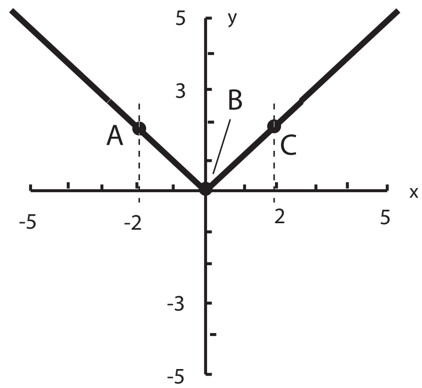

Figure 3.17 (Differentiability of $f(x) = |x|$)

(Type $(-X)(X < 0) + (0)(X = 0) + (X)(X > 0)$ into $Y_2 =$ (not Y_1 =!), WINDOW -5 5 1 -5 5 1.)

(a) Function f at point A (i) **is** (ii) **is not** differentiable because

i. **all conditions are satisfied for differentiability here,**
ii. **it is discontinuous (jump, hole, asymptote) here.**
iii. **it is not smooth (has a sharp corner) here.**
iv. **it has a vertical ("infinite slope") tangent line here.**

and $f'(-2) =$ (i) $\mathbf{-1}$ (ii) $\mathbf{0}$ (iii) $\mathbf{1}$

(2nd CALC dy/dx, -2 ENTER, gives $dy/dx = -1$.)

(b) Function f at point B (i) **is** (ii) **is not** differentiable because

i. **all conditions are satisfied for differentiability here,**
ii. **it is discontinuous (jump, hole, asymptote) here.**
iii. **it is not smooth (has a sharp corner) here,**
iv. **it has a vertical ("infinite slope") tangent line here.**

because, at $x = 0$, left and right limits of f' do not equal one another:

$$\begin{aligned}\lim_{h\to 0^-} f' &= \lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^-} \frac{|x+h|-|x|}{h} = \lim_{h\to 0^-} \frac{|0+h|-|0|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1 \\ &\neq \lim_{h\to 0^+} f' = \lim_{h\to 0^+} \frac{|x+h|-|x|}{h} = \lim_{h\to 0^+} \frac{|0+h|-|0|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1\end{aligned}$$

(c) Function f at point C (i) **is** (ii) **is not** differentiable because

i. **all conditions are satisfied for differentiability here,**
ii. **it is discontinuous (jump, hole, asymptote) here.**
iii. **it is not smooth (has a sharp corner) here.**
iv. **it has a vertical ("infinite slope") tangent line here.**

and $f'(2) =$ (i) $\mathbf{-1}$ (ii) $\mathbf{0}$ (iii) $\mathbf{1}$

(2nd CALC dy/dx, 2 ENTER, gives $dy/dx = 1$.)

(d) (i) **True** (ii) **False**
Function $f(x)$ is differentiable everywhere *except* at $x = 0$.

7. *Checking for differentiability: not smooth (sharp corner).*

$$f(x) = \begin{cases} x & \text{if } x < 1 \\ x^2 & \text{if } x \geq 1 \end{cases}$$

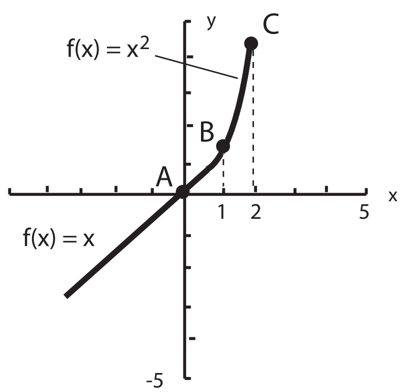

Figure 3.18 (Differentiability of $f(x)$)

(Type $(X)(X < 1) + (X^2)(X \geq 1)$ into Y_3 =, then WINDOW -5 5 1 -5 5 1.)

(a) Function f at point A (i) **is** (ii) **is not** differentiable because
   i. **all conditions are satisfied for differentiability here,**
   ii. **it is discontinuous (jump, hole, asymptote) here.**
   iii. **it is not smooth (has a sharp corner) here.**
   iv. **it has a vertical ("infinite slope") tangent line here.**

   and $f'(0) =$ (i) $\mathbf{-1}$ (ii) $\mathbf{0}$ (iii) $\mathbf{1}$
   (2nd CALC dy/dx, 0 ENTER, gives $dy/dx = 1$.)

(b) Function f at point B (i) **is** (ii) **is not** differentiable because
   i. **all conditions are satisfied for differentiability here.**
   ii. **it is discontinuous (jump, hole, asymptote) here.**
   iii. **it is not smooth (has a sharp corner) here,**
   iv. **it has a vertical ("infinite slope") tangent line here.**

   because, at $x = 1$, left and right limits of f' do not equal one another:

$$\begin{aligned}\lim_{h \to 1^-} f' &= \lim_{h \to 1^-} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 1^-} \frac{(x+h) - x}{h} = \lim_{h \to 1^-} \frac{(1+h) - 1}{h} = \lim_{h \to 1^-} \frac{h}{h} = 1 \\ &\neq \lim_{h \to 1^+} f' = \lim_{h \to 1^-} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 1^-} \frac{(1+h)^2 - 1}{h} = \lim_{h \to 1^-} \frac{2h + h^2}{h} = 2\end{aligned}$$

(c) Function f at point C (i) **is** (ii) **is not** differentiable because

i. **all conditions are satisfied for differentiability here,**
ii. **it is discontinuous (jump, hole, asymptote) here.**
iii. **it is not smooth (has a sharp corner) here.**
iv. **it has a vertical ("infinite slope") tangent line here.**

and $f'(2) =$ (i) **0.54** (ii) **1** (iii) **4**
(2nd CALC dy/dx, 2 ENTER, gives $dy/dx \approx 4$.)

(d) (i) **True** (ii) **False** Function $f(x)$ is differentiable everywhere *except* at $x = 0$.

8. *Checking for differentiability: vertical tangent, discontinuity (jump).*

$$f(x) = \begin{cases} x^{1/3} & \text{if } -1 < x < 1.5 \\ -2x^2 + 2x - 2 & \text{if } 1.5 \le x, \end{cases}$$

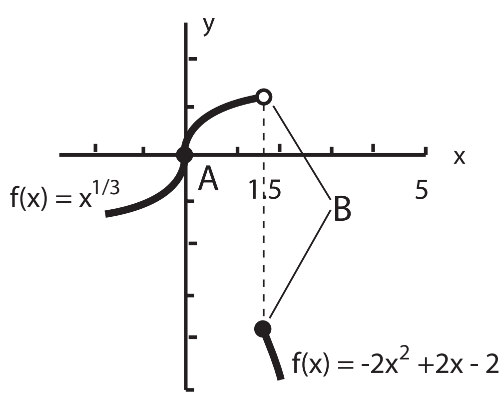

Figure 3.19 (Differentiability of $f(x)$)

Type $(x^{1/3})(x < 1.5) + (-2x^2 + 2x - 2)(x \geq 1.5)$ in $Y_4 =$ (use the dotted line), WINDOW -5 5 1 -5 5 1.

(a) Function f at point A (i) **is** (ii) **is not** differentiable because

i. **all conditions are satisfied for differentiability here.**
ii. **it is discontinuous (jump, hole, asymptote) here.**
iii. **it is not smooth (has a sharp corner) here.**
iv. **it has a vertical ("infinite slope") tangent line here,**

because, as $h \to 0$, limit of f' approaches infinity:
left limit, by table.

$h \to$	-0.1	-0.01	-0.001	-0.0001	-0.00001
$f'(x) = \frac{(0+h)^{1/3} - 0^{1/3}}{h} = \frac{h^{1/3}}{h} \to$	4.6	21.5	100	464.2	2154.4

(Type 2nd TBLSET 1 1 Ask Auto, then 2nd TABLE -0.1 ENTER -0.01 ENTER and so on.)
So $\lim_{x \to 0^-} f'(x) =$ (i) ∞ (ii) **0** (iii) $-\infty$.
right limit, by table.

2154.4	464.2	100	21.5	4.6	$\leftarrow x$
0.00001	0.0001	0.001	0.01	0.1	$\leftarrow f(x) = \frac{h^{1/3}}{h}$

(Type 2nd TBLSET 1 1 Ask Auto, then 2nd TABLE 0.1 ENTER 0.01 ENTER and so on.)
So $\lim_{x \to 0^+} f'(x) =$ (i) ∞ (ii) **0** (iii) $-\infty$.
In other words, $\lim_{x \to 0} f'(x) =$ (i) ∞ (ii) **0** (iii) $-\infty$.
(Notice calculator does *not* work: 2nd CALC dy/dx, 0 ENTER, gives $dy/dx = 100!$.)

(b) Function f at point B (i) **is** (ii) **is not** differentiable because

i. **all conditions are satisfied for differentiability here,**
ii. **it is discontinuous (jump, hole, asymptote) here,**
iii. **it is not smooth (has a sharp corner) here.**
iv. **it has a vertical ("infinite slope") tangent line here.**

because, at $x = 1.5$, left and right limits of f do not equal one another:
$\lim_{x \to 1.5^-} x^{1/3} = (1.5)^{1/3} \approx$ (i) **1.14** (ii) **1.13** (iii) **1.15**.
$\lim_{x \to 1.5^+} (-2x^2 + 2x - 2) =$ (i) **−3.5** (ii) **−4** (iii) **−5**.

9. *Checking for differentiability; discontinuity (asymptote).*

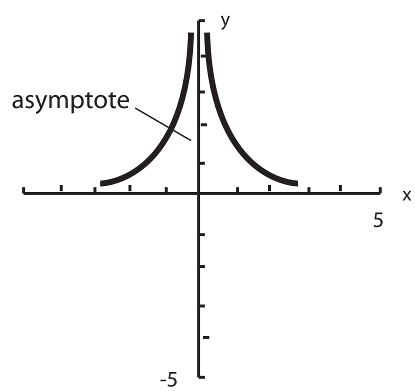

Figure 3.20 (Differentiability of $f(x) = \frac{1}{x^2}$)

Type $(x^{1/3})(x < 1.5) + (-2x^2 + 2x - 2)(x \geq 1.5)$ in Y_5 = (use the dotted line), WINDOW -5 5 1 -5 5 1.
Function f at $x = 0$ (i) **is** (ii) **is not** differentiable because

(a) **all conditions are satisfied for differentiability here,**
(b) **it is discontinuous (jump, hole, asymptote) here,**
(c) **it is not smooth (has a sharp corner) here.**
(d) **it has a vertical ("infinite slope") tangent line here.**

because, at $x = 0$, f does not exist (there is an vertical asymptote):
$\lim_{x\to 0} \frac{1}{x^2}$ = (i) $\mathbf{\infty}$ (ii) $\mathbf{-\infty}$ (iii) **0**.

3.5 Graphical Differentiation

Some things to remember when drawing derivatives of functions *graphical*: identify points of function where derivatives

- zero ("peaks and valleys" of a function); for example, at B and D in Figure,
- negative/positive (downward/upwarding sloping parts of a function); for example, positive derivative in region between points A and B, negative in region between points B and D in Figure,
- large (slope of function is either vertical or near-vertical) and also use previous positive/negative information to designate large positive/negative derivative; for example, large negative tangent at C gives minimum derivative in Figure

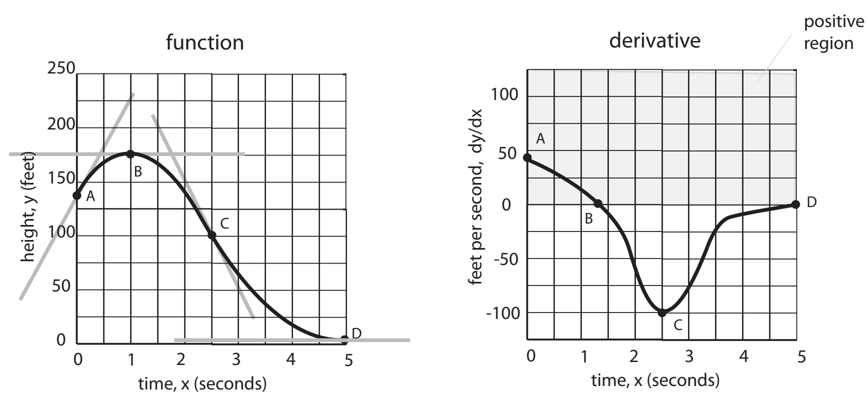

Figure 3.21 (Drawing graph of a derivative from graph of a function)

Exercise 3.5 (Graphical Differentiation)

1. *Graphing derivative function for* $f(x) = \frac{(2x-3)(0.5x+1)}{2x-3} = 0.5x + 1, x \neq 1.5.$

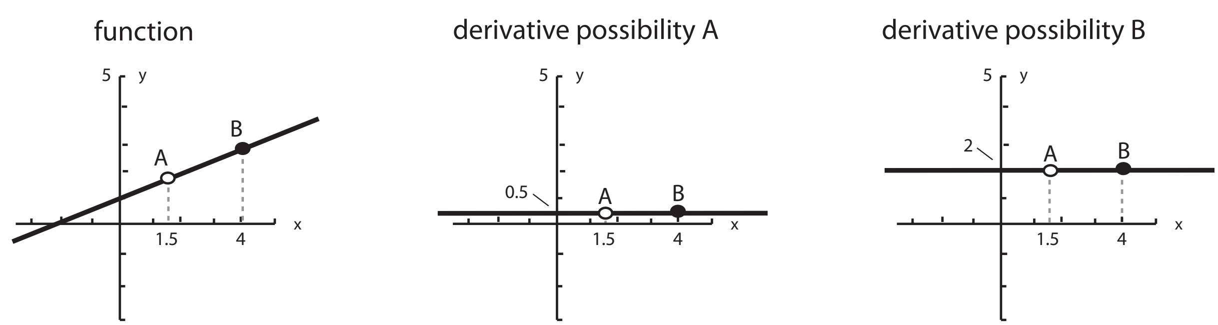

Figure 3.22 (Graphs of $f(x) = \frac{(2x-3)(0.5x+1)}{2x-3}$ and possible derivatives $f'(x)$)

Derivative of function is possibility (i) **A** (ii) **B**

Hint: What is slope of f? Does this slope (derivative) ever change; is it always constant?

2. *Graphing derivative function for* $f(x) = |x|$.

$$f(x) = |x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

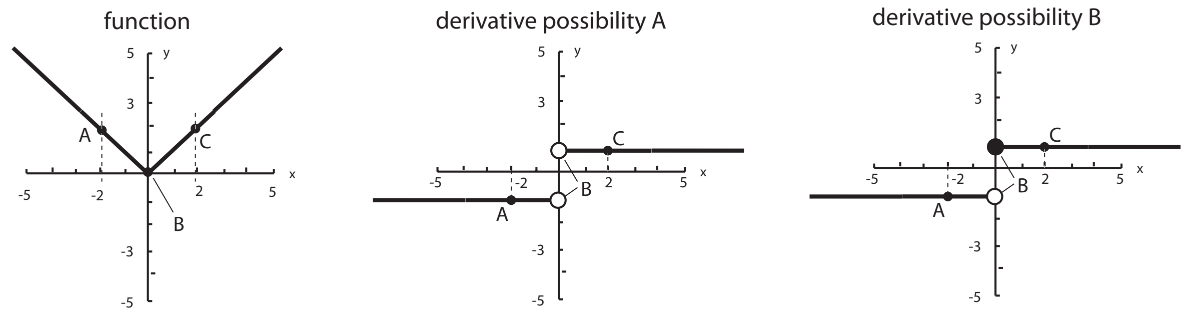

Figure 3.23 (Graphs of $f(x) = |x|$ and possible derivatives $f'(x)$)

Derivative of function is possibility (i) **A** (ii) **B**

Hint: What is (constant) derivative (slope) of f when $x < 1.5$, when $x > 1.5$? Can derivative exist at $x = 1.5$?

3. *Graphing derivative function for* $f(x)$.

$$f(x) = \begin{cases} x & \text{if } x < 1 \\ x^2 & \text{if } x \geq 1 \end{cases}$$

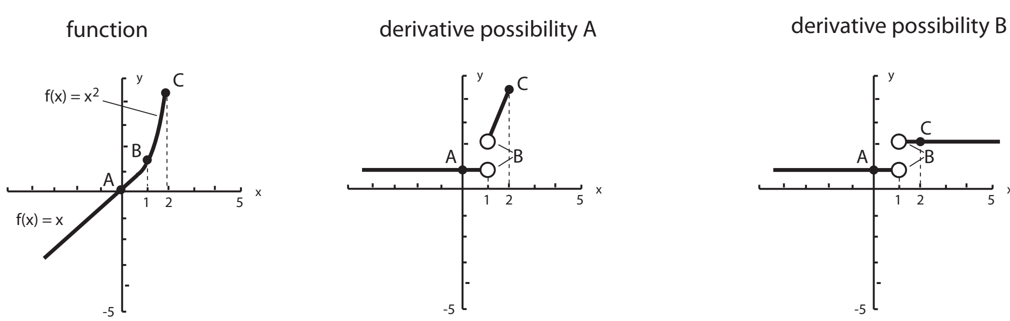

Figure 3.24 (Graphs of $f(x)$ and possible derivatives $f'(x)$)

(Type $(X)(X < 1) + (X^2)(X \geq 1)$ into Y_3 =, WINDOW -5 5 1 -5 5 1.)

Derivative of function is possibility (i) **A** (ii) **B**

Calculator: MATH nDeriv(ENTER X ENTER (VARS Y-VARS Function) Y_3 ENTER X ENTER into Y_6 =.

4. *Graphing derivative function for* $f(x)$.

$$f(x) = \begin{cases} x^{1/3} & \text{if } -1 < x < 1.5 \\ -2x^2 + 2x - 2 & \text{if } 1.5 \leq x, \end{cases}$$

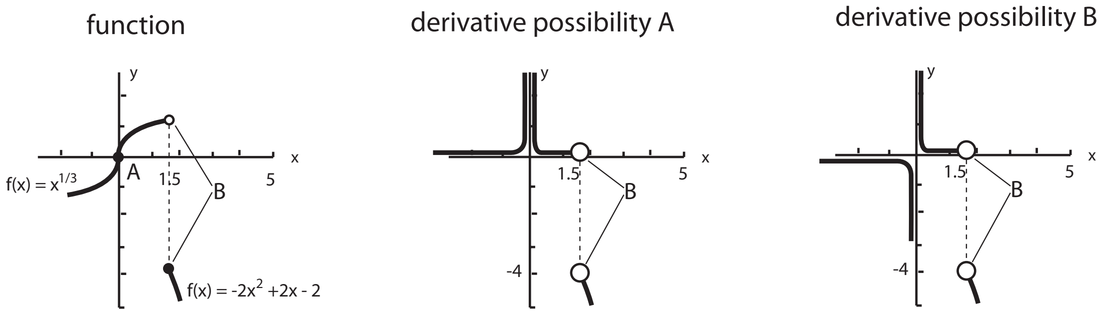

Figure 3.25 (Graphs of $f(x)$ and possible derivatives $f'(x)$)

(Type $(X^{1/3})(X < 1.5) + (-2X^2 + 2X - 2)(X \geq 1.5)$ into Y_4 =, WINDOW -5 5 1 -5 5 1.)

Derivative of function is possibility (i) **A** (ii) **B**

Calculator: MATH nDeriv(ENTER X ENTER (VARS Y-VARS Function) Y_4 ENTER X ENTER into Y_6 =.

5. *Graphing derivative function for* $f(x) = \frac{1}{x^2}$.

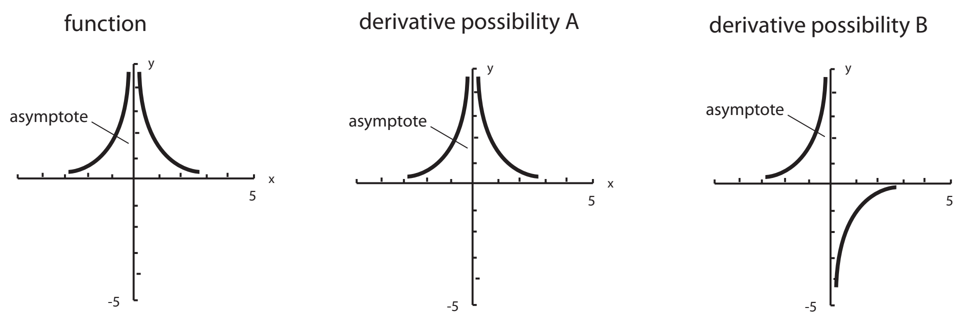

Figure 3.26 (Graphs of $f(x)$ and possible derivatives $f'(x)$)

(Type $1/X^2$ into Y_5 =, WINDOW -5 5 1 -5 5 1.)

Derivative of function is possibility (i) **A** (ii) **B**

Calculator: MATH nDeriv(ENTER X ENTER (VARS Y-VARS Function) Y_5 ENTER X ENTER into Y_6 =.