Chapter 9

Multivariable Calculus

We will look at the calculus of functions with several variables.

9.1 Functions of Several Variables

Equation $z = f(x, y)$ is a *function of two variables* if there is a unique z from each ordered pair (x, y) whose graph is an example of a *surface*. Pair (x, y) are *independent* variables; z is a *dependent* variable; set of all (x, y) is domain; set of all $z = f(x, y)$ is *range*. These definitions extend naturally to more than two dimensions. Graph

$$ax + by + cz = d$$

is a *plane* if a, b, c are all not 0. *Traces* take "*coordinate axes* plane slices" through surfaces; *level curves* are "slices" of planes parallel to coordinate axes" through surfaces. There are three types of traces for the $z = f(x, y)$ surface: *xy-trace*, *yx-trace* and *xz-trace*. Four common equations are

- *paraboloid*: $z = x^2 + y^2$
- *ellipsoid*: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
- *hyperbolic paraboloid*: $z = x^2 - y^2$
- *hyperboloid of two sheets*: $-x^2 - y^2 + z^2 = 1$

Although an *ellipsoid* is *not* a function, since there is more than one z for different (x, y), it is possible in this case to treat the ellipsoid as a *level surface* for a *function* of *three* variables,

$$w(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$$

where $w = 1$.

Exercise 9.1 (Functions of Several Variables)

1. *Multivariate function evaluation*

   (a) $f(x,y) = 3x + 4y$
   For $x = 3$, $y = 5$, $f(x,y) = f(3,5) = 3(3) + 4(5) =$ (i) **28** (ii) **29** (iii) **30**
   Multivariate function calculations not available on calculator, so awkward to deal with:
   $\text{Y}_1 = X$, $\text{Y}_2 = X$, 2nd QUIT, 3 VARS Y-VARS ENTER $\text{Y}_1(3)$ + 4 VARS Y-VARS ENTER $\text{Y}_2(5)$
   OR, easiest to just type $3 \times 3 + 4 \times 5 = 29$

   Different Notation.
   For $x = 3$, $y = 5$, $z = 3x + 4y =$ (i) **28** (ii) **29** (iii) **30**
   For $x = -3$, $y = 17$, $z = 3x + 4y =$ (i) **28** (ii) **29** (iii) **59**
   For $x = -3.2$, $y = -7.5$, $z = 3x + 4y =$ (i) **−28.3** (ii) **−39.6** (iii) **−59**

   (b) $f(x,y) = 3x^2 + 4y$
   $f(3,5) =$ (i) **38** (ii) **44** (iii) **47**

   (c) $f(x,y) = \sqrt{3x^2 + 4y}$
   $f(3,5) =$ (i) **3.86** (ii) **6.86** (iii) **7.32**

   (d) $f(x,y,z) = 3x^2 + 4y + 3z$.
   $f(3,5,-8) =$ (i) **22** (ii) **23** (iii) **24**

   (e) $f(x,y,z) = 3x^2 + \ln y + 3z$.
   $f(3,e^2,-8) =$ (i) **2** (ii) **4** (iii) **5**

   (f) $f(x,y,z) = 3x^2(\ln y)z$.
   $f(3,e^2,-8) =$ (i) **−245** (ii) **−432** (iii) **−1296**

   (g) $f(a,b,c) = 3a^2(\ln b)c$.
   $f(3,e^2,-8) =$ (i) **−245** (ii) **−432** (iii) **−1296**

   (h) $f(u,v,w) = 3$.
   $f(3,e^2,-8) =$ (i) **3** (ii) **−456** (iii) **−1296**

   (i) $f(x_1,x_2,x_3,x_4) = 3x_1^{x_2} + \frac{x_3}{3x_4^2}$.
   $f(3,2,8,5) =$ (i) **26.23** (ii) **27.11** (iii) **28.03**

   (j) $f(x_1,x_2,x_3,x_4) = 3x_1^{x_2} + 5$.
   $f(3,2,8,5) =$ (i) **26** (ii) **29** (iii) **32**

(k) Let $f(x,y) = 3x^2 + 2y^2$

$$\begin{aligned}\frac{f(x+h,y)-f(x,y)}{h} &= \frac{(3(x+h)^2+2y^2)-(3x^2+2y^2)}{h} \\ &= \frac{(3(x^2+2xh+h^2)+2y^2)-3x^2-2y^2}{h} \\ &= \frac{3x^2+6xh+3h^2+2y^2-3x^2-2y^2}{h} =\end{aligned}$$

(i) $\boldsymbol{6x+2h}$ (ii) $\boldsymbol{6x+3h}$ (iii) $\boldsymbol{6xh+3h^2}$

2. *Social Science Application: Teaching*
   A teacher's rating, f, is given by

$$f(n,p,a,t) = 3\frac{a}{n} + \sqrt{t}p^2$$

   where n is number of students, p is teacher preparedness, a is student attendance and t is teacher–student interaction.

   So, $f(30,5,0.85,5) =$ (i) **36.23** (ii) **40.05** (iii) **55.99**

3. *Biology Application: Virus*
   A virus's infection rate, f, is given by

$$f(L,p,R,r,v) = \left|\frac{p}{4Lv}\left(R-r^2\right)\right|$$

   where the L is length of incubation period, p is blood pressure, R is radius of virus, r is time between infections, and v is viscosity.

   So, $f(10,120,0.001,3,12) =$ (i) **2.25** (ii) **3.05** (iii) **8.03**

4. *Linear equations geometrically: planes in three–dimensional space.*

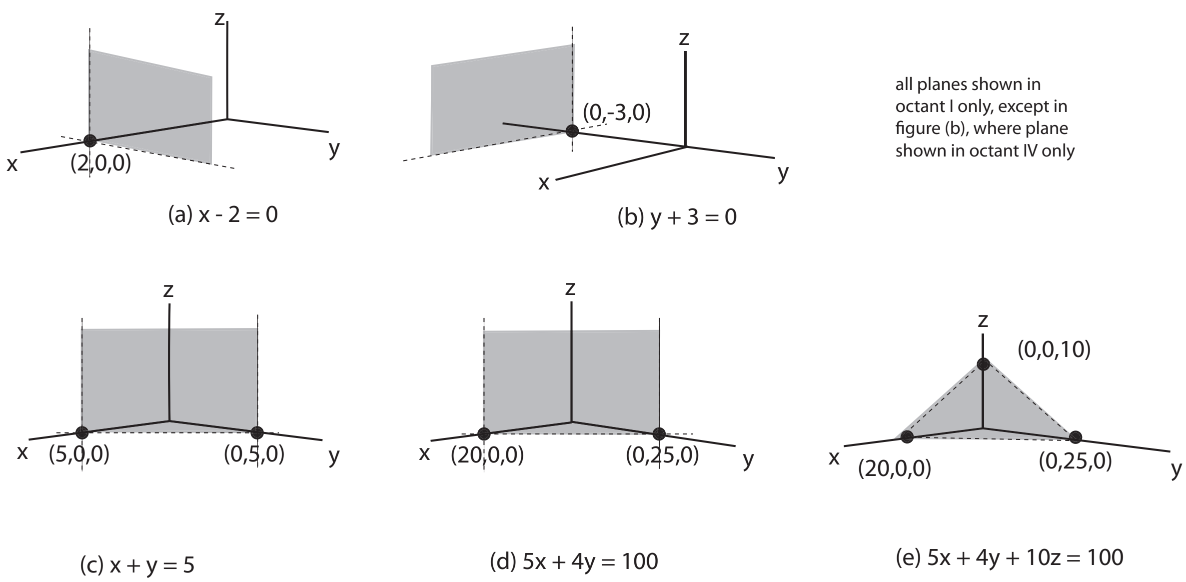

Figure 9.1 (Planes in three–dimensional space)

(a) *Figure (a).*
Equation $x = 2$ (i) **point** (ii) **line** (iii) **plane**, parallel z–y plane.
Equation $x = 2$ is equivalent to equation $x - 2 = 0$.
Plane $x - 2 = 0$ has x–intercept $x = 2$ but no y–intercept or z–intercept.

(b) *Figure (b).*
Equation $y + 3 = 0$ (i) **point** (ii) **line** (iii) **plane** parallel to z–x plane.
Plane $y + 3 = 0$ has y–intercept $y = -3$ but no x–intercept or z–intercept.

(c) *Figure (c).*
Equation $x + y = 5$ (i) **point** (ii) **line** (iii) **plane** parallel to z–axis
x–intercept (i) $\boldsymbol{x = 1}$ (ii) $\boldsymbol{x = 3}$ (iii) $\boldsymbol{x = 5}$ (Hint: What is x when $y = 0$?)
y–intercept (i) $\boldsymbol{y = 1}$ (ii) $\boldsymbol{y = 3}$ (iii) $\boldsymbol{y = 5}$ (Hint: What is y when $x = 0$?)

(d) *Figure (d).*
Equation $5x + 4y = 100$ describes **point** / **line** / **plane** parallel to z–axis
x–intercept (i) $\boldsymbol{x = 20}$ (ii) $\boldsymbol{x = 25}$ (iii) $\boldsymbol{x = 30}$
y–intercept (i) $\boldsymbol{y = 20}$ (ii) $\boldsymbol{y = 25}$ (iii) $\boldsymbol{y = 30}$

(e) *Figure (e)*
Equation $5x + 4y + 10z = 100$ describes a (i) **point** (ii) **line** (iii) **plane**
x–intercept (i) $\boldsymbol{x = 20}$ (ii) $\boldsymbol{x = 25}$ (iii) $\boldsymbol{x = 30}$ (Hint: Set $y = 0$ and $z = 0$.)
y–intercept (i) $\boldsymbol{y = 20}$ (ii) $\boldsymbol{y = 25}$ (iii) $\boldsymbol{y = 30}$ (Hint: Set $x = 0$ and $z = 0$.)
z–intercept (i) $\boldsymbol{z = 5}$ (ii) $\boldsymbol{z = 10}$ (iii) $\boldsymbol{z = 30}$ (Hint: Set $x = 0$ and $y = 0$.)

5. *yz-traces and level curves of* $3x + 2y + 6z = 6$.

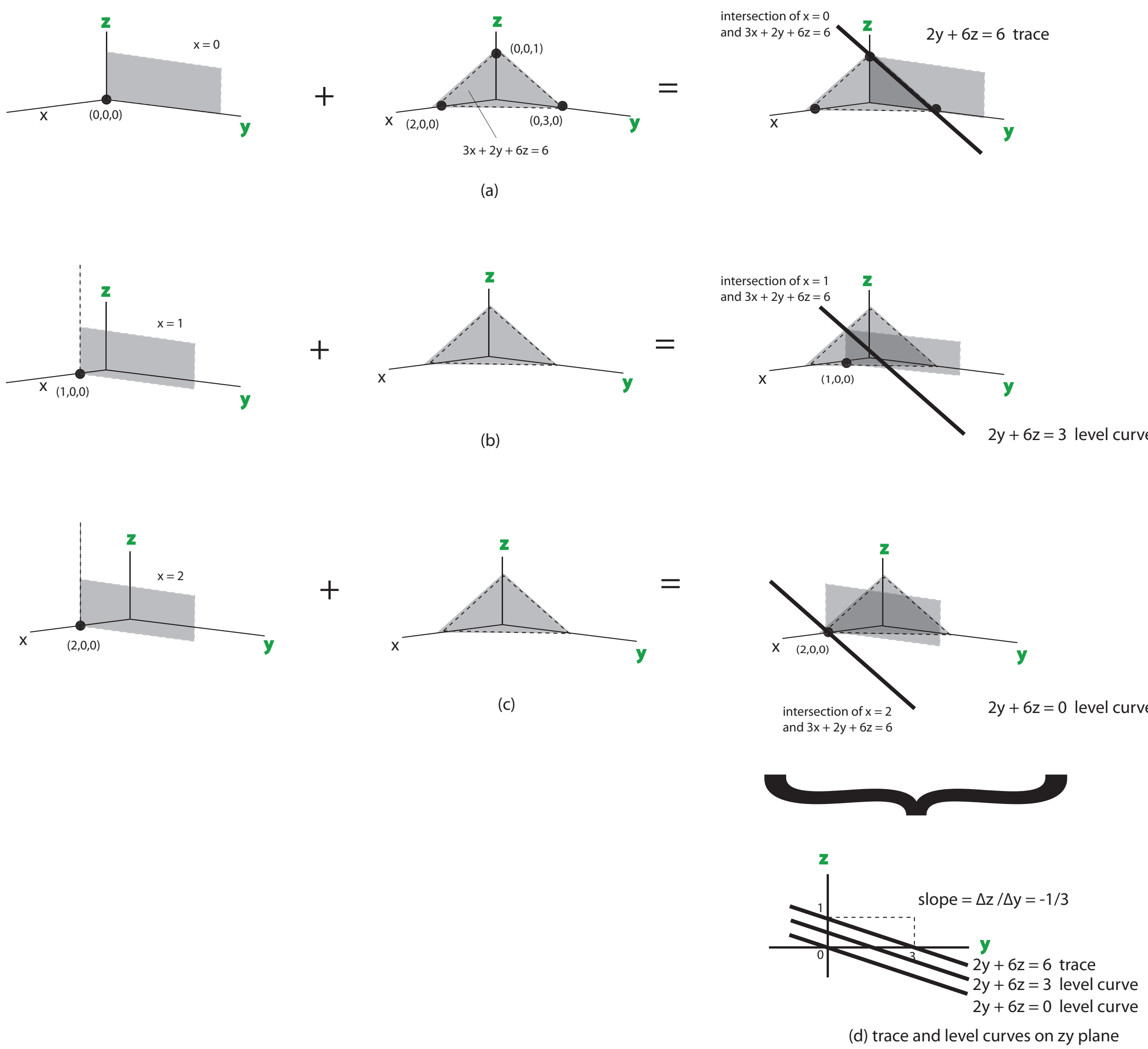

Figure 9.2 (yz-traces and level curves of $3x + 2y + 6z = 6$)

(a) *Figure (a)*
Plane $x = 0$ (yz-plane) intersects plane $3x + 2y + 6z = 6$ at line
(i) $\mathbf{2y + 6z = 6}$ (ii) $\mathbf{2y + 6z = 3}$ (iii) $\mathbf{2y + 6z = 0}$
since $x = 0$, $3x + 2y + 6z = 6$ becomes $3(0) + 2y + 6z = 6$ or $2y + 6z = 6$
The intersecting line an example of a yz-trace.

(b) *Figure (b)*
Plane $x = 1$ intersects plane $3x + 2y + 6z = 6$ at line

(i) $\mathbf{2y + 6z = 6}$ (ii) $\mathbf{2y + 6z = 3}$ (iii) $\mathbf{2y + 6z = 0}$

since $x = 1$, $3x + 2y + 6z = 6$ becomes $3(1) + 2y + 6z = 6$ or $2y + 6z = 3$

This intersecting line is another example of a yz-level curve.

(c) *Figure (c)*

Plane $x = 2$ intersects plane $3x + 2y + 6z = 6$ at line

(i) $\mathbf{2y + 6z = 6}$ (ii) $\mathbf{2y + 6z = 3}$ (iii) $\mathbf{2y + 6z = 0}$

since $x = 2$, $3x + 2y + 6z = 6$ becomes $3(2) + 2y + 6z = 6$ or $2y + 6z = 0$

This third intersecting line is yet another example of a yz-curve.

(d) *Figure (d)*

The yz-trace and two yz-level curves, all have the same slope:

(i) $-\frac{1}{3}$ (ii) $\frac{1}{3}$ (iii) $-\frac{2}{3}$

This slope is an example of a *partial derivative with respect to y*, explained in greater detail later.

6. *xz-traces, xy-traces and level curves of* $3x + 2y + 6z = 6$.

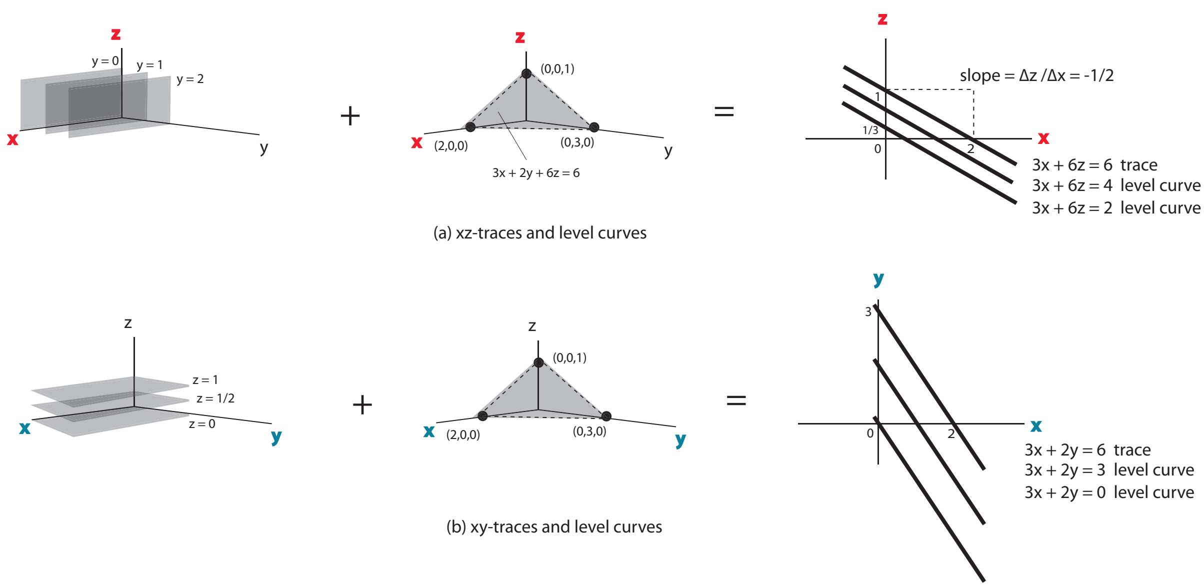

Figure 9.3 (xz-traces, xy-traces and level curves of $3x + 2y + 6z = 6$)

(a) *Figure (a)*

All the xz-trace and two xz-level curves al have the same slope:

(i) $-\frac{1}{3}$ (ii) $\frac{1}{2}$ (iii) $-\frac{1}{2}$

This slope is an example of a *partial derivative with respect to x*, explained in greater detail later.

(b) *Figure (b)*

The xy-trace and two xy-level curves are drawn on the

(i) xy**-plane** (ii) xz**-plane**

7. *xz-traces and xy-traces of other functions.*

   (a) $z = 3x + 4y$

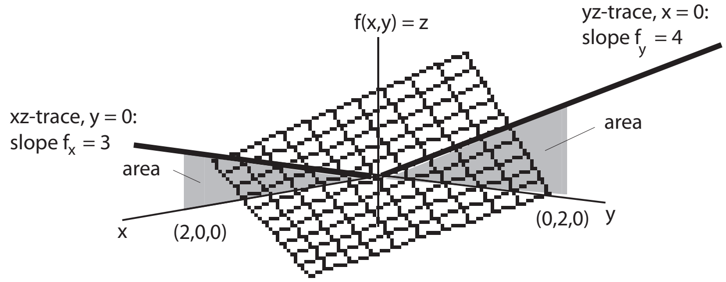

Figure 9.4 ($z = 3x + 4y$)

Slope of plane $z = 3x + 4y$ in y–axis direction (yz-trace, $x = 0$), since $f(0, y) = 3(0) + 4y = 4y$, is

$\frac{\partial f}{\partial y} = f_y(x, y) = 4 \cdot 1y^{1-1} =$ (i) **0** (ii) **3** (iii) **4**

Slope of plane $z = 3x + 4y$ in x–axis direction (xz-trace, $y = 0$), since $f(x, 0) = 3x + 4(0) = 3x$, is

$\frac{\partial f}{\partial x} = f_x(x, y) = 3 \cdot 1x^{1-1} =$ (i) **0** (ii) **3** (iii) **4**

Function $z = 3x + 4y$ increases *faster* in

(i) **positive y-axis** (ii) **positive x-axis** direction

slope of plane in y–axis direction, $f_y = 4$, is *steeper* than slope of plane in x–axis direction, $f_x = 3$

Maximum value of z of plane $z = 3x + 4y$ is

(i) **15** (ii) **20** (iii) **does not exist, is ∞**

as $x \to \infty$ and $y \to \infty$, $z = 3x + 4y \to \infty$

Minimum value of z of plane $z = 3x + 4y$ is

(i) **15** (ii) **20** (iii) **does not exist, is $-\infty$**

as $x \to -\infty$ and $y \to -\infty$, $z = 3x + 4y \to -\infty$

If $z = 3x + 4y$ is *constrained* by $-5 \leq x \leq 5$, $-5 \leq y \leq 5$, then maximum value of z of plane $z = 3x + 4y$ is

$f(5, 5) = 3(5) + 4(5) =$ (i) **−35** (ii) **20** (iii) **35**

If $z = 3x + 4y$ is constrained by $-5 \leq x \leq 5$, $-5 \leq y \leq 5$, then *minimum* value of z of plane $z = 3x + 4y$ is

$f(-5,-5) = 3(-5) + 4(-5) =$ (i) $\mathbf{-35}$ (ii) **20** (iii) **35**

Area under $z = 3x + 4y$ along y-axis direction (yz-trace, $x = 0$), between $y = 0$ and $y = 2$ is

$$\begin{aligned} \int_0^2 f(0,y)\,dy &= \int_0^2 (3(0) + 4y)\,dy \\ &= \left[4 \cdot \frac{1}{1+1} y^{1+1}\right]_{y=0}^{y=2} \\ &= \left[2y^2\right]_{y=0}^{y=2} = \end{aligned}$$

(i) **6** (ii) **8** (iii) **10**

also notice $\frac{1}{2} \times$ base $\times$ height $= \frac{1}{2} \times 2 \times f(0,2) = \frac{1}{2} \times 2 \times (3(0) + 4(2)) = 8$

Area under $z = 3x + 4y$ along x-axis direction (xz-trace, $y = 0$), between $x = 0$ and $x = 2$ is

$$\begin{aligned} \int_0^2 f(x,0)\,dx &= \int_0^2 (3x + 4(0))\,dx \\ &= \left[3 \cdot \frac{1}{1+1} x^{1+1}\right]_{x=0}^{x=2} \\ &= \left[\frac{3}{2} x^2\right]_{x=0}^{x=2} = \end{aligned}$$

(i) **6** (ii) **8** (iii) **10**

also notice $\frac{1}{2} \times$ base $\times$ height $= \frac{1}{2} \times 2 \times f(x,0) = \frac{1}{2} \times 2 \times (3(2) + 4(0)) = 6$

Volume under plane $z = 3x + 4y$, between $x = 0$ and $x = 2$ and between $y = 0$ and $y = 2$ is $\int_0^2 \int_0^2 f(x,y)\,dx\,dy =$

(i) $\boldsymbol{\int_0^2 \int (3x + 4y)\,dx\,dy}$
(ii) $\boldsymbol{\int \int_0^2 (3x + 4y)\,dx\,dy}$
(iii) $\boldsymbol{\int_0^2 \int_0^2 (3x + 4y)\,dx\,dy}$

we find out later $\int_0^2 \int_0^2 (3x + 4y\,dx\,dy = 28$

(b) $z = x^2$

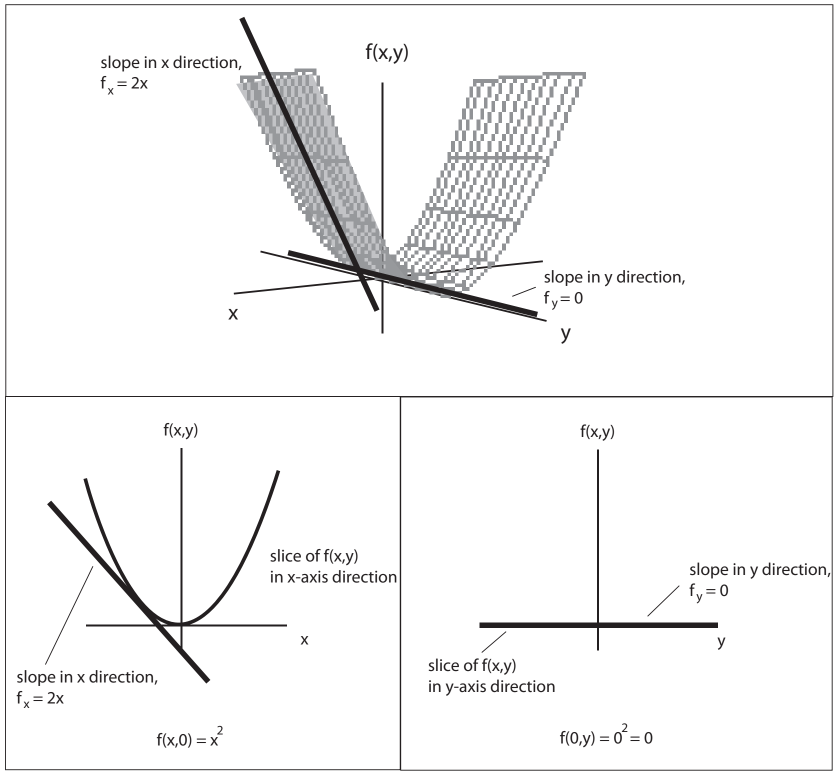

Figure 9.5 ($z = x^2$)

Slope of it surface $z = x^2$ of along y-axis direction (yz-trace, $x = 0$), since $f(0, y) = (0)^2 = 0$, is

$\frac{\partial f}{\partial y} = f_y(x, y) =$ (i) **0** (ii) **2** (iii) $\boldsymbol{2x}$
slope is *horizontal*, unchanging

Slope of surface $z = x^2$ along x-axis direction (xz-trace, $y = 0$), since $f(x, 0) = x^2$, is

$\frac{\partial f}{\partial x} = f_x(x, y) = 2x^{2-1} =$ (i) **0** (ii) **2** (iii) $\boldsymbol{2x}$
slope varies along this trace

Maximum of surface $z = x^2$ is

(i) **0** (ii) **20** (iii) **does not exist, is** ∞
as $x \to \infty$, $z = x^2 \to \infty$

Minimum of surface $z = x^2$ is

(i) **0** (ii) **20** (iii) **does not exist, is** $-\infty$
notice both slopes are zero, $f_y = 0$ and $f_x = 2(0) = 0$, at $(x, y) = (0, 0)$

If $z = x^2$ is constrained by $-5 \leq x \leq 5$, $-5 \leq y \leq 5$, then maximum value of z of surface $z = x^2$ is

$f(5,5) = (5)^2 =$ (i) **0** (ii) **−25** (iii) **25**

If $z = x^2$ is constrained by $-5 \leq x \leq 5$, $-5 \leq y \leq 5$, then minimum value of z of surface $z = x^2$ is, once again,

$f(0,0) = (0)^2 =$ (i) **0** (ii) **−25** (iii) **25**

(c) *Function* $f(x,y) = x^2 + y^2$

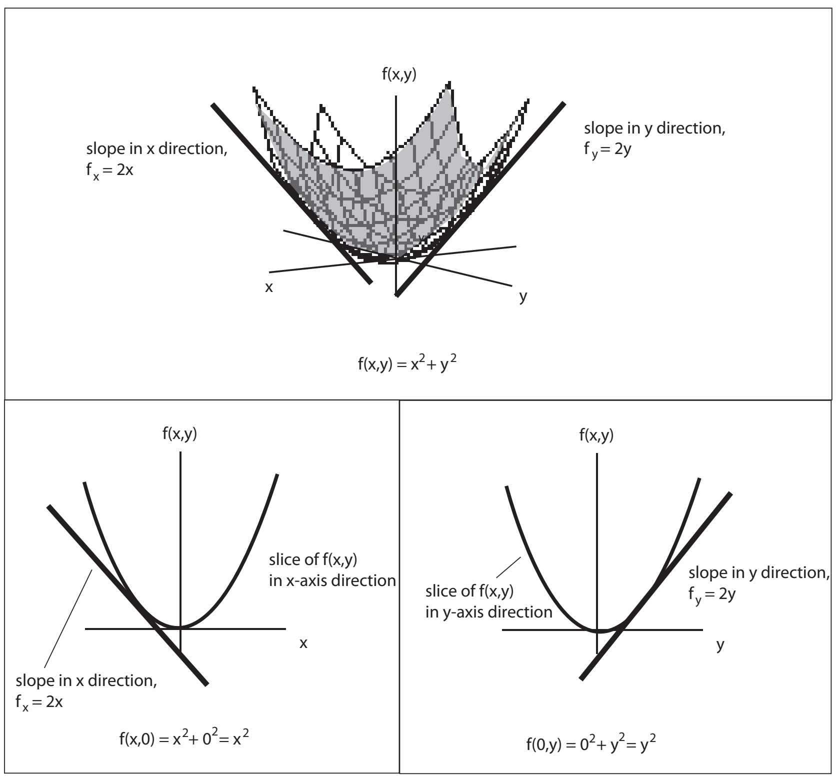

Figure 9.6 ($z = x^2 + y^2$)

Slope of it surface $z = x^2 + y^2$ of along y-axis direction (yz-trace, $x = 0$), since $f(0,y) = (0)^2 + y^2 = y^2$, is

$\frac{\partial f}{\partial y} = f_y(x,y) = 2y^{2-1} =$ (i) **0** (ii) $\boldsymbol{2y}$ (iii) $\boldsymbol{2x}$

Slope of surface $z = x^2 + y^2$ along x-axis direction (xz-trace, $y = 0$), since $f(x,0) = x^2$, is

$\frac{\partial f}{\partial x} = f_x(x, y) = 2x^{2-1} =$ (i) **0** (ii) $\boldsymbol{2y}$ (iii) $\boldsymbol{2x}$

Maximum of surface $z = x^2 + y^2$ is

(i) **0** (ii) **20** (iii) **does not exist, is** ∞
as $x \to \infty$ or $y \to \infty$, $z = x^2 + y^2 \to \infty$

Minimum of surface $z = x^2 + y^2$ is

(i) **0** (ii) **20** (iii) **does not exist, is** $-\infty$
notice both slopes are zero, $f_y = 2(0) = 0$ and $f_x = 2(0) = 0$, at $(x, y) = (0, 0)$

If $z = x^2 + y^2$ is constrained by $-5 \leq x \leq 5$, $-5 \leq y \leq 5$, then maximum value of z of surface $z = x^2 + y^2$ is

$f(5, 5) = (5)^2 + (5)^2 =$ (i) **0** (ii) **25** (iii) **50**

If $z = x^2 + y^2$ is constrained by $-5 \leq x \leq 5$, $-5 \leq y \leq 5$, then minimum value of z of surface $z = x^2 + y^2$ is,

$f(0, 0) = (0)^2 + (0)^2 =$ (i) **0** (ii) **25** (iii) **50**

(d) *Function* $f(x, y) = \frac{-4}{1+x^2+y^2}$

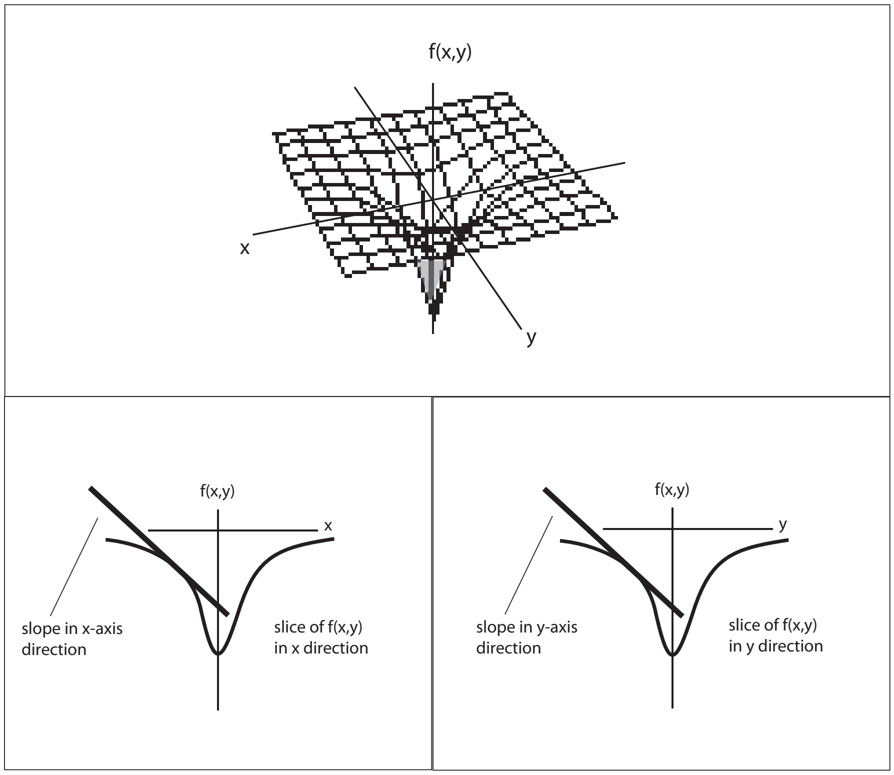

Figure 9.7 ($z = \frac{-4}{1+x^2+y^2}$)

Slope of it surface $z = \frac{-4}{1+x^2+y^2}$ of along y-axis direction (yz-trace, $x = 0$), since $f(0, y) = \frac{-4}{1+(0)^2+y^2} = \frac{-4}{1+y^2}$, is

$$\frac{\partial f}{\partial y} = f_y(x, y) = \frac{v(y) \cdot u_y(y) - u(y) \cdot v_y(y)}{\left[v(y)\right]^2} = \frac{(1 + y^2)(0) - (-4)(2y)}{(1 + y^2)^2} = \frac{8y}{1 + 2y^2 + y^4}$$

use quotient rule, where $u = -4$, $v = 1 + y^2$, so $u_y = 0$, $v_y = 2y^{2-1} = 2y$

so, at $y = -1$, $f_y = \frac{8(-1)}{1+2(-1)^2+(-1)^4} =$ (i) $\mathbf{-2}$ (ii) $\mathbf{-4}$ (iii) $\mathbf{4}$

Slope of surface $z = \frac{-4}{1+x^2+y^2}$ along x-axis direction (xz-trace, $y = 0$), since $f(x, 0) = \frac{-4}{1+x^2+(0)^2} = \frac{-4}{1+x^2}$, is

$$\frac{\partial f}{\partial x} = f_x(x, y) = \frac{v(x) \cdot u_x(x) - u(x) \cdot v_x(x)}{\left[v(x)\right]^2} = \frac{(1 + x^2)(0) - (-4)(2x)}{(1 + x^2)^2} = \frac{8x}{1 + 2x^2 + x^4}$$

use quotient rule, where $u = -4$, $v = 1 + x^2$, so $u_x = 0$, $v_x = 2x^{2-1} = 2x$

so, at $x = -1$, $f_x = \frac{8(-1)}{1+2(-1)^2+(-1)^4} =$ (i) $\mathbf{-2}$ (ii) $\mathbf{-4}$ (iii) $\mathbf{4}$

Maximum of surface $z = \frac{-4}{1+x^2+y^2}$ is

(i) **0** (ii) **20** (iii) **does not exist, is** ∞
as $x \to \infty$ or $y \to \infty$, $z = \frac{-4}{1+x^2+y^2} \to 0$

Minimum of surface $z = \frac{-4}{1+x^2+y^2}$ is

(i) $\mathbf{-4}$ (ii) **0** (iii) **does not exist, is** $-\infty$
as $x \to 0$ and $y \to 0$, $z = \frac{-4}{1+x^2+y^2} \to -4$

If $z = \frac{-4}{1+x^2+y^2}$ is constrained by $-5 \leq x \leq 5$, $-5 \leq y \leq 5$, then maximum value of z of surface $z = \frac{-4}{1+x^2+y^2}$ is

$f(5,5) = f(-5,-5) = \frac{-4}{1+5^2+5^2} =$ (i) $-\mathbf{\frac{4}{51}}$ (ii) $\mathbf{\frac{4}{51}}$ (iii) $-\mathbf{\frac{4}{50}}$

If $z = \frac{-4}{1+x^2+y^2}$ is constrained by $-5 \leq x \leq 5$, $-5 \leq y \leq 5$, then minimum value of z of surface $z = \frac{-4}{1+x^2+y^2}$ is,

(i) $\mathbf{-4}$ (ii) **0** (iii) **does not exist, is** $-\infty$
as $x \to 0$ or $y \to 0$, $z = \frac{-4}{1+x^2+y^2} \to -4$

9.2 Partial Derivatives

We look at *partial* derivatives in this section. A partial derivative is the slope of the tangent to the intersection of either the xz-trace or yz-trace to the $z = f(x,y)$ function, for example. More exactly, for $z = f(x,y)$,

$$\begin{aligned} \frac{\partial z}{\partial x} &= \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} \\ \frac{\partial z}{\partial y} &= \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h} \end{aligned}$$

We also look at *second–order* partial derivatives, including:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right), \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right), \quad \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right), \quad \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

which can also be written as:

$$f_{xx}(x,y) = z_{xx}, \quad f_{yx}(x,y) = z_{yx}, \quad f_{xy}(x,y) = z_{xy}, \quad f_{yy}(x,y) = z_{yy}.$$

Notice reversal in order of x and y between, for example, notation $\frac{\partial^2 z}{\partial x \partial y}$ and notation $f_{yx}(x,y) = z_{yx}$.
Second–order partial derivatives are useful in determining a general notion of concavity as well as, more importantly, in identifying maximum and minimum points of functions.

Exercise 9.2 (Partial Derivatives)

1. *Application: health.* Assume health, H, of an individual depends on both nutrition, N, and exercise, E, according to the following function:

$$H = 3N + 4E$$

   Health improves

$$\frac{\partial H}{\partial N} = 3 \cdot 1N^{1-1} + 0 =$$

   (i) **0** (ii) **3** (iii) **4** units when nutrition improves 1 unit.
   derivative of $3N$ with respect to N is 3, but $\frac{\partial}{\partial N}(4E) = 0$ because $4E$ is *constant* with respect to N

   Health improves

$$\frac{\partial H}{\partial E} = 0 + 4 \cdot 1E^{1-1} =$$

   (i) **0** (ii) **3** (iii) **4** units when exercise improves 1 unit.
   derivative of $4E$ with respect to E is 4, but $\frac{\partial}{\partial E}(3N) = 0$ because $3N$ is *constant* with respect to E

2. $z = 3x + 4y$

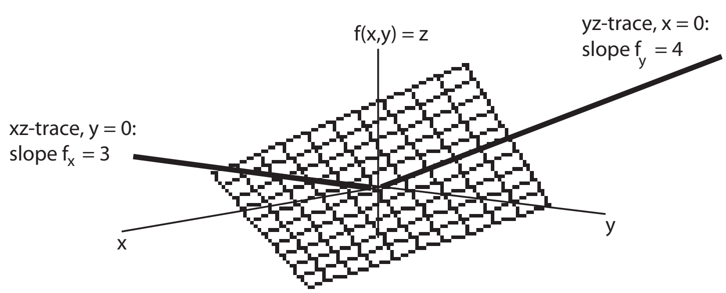

Figure 9.8 ($z = 3x + 4y$)

   first order derivatives

$$\frac{\partial z}{\partial x} = 3 \cdot 1x^{1-1} + 0 =$$

   (i) **0** (ii) **3** (iii) **4**
   derivative of $3x$ with respect to x is 3, but $4y$ is *constant* with respect to x, so $\frac{\partial}{\partial x}(4y) = 0$

   which can also be written,

(a) $\frac{\partial}{\partial x} f(x, y) =$ (i) **0** (ii) **3** (iii) **4**
(b) $\frac{\partial f}{\partial x} =$ (i) **0** (ii) **3** (iii) **4**
(c) $f_x(x, y) =$ (i) **0** (ii) **3** (iii) **4**

$$\frac{\partial z}{\partial y} = 0 + 4 \cdot 1y^{1-1} =$$

(i) **0** (ii) **3** (iii) **4**

derivative of $4y$ with respect to y is 4, but $3x$ is *constant* with respect to y, so $\frac{\partial}{\partial y}(3x) = 0$

which can also be written,

(a) $\frac{\partial}{\partial y} f(x, y) =$ (i) **0** (ii) **3** (iii) **4**
(b) $\frac{\partial f}{\partial y} =$ (i) **0** (ii) **3** (iii) **4**
(c) $f_y(x, y) =$ (i) **0** (ii) **3** (iii) **4**

second order partial derivatives

$$\frac{\partial^2 z}{\partial x \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}(3) =$$

(i) **0** (ii) **3** (iii) **4**

which also can be written

(a) $\frac{\partial^2 f}{\partial x \partial x} =$ (i) **0** (ii) **3** (iii) **4**
(b) $\frac{\partial^2 f}{\partial x^2} =$ (i) **0** (ii) **3** (iii) **4**
(c) $f_{xx}(x, y) =$ (i) **0** (ii) **3** (iii) **4**

$$f_{xy}(x, y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(3) =$$

(i) **0** (ii) **3** (iii) **4**

f_{xy} measures rate of change of slope in y–axis direction, in the x–axis direction

$$f_{yx}(x, y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(4) =$$

(i) **0** (ii) **3** (iii) **4**

f_{yx} does *not* always equal f_{xy}; however, in this course, we will *mostly* only use equations where $f_{yx} = f_{xy}$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}(4) =$$

(i) **0** (ii) **3** (iii) **4**

since both $f_{xx}(x, y) = 0$ and $f_{yy}(x, y) = 0$, $z = 3x + 4y$ is

(i) **concave up** (ii) **concave down** (iii) **neither–it is linear**

3. $z = x^2$

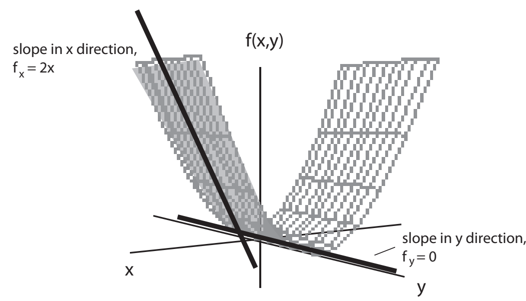

Figure 9.9 ($z = x^2$)

first order derivatives

$$f_x(x, y) = 2x^{2-1} =$$

(i) **2** (ii) $\boldsymbol{2x}$ (iii) $\boldsymbol{2x^2}$

(a) $f_x(1, 2) = 2(1) =$ (i) **0** (ii) **2** (iii) **4**
(b) $f_x(2, 1) = 2(2) =$ (i) **0** (ii) **2** (iii) **4**
(c) $f_x(2, y) = 2(2) =$ (i) **0** (ii) **3** (iii) **4**

$$f_y(x, y) = \frac{\partial}{\partial y}\left(x^2\right) =$$

(i) **0** (ii) **3** (iii) **4**

it is zero because x^2 is a *constant* with respect to y; that is, x^2 does not change when y changes

(a) $f_y(1, 2) =$ (i) **0** (ii) **2** (iii) **4**
(b) $f_y(2, 1) =$ (i) **0** (ii) **2** (iii) **4**
(c) $f_y(2, y) =$ (i) **0** (ii) **3** (iii) **4**

no matter what the values of (x, y), f_y is always zero in this case

second order partial derivatives

$$\frac{\partial^2 z}{\partial x \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}(2x) = 2 \cdot 1x^{1-1} =$$

(i) **2** (ii) $\boldsymbol{2x}$ (iii) $\boldsymbol{2x^2}$

$$f_{xy}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2x) =$$

(i) **0** (ii) **3** (iii) **4**

again, it is zero because $2x$ is a *constant* with respect to y

$$f_{yx}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(0) =$$

(i) **0** (ii) **3** (iii) **4**

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}(0) =$$

(i) **0** (ii) **3** (iii) **4**

since $f_{xx}(x,y) = 2$ but $f_{yy}(x,y) = 0$, $f(x,y) = x^2$ is

(i) **concave up in x-axis direction only**
(ii) **concave up in y-axis direction only**
(iii) **concave up in both x-axis and y-axis directions**

4. $z = x^2 + y^2$

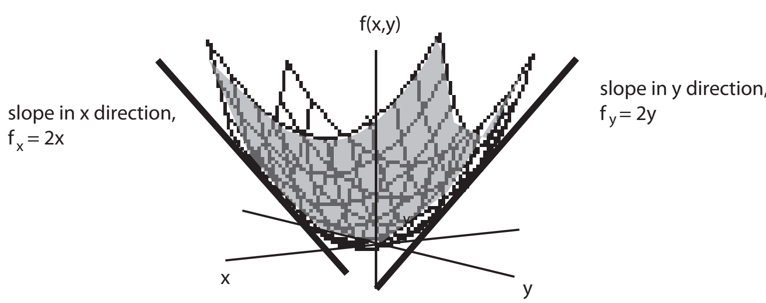

Figure 9.10 ($z = x^2 + y^2$)

first order derivatives

$$f_x(x,y) = 2x^{2-1} + 0 =$$

(i) $\boldsymbol{2x}$ (ii) $\boldsymbol{2y}$ (iii) $\boldsymbol{2x + 2y}$

derivative of x^2 with respect to x is $2x$, but derivative of y^2 is *zero* because y^2 *constant* with respect to x

(a) $f_x(1,2) = 2(1) =$ (i) **0** (ii) **2** (iii) **4**

(b) $f_x(2,1) = 2(2) =$ (i) **0** (ii) **2** (iii) **4**

$$f_y(x,y) = 0 + 2y^{2-1} =$$

(i) $\boldsymbol{2x}$ (ii) $\boldsymbol{2y}$ (iii) $\boldsymbol{2x+2y}$

derivative of y^2 with respect to y is $2y$, but derivative of x^2 is zero because x^2 *constant* with respect to y

(a) $f_y(1,2) = 2(2) =$ (i) **0** (ii) **2** (iii) **4**

(b) $f_y(2,1) = 2(1) =$ (i) **0** (ii) **2** (iii) **4**

determine values of x and y when $f_x(x,y) = 0$ and $f_y(x,y) = 0$

$$f_x(x,y) = 2x = 0, \quad f_y(x,y) = 2y = 0$$

when $(x,y) =$ (i) $\mathbf{(0,2)}$ (ii) $\mathbf{(2,0)}$ (iii) $\mathbf{(0,0)}$

this point $(x,y) = (0,0)$ is an example of a *critical* point, and so a possible *extrema* point

first order derivatives, using definition

$$\begin{aligned} f_x(x,y) &= \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2 + y^2 - (x^2+y^2)}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 + y^2 - x^2 - y^2}{h} \\ &= \lim_{h\to 0} (2x+h) = \end{aligned}$$

(i) $\boldsymbol{2x}$ (ii) $\boldsymbol{2y}$ (iii) $\boldsymbol{2x+2y}$

$$\begin{aligned} f_y(x,y) &= \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h} \\ &= \lim_{h\to 0} \frac{x^2 + (y+h)^2 - (x^2+y^2)}{h} \\ &= \lim_{h\to 0} \frac{x^2 + y^2 + 2yh + h^2 - x^2 - y^2}{h} \\ &= \lim_{h\to 0} (2y+h) = \end{aligned}$$

(i) $\boldsymbol{2x}$ (ii) $\boldsymbol{2y}$ (iii) $\boldsymbol{2x+2y}$

second order partial derivatives

$$f_{xx}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2x) = 2 \cdot 1x^{1-1} =$$

(i) **0** (ii) **2** (iii) **4**

$$f_{xy}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2x) =$$

(i) **0** (ii) **2** (iii) **4**

it is zero because $2x$ is a *constant* with respect to y

$$f_{yx}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(2y) =$$

(i) **0** (ii) **2** (iii) **4**

it is zero because $2y$ is a *constant* with respect to x

$$f_{yy}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(2y) = 2 \cdot 1y^{1-1} =$$

(i) **0** (ii) **2** (iii) **4**

since $f_{xx}(x,y) = 2$ and $f_{yy}(x,y) = 2$, $f(x,y) = x^2 + y^2$ is

(i) **concave up in x-axis direction only**
(ii) **concave up in y-axis direction only**
(iii) **concave up in both x-axis and y-axis directions**

5. $f(x,y) = \frac{-4}{1+x^2+y^2}$

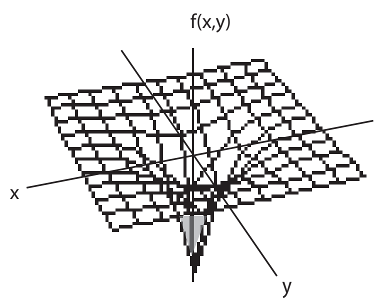

Figure 9.11 ($z = \frac{-4}{1+x^2+y^2}$)

first order derivative, $f_x(x,y)$

let $u(x,y) = -4$ and $v(x,y) = 1 + x^2 + y^2$,

then $u_x(x,y) =$ (i) **0** (ii) $\boldsymbol{2x}$ (iii) $\boldsymbol{2y}$

and $v_x(x,y) = 0 + 2x^{2-1} + 0 =$ (i) **0** (ii) $\boldsymbol{2x}$ (iii) $\boldsymbol{2y}$

and so

$$f_x(x,y) = \frac{v(x,y)\cdot u_x(x,y) - u(x,y)\cdot v_x(x,y)}{[v(x,y)]^2} = \frac{(1+x^2+y^2)(0)-(-4)(2x)}{(1+x^2+y^2)^2} =$$

(i) $\frac{\mathbf{8x}}{\mathbf{(1+x^2+y^2)^2}}$ (ii) $\frac{\mathbf{8y}}{\mathbf{(1+x^2+y^2)^2}}$ (iii) $\frac{\mathbf{8xy}}{\mathbf{(1+x^2+y^2)^2}}$

and so

(a) $f_x(1,2) = \frac{8(1)}{(1+(1)^2+(2)^2)^2} =$ (i) $\frac{\mathbf{2}}{\mathbf{9}}$ (ii) $\frac{\mathbf{4}}{\mathbf{9}}$ (iii) $\frac{\mathbf{6}}{\mathbf{9}}$

(b) $f_x(2,1) = \frac{8(2)}{(1+(2)^2+(1)^2)^2}$ (i) $\frac{\mathbf{2}}{\mathbf{9}}$ (ii) $\frac{\mathbf{4}}{\mathbf{9}}$ (iii) $\frac{\mathbf{6}}{\mathbf{9}}$

first order derivative, $f_y(x,y)$

let $u(x,y) = -4$ and $v(x,y) = 1+x^2+y^2$,

then $u_y(x,y) =$ (i) $\mathbf{0}$ (ii) $\mathbf{2x}$ (iii) $\mathbf{2y}$

and $v_y(x,y) = 0+0+2y^{2-1} =$ (i) $\mathbf{0}$ (ii) $\mathbf{2x}$ (iii) $\mathbf{2y}$

and so

$$f_y(x,y) = \frac{v(x,y)\cdot u_y(x,y) - u(x,y)\cdot v_y(x,y)}{[v(x,y)]^2} = \frac{(1+x^2+y^2)(0)-(-4)(2y)}{(1+x^2+y^2)^2} =$$

(i) $\frac{\mathbf{8x}}{\mathbf{(1+x^2+y^2)^2}}$ (ii) $\frac{\mathbf{8y}}{\mathbf{(1+x^2+y^2)^2}}$ (iii) $\frac{\mathbf{8xy}}{\mathbf{(1+x^2+y^2)^2}}$

second order partial derivative, $f_{xx}(x,y)$

$$f_{xx}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{8x}{(1+x^2+y^2)^2}\right)$$

so let $u(x,y) = 8x$ and $v(x,y) = (1+x^2+y^2)^2$,

then $u_x(x,y) = 8x^{1-1} =$ (i) $\mathbf{8}$ (ii) $\mathbf{4x(1+x^2+y^2)}$ (iii) $\mathbf{4y(1+x^2+y^2)}$

$v_x(x,y) = 2(1+x^2+y^2)(2x) =$ (i) $\mathbf{8}$ (ii) $\mathbf{4x(1+x^2+y^2)}$ (iii) $\mathbf{4y(1+x^2+y^2)}$

and so

$$f_{xx}(x,y) = \frac{v(x,y)\cdot u_x(x,y) - u(x,y)\cdot v_x(x,y)}{[v(x,y)]^2} = \frac{(1+x^2+y^2)(8)-(8x)(4x(1+x^2+y^2))}{(1+x^2+y^2)^4} =$$

(i) $\frac{\mathbf{8-32x^2}}{\mathbf{(1+x^2+y^2)^3}}$ (ii) $\frac{\mathbf{8-32y^2}}{\mathbf{(1+x^2+y^2)^2}}$ (iii) $\frac{\mathbf{8xy}}{\mathbf{(1+x^2+y^2)^2}}$

second order partial derivative, $f_{xy}(x,y)$

$$f_{xy}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{8x}{(1+x^2+y^2)^2}\right)$$

so let $u(x,y) = 8x$ and $v(x,y) = (1+x^2+y^2)^2$,

then $u_y(x,y) =$ (i) $\mathbf{0}$ (ii) $\mathbf{4x(1+x^2+y^2)}$ (iii) $\mathbf{4y(1+x^2+y^2)}$

$v_y(x,y) = 2(1+x^2+y^2)(2y) =$ (i) $\mathbf{8}$ (ii) $\mathbf{4x(1+x^2+y^2)}$ (iii) $\mathbf{4y(1+x^2+y^2)}$

and so

$$f_{xy}(x,y) = \frac{v(x,y)\cdot u_y(x,y) - u(x,y)\cdot v_y(x,y)}{[v(x,y)]^2} = \frac{(1+x^2+y^2)(0)-(8x)(4y(1+x^2+y^2))}{(1+x^2+y^2)^4} =$$

(i) $\frac{\mathbf{8-32x^2}}{\mathbf{(1+x^2+y^2)^3}}$ (ii) $\frac{\mathbf{8-32y^2}}{\mathbf{(1+x^2+y^2)^3}}$ (iii) $\frac{\mathbf{-32xy}}{\mathbf{(1+x^2+y^2)^3}}$

second order partial derivative, $f_{yx}(x,y)$

$$f_{yx}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{8y}{(1+x^2+y^2)^2}\right)$$

so let $u(x,y) = 8y$ and $v(x,y) = (1+x^2+y^2)^2$,

then $u_x(x,y) =$ (i) $\mathbf{0}$ (ii) $\mathbf{4x(1+x^2+y^2)}$ (iii) $\mathbf{4y(1+x^2+y^2)}$

$v_x(x,y) = 2(1+x^2+y^2)(2x) =$ (i) $\mathbf{8}$ (ii) $\mathbf{4x(1+x^2+y^2)}$ (iii) $\mathbf{4y(1+x^2+y^2)}$

and so

$$f_{yx}(x,y) = \frac{v(x,y)\cdot u_x(x,y) - u(x,y)\cdot v_x(x,y)}{[v(x,y)]^2} = \frac{(1+x^2+y^2)(0)-(8y)(4x(1+x^2+y^2))}{(1+x^2+y^2)^4} =$$

(i) $\frac{\mathbf{8-32x^2}}{\mathbf{(1+x^2+y^2)^3}}$ (ii) $\frac{\mathbf{8-32y^2}}{\mathbf{(1+x^2+y^2)^3}}$ (iii) $\frac{\mathbf{-32xy}}{\mathbf{(1+x^2+y^2)^3}}$

second order partial derivative, $f_{yy}(x,y)$

$$f_{yy}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{8y}{(1+x^2+y^2)^2}\right)$$

so let $u(x,y) = 8y$ and $v(x,y) = (1+x^2+y^2)^2$,

then $u_y(x,y) = 8y^{1-1} =$ (i) $\mathbf{8}$ (ii) $\boldsymbol{4x(1+x^2+y^2)}$ (iii) $\boldsymbol{4y(1+x^2+y^2)}$

$v_y(x,y) = 2(1+x^2+y^2)(2y) =$ (i) $\mathbf{8}$ (ii) $\boldsymbol{4x(1+x^2+y^2)}$ (iii) $\boldsymbol{4y(1+x^2+y^2)}$

and so

$$f_{yy}(x,y) = \frac{v(x,y)\cdot u_y(x,y) - u(x,y)\cdot v_y(x,y)}{[v(x,y)]^2} = \frac{(1+x^2+y^2)(8) - (8y)(4y(1+x^2+y^2))}{(1+x^2+y^2)^4} =$$

(i) $\boldsymbol{\frac{8-32x^2}{(1+x^2+y^2)^3}}$ (ii) $\boldsymbol{\frac{8-32y^2}{(1+x^2+y^2)^3}}$ (iii) $\boldsymbol{\frac{8xy}{(1+x^2+y^2)^2}}$

6. $z = 6x^2y + 4e^{xy}$

first order derivatives

$$f_x(x,y) = 2x^{2-1}(6y) + 4e^{xy}(x^{1-1}y) =$$

(i) $\boldsymbol{12xy + 4ye^{xy}}$ (ii) $\boldsymbol{6x^2 + 4xe^{xy}}$ (iii) $\boldsymbol{2x + 2y}$

derivative of $6x^2y = (6y)x^2$ with respect to x is $6y \cdot 2x^{2-1} = 12xy$ because $6y$ constant with respect to x

also derivative of e^{xy} with respect to x is $e^{xy} \cdot (x^{1-1}y) = ye^{xy}$ because y is constant with respect to x

$$f_y(x,y) = 6y^{1-1}x^2 + 4e^{xy}(xy^{1-1}) =$$

(i) $\boldsymbol{12xy + 4ye^{xy}}$ (ii) $\boldsymbol{6x^2 + 4xe^{xy}}$ (iii) $\boldsymbol{2x + 2y}$

second order partial derivatives

$$f_{xx}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(12xy + 4ye^{xy}) = 12x^{1-1}y + 0\cdot e^{xy} + 4y\cdot e^{xy}(x^{1-1}y) =$$

(i) $\mathbf{0}$ (ii) $\boldsymbol{12x + (4+4y)xe^{xy}}$ (iii) $\boldsymbol{12y + 4y^2e^{xy}}$

use product rule on $4ye^{xy}$, where $u = 4y$, $v = e^{xy}$, so $u_x = 0$, $v_x = e^{xy}(x^{1-1}y) = ye^{xy}$

so $v \cdot u_x + u \cdot v_x = e^{xy} \cdot 0 + 4y \cdot ye^{xy} = 4y^2e^{xy}$

$$f_{xy}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(12xy + 4ye^{xy}) = 12xy^{1-1} + e^{xy}\cdot 4 + 4y \cdot xe^{xy} =$$

(i) $\mathbf{0}$ (ii) $\boldsymbol{12x + (4+4xy)e^{xy}}$ (iii) $\boldsymbol{12y + 4y^2e^{xy}}$

use product rule on $4ye^{xy}$, where $u = 4y$, $v = e^{xy}$, so $u_y = 4y^{1-1} = 4$, $v_y = e^{xy}(xy^{1-1}) = xe^{xy}$

so $v \cdot u_x + u \cdot v_x = e^{xy} \cdot 4 + 4y \cdot xe^{xy} = (4+4y)xe^{xy}$

$$f_{yx}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(6x^2 + 4xe^{xy}\right) = 6\cdot 2x^{2-1} + e^{xy}\cdot 4 + 4x\cdot ye^{xy} =$$

(i) $\boldsymbol{12y + (4 + 4xy)e^{xy}}$ (ii) $\boldsymbol{12x + (4 + 4y)xe^{xy}}$ (iii) $\boldsymbol{12y + 4y^2e^{xy}}$

use product rule on $4xe^{xy}$, where $u = 4x$, $v = e^{xy}$, so $u_x = 4$, $v_x = e^{xy}(x^{1-1}y) = ye^{xy}$

so $v \cdot u_x + u \cdot v_x = e^{xy} \cdot 4 + 4x \cdot ye^{xy} = (4 + 4x)ye^{xy}$

also, notice once again, $f_{xy} = f_{yx}$; although not always true, this will always be true in this course

$$f_{yy}(x, y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(6x^2 + 4xe^{xy}\right) = 0 + e^{xy} \cdot 0 + 4x \cdot xe^{xy} =$$

(i) $\boldsymbol{4x^2e^{xy}}$ (ii) $\boldsymbol{12x + (4 + 4y)xe^{xy}}$ (iii) $\boldsymbol{12y + 4y^2e^{xy}}$

use product rule on $4xe^{xy}$, where $u = 4x$, $v = e^{xy}$, so $u_y = 0$, $v_y = e^{xy}(xy^{1-1}) = xe^{xy}$

so $v \cdot u_x + u \cdot v_x = e^{xy} \cdot 0 + 4x \cdot xe^{xy} = 4x^2e^{xy}$