Chapter 24

Random and Mixed Effects Models

Random factor levels are factor levels that have been assumed to have been sampled from (represent) a larger population of factor levels. Two types of ANOVA models involving random factors are discussed: ANOVA model II and ANOVA model III. An ANOVA model II is one in which *all* of the factors are random; an ANOVA model III is one is which some of the factors are random and some are fixed. Inference (tests and confidence intervals) procedures are somewhat different for the three ANOVA models.

24.1 Single–Factor Studies—ANOVA Model II

SAS program: att6-24-1-drugs-ANOVAII

In the *cell means* version of the single–factor ANOVA model II,

$$Y_{ij} = \mu_i + \varepsilon_{ij},$$

both μ_i and ε_{ij} are random variables. Equivalently, in the *factor effects* model formulation of the single–factor ANOVA model II,

$$Y_{ij} = \mu_{\cdot} + \tau_i + \varepsilon_{ij},$$

$\mu_{\cdot}$ is fixed (constant, non–random), but both τ_i and ε_{ij} are random variables. By way of comparison, for a single–factor ANOVA model I (I, not II!), the parameters μ_i, $\mu_{\cdot}$ and τ_i are all fixed and only ε_{ij} is a random variable.

Exercise 24.1 (Single–Factor Studies—ANOVA Model II: Drug Types) Patients responses are recorded for three drug types.

drug 1	5.90	5.92	5.91	5.89	5.88	$\overline{Y}_{1\cdot} \approx 5.90$
drug 2	5.50	5.50	5.50	5.49	5.50	$\overline{Y}_{2\cdot} = 5.50$
drug 3	5.01	5.00	4.99	4.98	5.02	$\overline{Y}_{3\cdot} \approx 5.00$

1. *95% Confidence Interval of $\mu_{\cdot}$.*
   Using the SAS output,

$$\begin{aligned}
\bar{Y}_{\cdot\cdot} &= 5.4667 \\
s\{\bar{Y}_{\cdot\cdot}\} &= \sqrt{\frac{MSTR}{rn}} \\
&= \sqrt{\frac{1.0166666}{(3)(5)}} \\
&= 0.260 \\
t(1-\frac{\alpha}{2}; r-1) &= t(1-\frac{0.05}{2}; 3-1) \\
&= t(0.975; 2) \\
&= 4.303
\end{aligned}$$

   the confidence interval for $\mu_{\cdot}$ is given by

$$\bar{Y}_{\cdot\cdot} \pm t(1-\frac{\alpha}{2}; r-1)s\{\bar{Y}_{\cdot\cdot}\} = 5.4667 \pm (4.303)(0.260) =$$

   (choose one) $\mathbf{(-6.466, -2.494)}$ / $\mathbf{(-5.98, -2.98)}$ / $\mathbf{(4.348, 6.585)}$

2. *Test*[1] $\sigma_\mu^2 = 0$*: factor levels μ_i are the same in an ANOVA II model.*

   (a) *Statement*
   The statement of the test is (choose *best* one):
      i. $H_0 : \sigma_\mu^2 = 0$ versus $H_a : \sigma_\mu^2 > 0$.
      ii. $H_0 : \mu_1 = \mu_2 = \mu_3$ versus $H_a :$ at least one $\mu_i \neq \mu_j$.
      iii. $H_0 : \alpha_1 = \alpha_2 = 0$ versus $H_a :$ at least one $\alpha_i \neq 0$.
      iv. $H_0 : \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ versus $H_a : \alpha_1 \neq \alpha_3, \alpha_1 \neq \alpha_2$.

   (b) *Test*
   Since the test statistic is

$$F^* = \frac{MSTR}{MSE} = \frac{1.01666667}{0.0001833} = 5545.45$$

   the p–value, with $(2, 12)$ degrees of freedom, is given by

$$\text{p–value} = P(f \geq 5545.45)$$

   which equals (circle one) **0.00** / **0.37** / **0.40**.
   The level of significance is 0.05.

[1] To test $\sigma_\mu^2 = 0$ is equivalent, in the fixed case, to testing $\mu_i = \mu_{\cdot}$, in other words, that all the μ_i are the same, since the μ_i are $N(\mu_{\cdot}, \sigma_\mu^2)$. Furthermore, to test $\sigma_\mu^2 = 0$ is also equivalent, in the fixed case, to testing $\alpha_i = 0$, since τ_i are $N(0, \sigma_\mu^2)$.

(c) *Conclusion*
Since the p–value, 0, is smaller than the level of significance, 0.05, we (circle one) **accept** / **reject** the null hypothesis that $\sigma_\mu^2 = 0$, in other words, the drug type factor *is* significant.

3. *Point estimate of σ_μ^2, variance of the random factor effect*
A point estimate of σ_μ^2 is

$$s_\mu^2 = \frac{MSTR - MSE}{n} = \frac{1.01666667 - 0.0001833}{5} =$$

(circle one) **0.00** / **0.203** / **0.40**.

4. *95% Confidence interval of σ_μ^2, modified large sample (MLS) procedure*
Since

$$\begin{aligned} c_1 &= 1/n = 1/5 = 0.2 \\ c_2 &= -1/n = -1/5 = -0.2 \\ df_1 &= r - 1 = 3 - 1 = 2 \\ df_2 &= r(n-1) = 3(5-1) = 12 \\ MS_1 &= MSTR = 1.01666667 \\ MS_2 &= MSE = 0.0001833 \end{aligned}$$

and[2]

$$\begin{aligned} F_1(1-\alpha/2; df_1, \infty) &= F(1-0.05/2; 2, \infty) = 3.69 \\ F_2(1-\alpha/2; df_2, \infty) &= F(1-0.05/2; 12, \infty) = 1.95 \\ F_3(1-\alpha/2; \infty, df_1) &= F(1-0.05/2; \infty, 2) = 39.50 \\ F_4(1-\alpha/2; \infty, df_2) &= F(1-0.05/2; \infty, 12) = 2.75 \\ F_5(1-\alpha/2; df_1, df_1) &= F(1-0.05/2; 2, 12) = 5.10 \\ F_6(1-\alpha/2; df_2, df_1) &= F(1-0.05/2; 12, 2) = 39.42 \end{aligned}$$

[2]Use INVF on your calculator; use 100000 (not "E99") when asked to give ∞ df. The program INVF does not deal with ∞ very well in this case. The answer will be incorrect in at least the third decimal place.

and

$$\begin{aligned}
G_1 &= 1 - 1/F_1 \;=\; 1 - 1/3.69 \;=\; 0.7290 \\
G_2 &= 1 - 1/F_2 \;=\; 1 - 1/1.95 \;=\; 0.4872 \\
G_3 &= \frac{(F_5 - 1)^2 - (G_1 F_5)^2 - (F_4 - 1)^4}{F_5} \\
&= \frac{(5.10 - 1)^2 - ((0.7290)(5.10))^2 - (2.75 - 1)^4}{5.10} \\
&= -0.0148 \\
G_4 &= F_6 \left(\left(\frac{F_6 - 1}{F_6} \right)^2 - \left(\frac{F_3 - 1}{F_6} \right)^2 - G_2^2 \right) \\
&= 39.42 \left(\left(\frac{39.42 - 1}{39.42} \right)^2 - \left(\frac{39.5 - 1}{39.42} \right)^2 - 0.4872^2 \right) \\
&= -9.513
\end{aligned}$$

and

$$\begin{aligned}
H_L &= \left((G_1 c_1 MS_1)^2 + ((F_4 - 1)c_2 MS_2)^2 - \; G_3 c_1 c_2 MS_1 MS_2\right)^{1/2} \\
&= \left((0.7290)(0.2)(1.01666667))^2 + \cdots\right)^{1/2} \\
&= 0.1482 \\
H_U &= \sqrt{(((F_3 - 1)c_1 MS_1)^2 + G_2 c_2 MS_2)^2 - G_4 c_1 c_2 MS_1 MS_2} \\
&= \sqrt{(((39.5 - 1)(0.2)(1.01666667))^2 + \cdots} \\
&= 7.828
\end{aligned}$$

and since $s_\mu^2 = 0.203$,
$s_\mu^2 - H_L = 0.203 - 0.1482 = 0.0548$ and $s_\mu^2 + H_U = 0.203 + 7.828 = 8.031$
and so the approximate confidence interval for σ_μ^2 is
(choose one) **(0.995, 0.999)** / **(0.0548, 2.8031)** / **(4.348, 6.585)**

5. *95% Confidence interval of σ_μ^2, Satterthwaite procedure*
   Since

$$\begin{aligned}
df &= \frac{(n \times s_\mu^2)^2}{\frac{(MSTR)^2}{r-1} + \frac{(MSE)^2}{r(n-1)}} \\
&= \frac{((5)(0.203))^2}{\frac{(1.01666667)^2}{3-1} + \frac{(0.0001833)^2}{3(5-1)}} \\
&= 1.993 \\
\chi^2(1 - \alpha/2; df) &= \chi^2(1 - 0.05/2; 200) \;=\; 241.06 \\
\chi^2(\alpha/2; df) &= \chi^2(0.05/2; 200) \;=\; 162.73
\end{aligned}$$

and so the approximate confidence interval for σ_μ^2 is

$$\frac{(df)(\sigma_\mu^2)}{\chi^2(1-\alpha/2;df)} \leq \sigma_\mu^2 \leq \frac{(df)(\sigma_\mu^2)}{\chi^2(\alpha/2;df)} =$$

or

$$\frac{(199.345)(0.203)}{241.06} \leq \sigma_\mu^2 \leq \frac{(199.345)(0.203)}{162.73} =$$

(choose one) **(0.0016, 0.00248)** / **(1.882, 9.858)** / **(4.348, 6.585)**

6. *95% CI of* $\frac{\sigma_\mu^2}{\sigma_\mu^2+\sigma^2}$*, fraction of factor effect variance to total variance.*
   Since

$$\begin{aligned}
L &= \frac{1}{n}\left(\frac{MSTR}{MSE}\left(\frac{1}{F(1-\alpha/2;r-1,r(n-1))}\right)-1\right) \\
&= \frac{1}{5}\left(\frac{1.01666667}{0.0001833}\left(\frac{1}{F(1-0.05/2;3-1,3(5-1))}\right)-1\right) \\
&= \frac{1}{5}\left(\frac{1.01666667}{0.0001833}\left(\frac{1}{5.096}\right)-1\right) \\
&= 217.44 \\
U &= \frac{1}{n}\left(\frac{MSTR}{MSE}\left(\frac{1}{F(\alpha/2;r-1,r(n-1))}\right)-1\right) \\
&= \frac{1}{5}\left(\frac{1.01666667}{0.0001833}\left(\frac{1}{F(0.05/2;3-1,3(5-1))}\right)-1\right) \\
&= \frac{1}{5}\left(\frac{1.01666667}{0.0001833}\left(\frac{1}{0.0254}\right)-1\right) \\
&= 43837.35 \\
L^* &= \frac{L}{1+L} \\
&= \frac{217.44}{1+217.44} \\
&= 0.995 \\
U^* &= \frac{U}{1+U} \\
&= \frac{43837.35}{1+43837.35} \\
&= 0.999
\end{aligned}$$

and so the confidence interval for $\frac{\sigma_\mu^2}{\sigma_\mu^2+\sigma^2}$ is given by
(choose one) **(0.995, 0.999)** / **(−5.98, −2.98)** / **(4.348, 6.585)**

7. *95% Confidence Interval of σ^2, variance of model error.*
   Since

$$\begin{aligned}
\frac{r(n-1)MSE}{\chi^2(1-\alpha/2; r(n-1))} &= \frac{3(5-1)(0.0001833)}{\chi^2(1-0.05/2; 3(5-1))} \\
&= \frac{0.0021996}{\chi^2(0.975; 12)} \\
&= \frac{0.0021996}{23.3367} \\
&= 0.00009425 \\
\frac{r(n-1)MSE}{\chi^2(\alpha/2; r(n-1))} &= \frac{3(5-1)(0.0001833)}{\chi^2(0.05/2; 3(5-1))} \\
&= \frac{0.0021996}{\chi^2(0.025; 12)} \\
&= \frac{0.0021996}{4.4038} \\
&= 0.0004995
\end{aligned}$$

   and so the confidence interval for σ^2 is given by
   (choose one) $\mathbf{(0.995, 0.999)}$ / $\mathbf{(0.00009425, 0.0004995)}$ / $\mathbf{(4.348, 6.585)}$

8. *Summary of inference procedures for single–factor ANOVA II models*
   True / False
   A summary of the inference procedures for single–factor ANOVA II models is given below.

Confidence interval $\mu_{\cdot}$; variability in grand average $\bar{Y}_{\cdot\cdot} \pm t(1-\frac{\alpha}{2}; r-1)s\{\bar{Y}_{\cdot\cdot}\}$, $s\{\bar{Y}_{\cdot\cdot}\} = \sqrt{\frac{MSTR}{rn}}$
Test $\sigma_\mu^2 = 0$; equivalent to testing $\mu_i = 0$ $F^* = \frac{MSTR}{MSE}$
Point Estimate σ_μ^2; variability in factor effect $s_\mu^2 = \frac{MSTR-MSE}{n}$
Confidence interval σ_μ^2; modified large sample (MLS) procedure $(s_\mu^2 - H_L, s_\mu^2 + H_U)$
Confidence interval σ_μ^2; Satterthwaite procedure $\frac{(df)(\sigma_\mu^2)}{\chi^2(1-\alpha/2;df)} \leq \sigma_\mu^2 \leq \frac{(df)(\sigma_\mu^2)}{\chi^2(\alpha/2;df)}$
Confidence interval $\frac{\sigma_\mu^2}{\sigma_\mu^2+\sigma^2}$; fraction of total variability due to factor effect $L^* \leq \frac{\sigma_\mu^2}{\sigma_\mu^2+\sigma^2} \leq U^*$
Confidence interval σ^2; variability due to error $\frac{r(n-1)MSE}{\chi^2(1-\alpha/2;r(n-1))} \leq \sigma^2 \leq \frac{r(n-1)MSE}{\chi^2(\alpha/2;r(n-1))}$

9. *Understanding the ANOVA II model, cell means version*
   In the cell means version of the single–factor ANOVA model II,

$$Y_{ij} = \mu_i + \varepsilon_{ij},$$

   to say the random variables μ_i are independent $N(\mu_., \sigma_\mu^2)$ means the (*choose none, one or more*!)

   (a) μ_i are normally distributed.
   (b) mean (average) of the μ_i is $\mu_.$.
   (c) variance (variability) of the μ_i is σ_μ^2.
   (d) each μ_i is independent of any other μ_j.

10. *More understanding the ANOVA II model, cell means version*
    True / False
    To say the factor level μ_i are independent $N(\mu_., \sigma_\mu^2)$ means, roughly, they can vary both more or less than $\mu_.$, but, on average, are equal to $\mu_.$. In particular, if $\sigma_\mu^2 = 0$, then the μ_i would *all* equal $\mu_.$: they would be all the *same*!

11. *Understanding the ANOVA II model, factor effects version*
    In the factor effects version of the single–factor ANOVA model II,

$$Y_{ij} = \mu_. + \tau_i + \varepsilon_{ij},$$

    to say the random variables τ_i are independent $N(0, \sigma_\mu^2)$ means the (*choose none, one or more*!)

    (a) τ_i are normally distributed.
    (b) mean (average) of the τ_i is zero (0).
    (c) variance (variability) of the τ_i is σ_μ^2.

12. *More understanding the ANOVA II model, factor effects version*
    True / False
    To say the factor effects τ_i are independent $N(0, \sigma_\mu^2)$ means, roughly, they can be both positive or negative, but, on average, are equal to zero (0). In particular, if $\sigma_\mu^2 = 0$, then the τ_i would *all* equal zero and this would imply all μ_i are *same*.

13. *More ANOVA II model*
    True / False In both versions of the ANOVA models, the error ε_{ij} are independent $N(0, \sigma^2)$; that is, roughly, the error can be positive or negative, but, on average, is equal to zero. In particular, if $\sigma^2 = 0$, then the error would be zero–there would be no error!

24.2 Two–Factor Studies—ANOVA Models II and III

In two–factor studies, the ANOVA can either (fixed) model I, (random) model II or (mixed) model III. These three models are compared in this section. By way of comparison, in a single–factor ANOVA, the model can be either (fixed) model I or (random) model II, but not (mixed) model III.

Exercise 24.2 (Two–Factor Study—ANOVA Models II and III)

1. *Different Models.*
   The possible different mixed models for two–factor ANOVA models include (choose none, one or more)

   (a) Both factor A and factor B are fixed (constant, non–random), ANOVA I.

   (b) Both factor A and factor B are random, ANOVA II.

   (c) Factor A is fixed and factor B is random, ANOVA III.

   (d) Factor A is random and factor B is fixed, ANOVA III.

2. *Examples of calculating F statistic for different models.*
   Consider the following two–factor ANOVA table.

Source	*df*	*SS*	*MS*
Factor A	4	440	110
Factor B	2	300	150
Interaction AB	8	80	10
Error	45	180	4
Total	59	1000	

   (a) The F statistic to test factor A, assuming fixed model ANOVA I, is

$$F = \frac{MSA}{MSE} = \frac{110}{4} =$$

   (choose one) **11** / **15.5** / **27.5**

   (b) The F statistic to test factor A, assuming random model ANOVA II, is

$$F = \frac{MSA}{MSAB} = \frac{110}{10} =$$

   (choose one) **11** / **15.5** / **27.5**

(c) The F statistic to test factor A, assuming mixed model ANOVA III (A fixed, B random), is

$$F = \frac{MSA}{MSAB} = \frac{110}{10} =$$

(choose one) **11 / 15.5 / 27.5**

(d) The F statistic to test factor B, assuming fixed model ANOVA I, is

$$F = \frac{MSB}{MSE} = \frac{150}{4} =$$

(choose one) **11 / 15.5 / 37.5**

(e) The F statistic to test interaction AB, assuming random model ANOVA II, is

$$F = \frac{MSAB}{MSE} = \frac{10}{4} =$$

(choose one) **2.5 / 15.5 / 27.5**

(f) The F statistic to test interaction AB, assuming mixed model ANOVA III (A fixed, B random), is

$$F = \frac{MSAB}{MSE} = \frac{10}{4} =$$

(choose one) **2.5 / 15.5 / 27.5**

3. *Calculating various F test statistics.*
The calculation of the F test statistics for the different models are given below.

Source	Fixed ANOVA Model I (A and B Fixed)	Random ANOVA Model II (A and B Random)	Mixed ANOVA Model III (A fixed, B random)
Factor A	MSA/MSE	$MSA/MSAB$	$MSA/MSAB$
Factor B	MSB/MSE	$MSB/MSAB$	MSB/MSE
Interaction AB	$MSAB/MSE$	$MSAB/MSE$	$MSAB/MSE$

The F statistics are calculated
(choose one) **in the same way / in a different way**
for each of the three models.

4. *Understanding how the F test statistics are created*
Consider the following table of expected mean squares.

Mean Square	df	Fixed ANOVA Model I (A and B Fixed)	Random ANOVA Model II (A and B Random)	Mixed ANOVA Model II (A fixed, B random)
Factor A	$a-1$	$\sigma^2 + nb\frac{\sum \alpha_i^2}{a-1}$	$\sigma^2 + nb\sigma_\alpha^2 + n\sigma_{\alpha\beta}^2$	$\sigma^2 + nb\frac{\sum \alpha_i^2}{a-1} + n\sigma_{\alpha\beta}^2$
Factor B	$b-1$	$\sigma^2 + na\frac{\sum \beta_j^2}{b-1}$	$\sigma^2 + na\sigma_\beta^2 + n\sigma_{\alpha\beta}^2$	$\sigma^2 + na\sigma_\beta^2$
Interaction AB	$(a-1)(b-1)$	$\sigma^2 + n\frac{\sum\sum(\alpha\beta)_{ij}^2}{(a-1)(b-1)}$	$\sigma^2 + n\sigma_{\alpha\beta}^2$	$\sigma^2 + n\sigma_{\alpha\beta}^2$
Error	$(n-1)ab$	σ^2	σ^2	σ^2

(a) **True / False**
The F statistic to test factor A, assuming fixed model ANOVA I, is

$$F = \frac{MSA}{MSE}$$

because not only is σ^2 common to both
MSE (σ^2) and MSA ($\sigma^2 + nb\frac{\sum \alpha_i^2}{a-1}$)
but also the factor A effect ($nb\frac{\sum \alpha_i^2}{a-1}$) has been isolated for testing.

(b) **True / False**
The F statistic to test factor A, assuming random model ANOVA II, is

$$F = \frac{MSA}{MSAB} = \frac{110}{10} =$$

because not only is $\sigma^2 + n\sigma_{\alpha\beta}^2$ is common to both
$MSAB$ ($\sigma^2 + n\sigma_{\alpha\beta}^2$) and MSA ($\sigma^2 + nb\sigma_\alpha^2 + n\sigma_{\alpha\beta}^2$).
but also the factor A effect ($nb\sigma_\alpha^2$) has been isolated for testing.

(c) **True / False**
The F statistic to test factor A, assuming mixed model ANOVA III (A fixed, B random), is

$$F = \frac{MSA}{MSAB} = \frac{110}{10} =$$

because not only is $\sigma^2 + n\sigma_{\alpha\beta}^2$ is common to both
$MSAB$ ($\sigma^2 + n\sigma_{\alpha\beta}^2$) and MSA ($\sigma^2 + nb\frac{\sum \alpha_i^2}{a-1} + n\sigma_{\alpha\beta}^2$).
but also the factor A effect ($nb\frac{\sum \alpha_i^2}{a-1}$) has been isolated for testing.

(d) **True / False**
The F statistic to test factor B, assuming fixed model ANOVA I, is

$$F = \frac{MSB}{MSE} = \frac{150}{4} =$$

because not only is σ^2 is common to both
MSE (σ^2) and MSB ($\sigma^2 + na\frac{\sum \beta_j^2}{b-1}$).
but also the factor B effect ($na\frac{\sum \beta_j^2}{b-1}$) has been isolated for testing.

(e) **True / False**
The F statistic to test interaction AB, assuming random model ANOVA II, is

$$F = \frac{MSAB}{MSE} = \frac{10}{4} =$$

because not only is σ^2 is common to both
MSE (σ^2) and $MSAB$ $(\sigma^2 + n\sigma^2_{\alpha\beta})$.
but also the interaction AB $(n\sigma^2_{\alpha\beta})$ has been isolated for testing.

(f) **True / False**
The F statistic to test interaction AB, assuming mixed model ANOVA III (A fixed, B random), is

$$F = \frac{MSAB}{MSE} = \frac{10}{4} =$$

because not only is σ^2 is common to both
MSE (σ^2) and $MSAB$ $(\sigma^2 + n\sigma^2_{\alpha\beta})$.
but also the interaction AB $(n\sigma^2_{\alpha\beta})$ has been isolated for testing.

5. *Which ANOVA model?*
The factor effects model,

$$Y_{ijk} = \mu_{\cdot\cdot} + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk},$$

where

$\mu_{\cdot\cdot}$ is a constant component of all observations

α_i are constants for the main effect factor A at ith level

β_j are random variables for the main effect factor B at jth level;
and are independent $N(0, \sigma^2_\beta)$

$\alpha\beta_{ij}$ are random variables for the interactions;
and are independent $N(0, \sigma^2_\mu)$, and are *random variables*!

ε_{ij} are independent $N(0, \sigma^2)$

$i = 1, \ldots, a; j = 1, \ldots, b;\quad k = 1, \ldots, n$

is an example of a (choose one)
fixed, ANOVA I / random, ANOVA II / mixed, ANOVA III model

24.3 Two–Factor Studies—ANOVA Tests for Models II and III

SAS program: att6-24-3-roc-twoANOVAII,III

Exercise 24.3 (Two–Factor Studies—ANOVA Tests for Models II and III) Consider the following two–factor study on the effect of air temperature *and* noise on the ROC of deer mice.

	Factor B, noise →	$j = 1$, low	$j = 2$, medium	$j = 3$, high	column ave
Factor A,	$i = 1$, 0^o F	10.3, 7.2	9.1, 5.4	6.1, 2.1	$\overline{Y}_{1\cdot\cdot} = 6.7$
temperature	$i = 2$, 10^o F	1.8, 9.8	12.1, 4.2	5.1, 6.2	$\overline{Y}_{2\cdot\cdot} = 6.5$
	$i = 3$, 20^o F	1.2, 8.1	6.5, 4.1	1.2, 2.1	$\overline{Y}_{3\cdot\cdot} = 3.9$
	$i = 4$, 30^o F	12.4, 15.1	16.1, 17.2	18.1, 19.1	$\overline{Y}_{1\cdot\cdot} = 16.3$
	column ave	$\overline{Y}_{\cdot 1 \cdot} = 8.2$	$\overline{Y}_{\cdot 2 \cdot} = 9.3$	$\overline{Y}_{\cdot 3 \cdot} = 7.5$	$\overline{Y}_{\cdot\cdot\cdot} = 8.4$

Test if either of the two main effects or the interaction are significant at $\alpha = 0.05$, assuming, for the model,

$$Y_{ijk} = \mu_{\cdot\cdot} + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk},$$

- Both Temperature, α_i, and Noise, β_j, is random, ANOVA II.
- Temperature, α_i, is fixed and Noise, β_j, is random, ANOVA III.

1. *Both Temperature and Noise are random, ANOVA II.*

Source	*df*	*SS*	*MS*
Factor A (Temperature)	3	539.138	179.71277
Factor B (Noise)	2	13.6808	6.8404
Interaction AB	6	13.6992	9.28319
Error	12	115.3	9.60833
Total	23	723.8183	

(a) *Test Random Interaction AB.*

i. *Statement*
The statement of the test is (choose one):
A. $H_0 : \sigma^2_{\alpha\beta} = 0$ versus $H_a : \sigma^2_{\alpha\beta} > 0$.
B. $H_0 :$ all $(\alpha\beta)_{ij} = 0$ versus $H_a :$ not all $(\alpha\beta)_{ij} = 0$.
C. $H_0 : \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ versus $H_a : \alpha_1 \neq \alpha_3, \alpha_1 \neq \alpha_2$.
D. $H_0 : \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ versus
$H_a :$ at least one $\alpha_i \neq 0$, $i = 1, 2, 3, 4$.

ii. *Test*
From SAS, since

$$F = \frac{MSAB}{MSE} = \frac{9.2831}{9.6083} = 0.97$$

the p–value is

$$\text{p–value} = P(F \geq 0.97; 6, 12) =$$

(circle one) **0.00** / **0.34** / **0.49**.
The level of significance is 0.05.

iii. *Conclusion*
Since the p–value, 0.49, is larger than the level of significance, 0.05, we (circle one) **accept** / **reject** the null hypothesis that the interaction effect is zero.

iv. In fact, since the interaction is zero, we (choose one) **can** / **cannot** test the main effects.

(b) *Test random factor A (Temperature)*

i. *Statement*
The statement of the test is (check none, one or more):
A. $H_0 : \alpha_1 = \alpha_2 = 0$ versus $H_a : \alpha_1 \neq \alpha_2$.
B. $H_0 : \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ versus $H_a : \alpha_1 \neq \alpha_3, \alpha_1 \neq \alpha_2$.
C. $H_0 : \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ versus
$H_a :$ at least one $\alpha_i \neq 0,\ i = 1, 2, 3, 4$.
D. $H_0 : \sigma^2_\alpha = 0$ versus $H_a : \sigma^2_\alpha > 0$.

ii. *Test*
From SAS, since

$$F = \frac{MSA}{MSAB} = \frac{179.71277}{9.28319} = 19.36$$

the p–value is

$$\text{p–value} = P(F \geq 19.36; 3, 6) =$$

(circle one) **0.002** / **0.34** / **0.49**.
The level of significance is 0.05.

iii. *Conclusion*
Since the p–value, 0.002, is smaller than the level of significance, 0.05, we (circle one) **accept** / **reject** the null hypothesis that the main Temperature effect A is zero.

(c) *Test random factor B (Noise)*

i. *Statement*
The statement of the test is (check none, one or more):

A. $H_0 : \beta_1 = \beta_2 = \beta_3 = 0$ versus $H_a : \beta_i \neq \beta_j$.
B. $H_0 : \beta_1 = \beta_2 = \beta_3 = 0$ versus $H_a : \beta_1 \neq \beta_3, \beta_1 \neq \beta_2$.
C. $H_0 : \beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = 0$ versus
$H_a :$ at least one $\beta_i \neq 0,\ i = 1, 2, 3$.
D. $H_0 : \sigma^2_\beta = 0$ versus $H_a : \sigma^2_\beta > 0$.

ii. *Test*
From SAS, since

$$F = \frac{MSB}{MSAB} = \frac{6.8404}{9.28319} = 0.737$$

the p–value is

$$\text{p–value} = P(F \geq 0.737; 2, 6) =$$

(circle one) **0.002** / **0.34** / **0.52**.
The level of significance is 0.05.

iii. *Conclusion.*
Since the p–value, 0.52, is larger than the level of significance, 0.05, we (circle one) **accept** / **reject** the null hypothesis that the main Noise effect is zero.

2. *Mixed: Temperature is fixed and Noise is random.*

Source	*df*	*SS*	*MS*
Factor A (Temperature)	3	539.138	179.71277
Factor B (Noise)	2	13.6808	6.8404
Interaction AB	6	13.6992	9.28319
Error	12	115.3	9.60833
Total	23	723.8183	

(a) *Test Random Interaction AB*

i. *Statement*
The statement of the test is (check none, one or more):
A. $H_0 :$ all $(\alpha\beta)_{ij} = 0$ versus $H_a :$ not all $(\alpha\beta)_{ij} = 0$.
B. $H_0 : \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ versus $H_a : \alpha_1 \neq \alpha_3, \alpha_1 \neq \alpha_2$.
C. $H_0 : \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ versus
$H_a :$ at least one $\alpha_i \neq 0,\ i = 1, 2, 3, 4$.
D. $H_0 : \sigma^2_{\alpha\beta} = 0$ versus
$H_a : \sigma^2_{\alpha\beta} > 0$.

ii. *Test*
From SAS, since

$$F = \frac{MSAB}{MSE} = \frac{9.2831}{9.6083} = 0.97$$

the p–value is

$$\text{p–value} = P(F \geq 0.97; 6, 12) =$$

(circle one) **0.00** / **0.34** / **0.49**.
The level of significance is 0.05.

iii. *Conclusion*
Since the p–value, 0.49, is larger than the level of significance, 0.05, we (circle one) **accept** / **reject** the null hypothesis that the interaction effect is zero.

iv. In fact, since the interaction is zero, we (choose one) **can** / **cannot** test the main effects.

(b) *Test fixed factor A (Temperature)*

i. *Statement*
The statement of the test is (check none, one or more):

A. $H_0 : \alpha_1 = \alpha_2 = 0$ versus $H_a : \alpha_1 \neq \alpha_2$.
B. $H_0 : \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ versus $H_a : \alpha_1 \neq \alpha_3, \alpha_1 \neq \alpha_2$.
C. $H_0 : \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ versus
$H_a :$ at least one $\alpha_i \neq 0,\ i = 1, 2, 3, 4$.
D. $H_0 : \sigma^2_\alpha = 0$ versus
$H_a : \sigma^2_\alpha > 0$.

ii. *Test*
From SAS, since

$$F = \frac{MSA}{MSAB} = \frac{179.71277}{9.28319} = 19.36$$

the p–value is

$$\text{p–value} = P(F \geq 19.36; 3, 6) =$$

(circle one) **0.002** / **0.34** / **0.49**.
The level of significance is 0.05.

iii. *Conclusion*
Since the p–value, 0.002, is smaller than the level of significance, 0.05, we (circle one) **accept** / **reject** the null hypothesis that the main Temperature effect A is zero.

(c) *Test random factor B (Noise)*

i. *Statement*
The statement of the test is (check none, one or more):

A. $H_0 : \beta_1 = \beta_2 = \beta_3 = 0$ versus $H_a : \beta_i \neq \beta_j$.
B. $H_0 : \beta_1 = \beta_2 = \beta_3 = 0$ versus $H_a : \beta_1 \neq \beta_3, \beta_1 \neq \beta_2$.
C. $H_0 : \beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = 0$ versus
$H_a :$ at least one $\beta_i \neq 0,\ i = 1, 2, 3$.

D. $H_0 : \sigma_\beta^2 = 0$ versus
$H_a : \sigma_\beta^2 > 0$.

ii. *Test*
From SAS, since

$$F = \frac{MSB}{MSE} = \frac{6.804}{9.60833} = 7.08$$

the p–value is

$$\text{p–value} = P(F \geq 7.08; 2, 12) =$$

(circle one) **0.009** / **0.34** / **0.51**.
The level of significance is 0.05.

iii. *Conclusion*
Since the p–value, 0.51, is larger than the level of significance, 0.05, we (circle one) **accept** / **reject** the null hypothesis that the main Noise effect is zero.

24.4 Two–Factor Studies—ANOVA Estimation of Factor Effects for Models II and III

SAS program: att6-24-4-roc-twoANOVAII,III-effects

Detailed inference of factor effects of Models II and III is discussed in this section.

Exercise 24.4 (Two–Factor Studies—ANOVA Estimation of Factor Effects for Models II and III)
Consider the effect of air temperature *and* noise on the ROC of deer mice.

	Factor B, noise →	$j = 1$, low	$j = 2$, medium	$j = 3$, high	column ave
Factor A,	$i = 1$, 0^o F	10.3, 7.2	9.1, 5.4	6.1, 2.1	$\overline{Y}_{1..} = 6.7$
temperature	$i = 2$, 10^o F	1.8, 9.8	12.1, 4.2	5.1, 6.2	$\overline{Y}_{2..} = 6.5$
	$i = 3$, 20^o F	1.2, 8.1	6.5, 4.1	1.2, 2.1	$\overline{Y}_{3..} = 3.9$
	$i = 4$, 30^o F	12.4, 15.1	16.1, 17.2	18.1, 19.1	$\overline{Y}_{1..} = 16.3$
	column ave	$\overline{Y}_{.1.} = 8.2$	$\overline{Y}_{.2.} = 9.3$	$\overline{Y}_{.3.} = 7.5$	$\overline{Y}_{...} = 8.4$

1. *Random model ANOVA II*

Source	*df*	*SS*	*MS*
Factor A (Temperature)	3	539.138	179.71277
Factor B (Noise)	2	13.6808	6.8404
Interaction AB	6	13.6992	9.28319
Error	12	115.3	9.60833
Total	23	723.8183	

(a) *Random model ANOVA II*
For the factor effects version of random model ANOVA II,

$$Y_{ijk} = \mu_{\cdot\cdot} + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk},$$

(choose one)

i. both Factor A (Temperature), α_i, and Factor B (Noise), β_j, is random.
ii. Factor A (Temperature), α_i, is fixed and Factor B (Noise), β_j, is random.

and also interaction $(\alpha\beta)_{ij}$ is random.

(b) *Point estimate of σ^2_α.*

$$\sigma^2_\alpha \approx s^2_\alpha = \frac{MSA - MSE}{nb} = \frac{179.71277 - 9.60833}{2(3)} =$$

(circle one) **10.0** / **12.34** / **28.35**.

(c) *Point estimates of σ^2_β.*

$$\sigma^2_\beta \approx s^2_\beta = \frac{MSB - MSE}{na} = \frac{6.8404 - 9.60833}{2(4)} =$$

(circle one) **−0.346** / **−2.34** / **−8.35**
which is *not* possible because σ^2_β cannot be negative; these estimates can only be used for *significant* random main effects.

(d) *95% Confidence interval for σ^2_α, using MLS method.*
Since

$$\begin{aligned} c_1 &= 1/bn = 1/((3)(2)) = 1/6 \\ c_2 &= -1/bn = -1/6 \\ df_1 &= a-1 = 4-1 = 3 \\ df_2 &= (a-1)(b-1) = (4-1)(3-1) = 6 \\ MS_1 &= MSA = 179.71277 \\ MS_2 &= MSE = 9.60833 \end{aligned}$$

and

$$\begin{aligned} F_1(1-\alpha/2; df_1, \infty) &= F(1-0.05/2; 3, \infty) = 3.12 \\ F_2(1-\alpha/2; df_2, \infty) &= F(1-0.05/2; 6, \infty) = 2.41 \\ F_3(1-\alpha/2; \infty, df_1) &= F(1-0.05/2; \infty, 3) = 13.9 \\ F_4(1-\alpha/2; \infty, df_2) &= F(1-0.05/2; \infty, 6) = 2.41 \\ F_5(1-\alpha/2; df_1, df_1) &= F(1-0.05/2; 3, 6) = 6.60 \\ F_6(1-\alpha/2; df_2, df_1) &= F(1-0.05/2; 6, 3) = 14.73 \end{aligned}$$

and

$$\begin{aligned}
G_1 &= 1 - 1/F_1 \;=\; 1 - 1/3.12 \;=\; 0.6795 \\
G_2 &= 1 - 1/F_2 \;=\; 1 - 1/2.41 \;=\; 0.5851 \\
G_3 &= \frac{(F_5-1)^2 - (G_1F_5)^2 - (F_4-1)^2}{F_5} \\
&= \frac{(6.60-1)^2 - ((0.6795)(6.60))^2 - (2.41-1)^2}{6.60} \\
&= 1.403 \\
G_4 &= F_6\left(\left(\frac{F_6-1}{F_6}\right)^2 - \left(\frac{F_3-1}{F_6}\right)^2 - G_2^2\right) \\
&= 14.73\left(\left(\frac{14.73-1}{14.73}\right)^2 - \left(\frac{13.9-1}{14.73}\right)^2 - 0.5851^2\right) \\
&= -3.542
\end{aligned}$$

and

$$\begin{aligned}
H_L &= \sqrt{(G_1c_1MS_1)^2 + ((F_4-1)c_2MS_2)^2 - G_3c_1c_2MS_1MS_2} \\
&= \sqrt{((0.6795)(1/6)(179.71277))^2 + ((2.41-1)(-1/6)(9.60833))^2 - (1.403)(1/6)(-1/6)(179.71277)(9.60833)} \\
&= 22.06 \\
H_U &= \sqrt{(((F_3-1)c_1MS_1)^2 + G_2c_2MS_2)^2 - G_4c_1c_2MS_1MS_2} \\
&= \sqrt{(((13.9-1)(1/6)(179.71277))^2 + ((0.5851)(-1/6)(9.60833))^2 - (-3.542)(1/6)(-1/6)(179.71277)(9.60833)} \\
&= 386.16
\end{aligned}$$

and since $s_\alpha^2 = 28.35$,
$28.35 - 22.06 = 6.29$ and $28.35 + 386.16 = 414.51$
and so the approximate confidence interval for σ_α^2 is
(choose one) $\mathbf{(0.995, 0.999)}$ / $\mathbf{(6.29, 414.51)}$ / $\mathbf{(4.348, 658.5)}$

(e) *95% Confidence interval for σ_α^2, using Satterthwaite method.*
Since

$$\begin{aligned}
df &= \frac{(bn \times s_\alpha^2)^2}{\frac{(MSA)^2}{a-1} + \frac{(MSE)^2}{(a-1)(b-1)}} \\
&= \frac{((3)(2)(28.35))^2}{\frac{(179.71277)^2}{4-1} + \frac{(9.60833)^2}{(4-1)(3-1)}} \\
&= 2.68 \\
\chi^2(1-\alpha/2; df) &= \chi^2(1-0.05/2; 3) \;=\; 9.35 \\
\chi^2(\alpha/2; df) &= \chi^2(0.05/2; 3) \;=\; 0.216
\end{aligned}$$

and so the approximate confidence interval for σ_α^2 is

$$\frac{(df)(\sigma_\alpha^2)}{\chi^2(1-\alpha/2; df)} \le \sigma_\alpha^2 \le \frac{(df)(\sigma_\mu^2)}{\chi^2(\alpha/2; df)} =$$

or

$$\frac{(2.68)(28.35)}{9.35} \leq \sigma_\alpha^2 \leq \frac{(2.68)(28.35)}{0.216}$$

(choose one) **(0.995, 0.999)** / **(6.29, 414.51)** / **(8.13, 351.75)**

2. *Mixed model ANOVA III*

Source	*df*	*SS*	*MS*
Factor A (Temperature)	3	539.138	179.71277
Factor B (Noise)	2	13.6808	6.8404
Interaction AB	6	13.6992	9.28319
Error	12	115.3	9.60833
Total	23	723.8183	

(a) *Mixed model ANOVA III*
For the factor effects version of mixed model ANOVA III,

$$Y_{ijk} = \mu_{\cdot\cdot} + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk},$$

(choose one)

i. both Factor A (Temperature), α_i, and Factor B (Noise), β_j, is random.
ii. Factor A (Temperature), α_i, is fixed and Factor B (Noise), β_j, is random.

and also interaction $(\alpha\beta)_{ij}$ is random.

(b) *Point estimate of $\sigma^2\{\hat{\alpha}_i\}$, α_i fixed.*

$$\sigma^2\{\hat{\alpha}_i\} \approx s^2\{\hat{\alpha}_i\} = \frac{MSAB}{nb}$$

(circle one) **1.55** / **2.32** / **28.35**

(c) *Point estimate of σ_β^2, β_i random.*

$$\sigma_\beta^2 \approx s_\beta^2 = \frac{MSB - MSE}{na} = \frac{6.8404 - 9.60833}{2(4)} =$$

(circle one) **−0.346** / **−2.34** / **−8.35**
which, notice, is the same (bad, unusable) estimate used for the random ANOVA II model.

(d) *Point estimate of $\sigma^2\{\hat{\mu}_{i\cdot}\}$, marginal mean $\hat{\mu}_{i\cdot}$.*

$$\begin{aligned} \sigma^2\{\hat{\mu}_{i\cdot}\} \approx s^2\{\hat{\mu}_{i\cdot}\} &= \frac{a-1}{nab}MSAB + \frac{1}{nab}MSB \\ &= \frac{4-1}{(2)(4)(3)}(9.28319) + \frac{1}{(2)(4)(3)}(6.8404) \\ &= \end{aligned}$$

(circle one) **0.346** / **1.45** / **8.35**

(e) *95% Confidence interval for $\mu_{2\cdot}$, using Satterthwaite method.*
Since,

$$\begin{aligned}
\hat{\mu}_{2\cdot} &= 6.5 \\
c_1 &= \frac{a-1}{nab} = \frac{4-1}{(2)(4)(2)} = 3/24 \\
c_2 &= \frac{1}{nab} = \frac{1}{(2)(4)(2)} = 1/24 \\
s^2\{\hat{\mu}_{2\cdot}\} &= 1.45 \\
df &= \frac{(c_1 MSAB + c_2 MSB)^2}{\frac{(c_1 MSAB)^2}{(a-1)(b-1)} + \frac{(c_2 MSB)^2}{b-1}} \\
&= \frac{(1.45)^2}{\frac{((3/24)(9.28319))^2}{(4-1)(3-1)} + \frac{((1/24)(6.8404))^2}{3-1}} \\
&= 7.93 \\
t(1-\alpha/2; df) &= t(1-0.05/2; 8) = 2.31
\end{aligned}$$

then

$$\hat{\mu}_{2\cdot} \pm t(1-\alpha/2; df)s\{\hat{\mu}_{2\cdot}\} = 6.5 \pm (2.31)\sqrt{1.45} =$$

(choose one) **(3.15, 9.85)** / **(3.72, 9.28)** / **(8.13, 351.75)**

(f) *Test of Contrasts For Fixed Effects.*
The test statistic for contrast L is given by

$$\frac{\hat{L} - L}{s\{\hat{L}\}}, \quad s\{\hat{L}\} = \sqrt{\frac{MSAB}{bn}\sum c_i^2}$$

and is distributed as $t[(a-1)(b-1)]$.
Test $D = L = \alpha_1 - \alpha_2 = \mu_{1\cdot} - \mu_{2\cdot}$ at $\alpha = 0.05$ (where $c_1 = 1$ and $c_2 = -1$)

i. *Statement.*
The statement of the test is (check none, one or more):
A. $H_0: \beta_1 = \beta_2 = \beta_3 = 0$ versus $H_a: \beta_i \neq \beta_j,\ i \neq j,\ i,j = 1,2,3.$
B. $H_0: \beta_1 = \beta_2 = \beta_3 = 0$ versus $H_a: \beta_1 \neq \beta_3, \beta_1 \neq \beta_2.$
C. $H_0: L = 0$ versus $H_a: L \neq 0.$
D. $H_0: \sigma_\beta^2 = 0$ versus $H_a: \sigma_\beta^2 > 0.$

ii. *Test*
Since

$$s\{\hat{L}\} = \sqrt{\frac{MSAB}{bn}\sum c_i^2} = \sqrt{\frac{9.28319}{(3)(2)}(1^2 + (-1)^2)} = 1.76$$

and

$$\hat{L} = \bar{Y}_{1\cdot\cdot} - \bar{Y}_{2\cdot\cdot} = 6.7 - 6.5 = -0.2$$

and so the test statistic is

$$\frac{\hat{L} - L}{s\{\hat{L}\}} = \frac{-0.2 - 0}{1.76}$$

which equals[3] (circle one) **−0.11** / **0.34** / **0.52**.
The level of significance is 0.05.

iii. *Conclusion*
Since the p–value, 0.46, is larger than the level of significance, 0.05, we (circle one) **accept** / **reject** the null hypothesis that the contrast L is zero.

24.5 Three–Factor Studies—ANOVA Models II and III

SAS program: att6-24-5-roc-threeANOVAII,III

An example of the ANOVA Model II is given in this section.

Mean Square	df	Random ANOVA Model II (A, B and C Random)	Mixed ANOVA Model II (A fixed, B and C random)
Factor A	$a-1$	$\sigma^2 + nbc\sigma^2_\alpha + nc\sigma^2_{\alpha\beta} + nb\sigma^2_{\alpha\gamma} + n\sigma^2_{\alpha\beta\gamma}$	$\sigma^2 + nbc\frac{\alpha_i^2}{a-1} + nc\sigma^2_{\alpha\beta} + nb\sigma^2_{\alpha\gamma} + n\sigma^2_{\alpha\beta\gamma}$
Factor B	$b-1$	$\sigma^2 + nac\sigma^2_\beta + nc\sigma^2_{\alpha\beta} + na\sigma^2_{\beta\gamma} + n\sigma^2_{\alpha\beta\gamma}$	$\sigma^2 + nac\sigma^2_\beta + na\sigma^2_{\beta\gamma}$
Factor B	$c-1$	$\sigma^2 + nab\sigma^2_\gamma + nb\sigma^2_{\alpha\gamma} + na\sigma^2_{\beta\gamma} + n\sigma^2_{\alpha\beta\gamma}$	$\sigma^2 + nab\sigma^2_\gamma + na\sigma^2_{\beta\gamma}$
Interaction AB	$(a-1)(b-1)$	$\sigma^2 + nc\sigma^2_{\alpha\beta} + n\sigma^2_{\alpha\beta\gamma}$	$\sigma^2 + nc\sigma^2_{\alpha\beta} + n\sigma^2_{\alpha\beta\gamma}$
Interaction AC	$(a-1)(c-1)$	$\sigma^2 + nb\sigma^2_{\alpha\gamma} + n\sigma^2_{\alpha\beta\gamma}$	$\sigma^2 + nb\sigma^2_{\alpha\gamma} + n\sigma^2_{\alpha\beta\gamma}$
Interaction BC	$(b-1)(c-1)$	$\sigma^2 + na\sigma^2_{\beta\gamma} + n\sigma^2_{\alpha\beta\gamma}$	$\sigma^2 + na\sigma^2_{\beta\gamma}$
Interaction ABC	$(a-1)(b-1)(c-1)$	$\sigma^2 + n\sigma^2_{\alpha\beta\gamma}$	$\sigma^2 + n\sigma^2_{\alpha\beta\gamma}$
Error	$(n-1)abc$	σ^2	σ^2

Exercise 24.5 (Three–Factor Studies—ANOVA Models II and III)
Data for the Male mice level of Factor A, (when $i = 1$) is given by

[3]The p–value is

$$P(t < -0.11; (3-1)(2-1)) = P(t < -0.11; 2) = 0.46$$

	Factor C, noise →	$k = 1$, low	$k = 2$, medium	$k = 3$, high	column ave
Factor B,	$j = 1$, 0^o F	10.3, 7.2	9.1, 5.4	6.1, 2.1	$\overline{Y}_{11\cdot\cdot} = 6.7$
temperature	$j = 2$, 10^o F	1.8, 9.8	12.1, 4.2	5.1, 6.2	$\overline{Y}_{12\cdot\cdot} = 6.5$
	$j = 3$, 20^o F	1.2, 8.1	6.5, 4.1	1.2, 2.1	$\overline{Y}_{13\cdot\cdot} = 3.9$
	$j = 4$, 30^o F	12.4, 15.1	16.1, 17.2	18.1, 19.1	$\overline{Y}_{14\cdot\cdot} = 16.3$
	column ave	$\overline{Y}_{1\cdot 1\cdot} = 8.2$	$\overline{Y}_{1\cdot 2\cdot} = 9.3$	$\overline{Y}_{1\cdot 3\cdot} = 7.5$	$\overline{Y}_{1\cdot\cdot\cdot} = 8.4$

and data for the Female mice level of Factor A, $(i = 2)$ is given by,

	Factor C, noise →	$k = 1$, low	$k = 2$, medium	$k = 3$, high	column ave
Factor B,	$j = 1$, 0^o F	9.3, 6.2	8.1, 5.4	8.1, 7.1	$\overline{Y}_{21\cdot\cdot} = 7.4$
temperature	$j = 2$, 10^o F	2.8, 9.8	10.1, 4.2	5.1, 6.2	$\overline{Y}_{22\cdot\cdot} = 6.4$
	$j = 3$, 20^o F	4.2, 5.1	7.5, 4.1	6.2, 7.1	$\overline{Y}_{23\cdot\cdot} = 5.7$
	$j = 4$, 30^o F	5.4, 5.1	11.1, 12.2	8.1, 9.1	$\overline{Y}_{24\cdot\cdot} = 8.5$
	column ave	$\overline{Y}_{2\cdot 1\cdot} = 6.0$	$\overline{Y}_{2\cdot 2\cdot} = 7.8$	$\overline{Y}_{2\cdot 3\cdot} = 7.1$	$\overline{Y}_{2\cdot\cdot\cdot} = 7.0$

Source	*df*	*SS*	*MS*
Factor A (Gender)	1	6.02	6.02
Factor B (Temperature)	3	257.22	85.74
Factor C (Noise)	2	1.82	0.91
Interaction AB	3	85.56	28.52
Interaction AC	2	15.79	7.90
Interaction BC	6	61.54	10.26
Interaction ABC	6	113.88	18.98
Error	24	284.9	11.87
Total	47	826.73	

Test whether a main effect and an interaction are significant at $\alpha = 0.05$, assuming Model II, where all three factors, gender, temperature and noise, are random.

1. *Model II: All three factors, gender, temperature and noise, are random; Test random interaction BC (Temperature, Noise)*

   (a) *Statement*
   The statement of the test is (check none, one or more):

   i. $H_0 : \sigma^2_\alpha = 0$ versus $H_a :\ \sigma^2_\alpha \neq 0$.

   ii. $H_0 : \beta_1 = \beta_2 = \beta_3 = 0$ versus $H_a :\ \beta_1 \neq \beta_3, \beta_1 \neq \beta_2$.

   iii. $H_0 : \beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = 0$ versus
   $H_a :$ at least one $\beta_i \neq 0,\ i = 1, 2, 3$.

   iv. $H_0 : \sigma^2_{\beta\gamma} = 0$ versus $H_a :\ \sigma^2_{\beta\gamma} > 0$.

(b) *Test*
Since the test statistic is

$$F = \frac{MSBC}{MSABC} = \frac{10.25667}{18.9791} = 0.54$$

the p–value, with 6 and 6 degrees of freedom, is given by

$$\text{p–value} = P(F \geq 0.54)$$

which equals (circle one) **0.00** / **0.34** / **0.76**.
The level of significance is 0.05.

(c) *Conclusion*
Since the p–value, 0.76, is larger than the level of significance, 0.05, we (circle one) **accept** / **reject** the null hypothesis that the interaction BC is zero.

2. *Model II: All three factors, gender, temperature and noise, are random; test factor A (Gender), Satterthwaite procedure*

(a) *Statement*
The statement of the test is (check none, one or more):

i. $H_0 : \sigma^2_\alpha = 0$ versus $H_a : \sigma^2_\alpha > 0$.
ii. $H_0 : \beta_1 = \beta_2 = \beta_3 = 0$ versus $H_a : \beta_1 \neq \beta_3, \beta_1 \neq \beta_2$.
iii. $H_0 : \beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = 0$ versus
$H_a :$ at least one $\beta_i \neq 0, \ i = 1, 2, 3$.
iv. $H_0 : \sigma^2_\beta = 0$ versus
$H_a : \sigma^2_\beta > 0$.

(b) *Test*
Since the test statistic is

$$F^* = \frac{MSA}{MSAB + MSAC - MSABC} = \frac{6.020833}{28.52080 + 7.895833 - 18.9791667} = 0.345$$

the p–value, with 1 degree of freedom (associated with factor S) and

$$df = \frac{(28.52080 + 7.895833 - 18.9791667)^2}{28.52^2/3 + 7.896^2/2 + 18.979^2/6} \approx 0.84$$

(associated with AB, AC and ABC) which is "rounded" to one (1) degree of freedom, is given by

$$\text{p–value} = P(F \geq 0.345; 1, 1)$$

which equals (circle one) **0.00** / **0.34** / **0.66**.
The level of significance is 0.05.

(c) *Conclusion.*
Since the p–value, 0.66, is larger than the level of significance, 0.05, we (circle one) **accept** / **reject** the null hypothesis that the main Gender effect is zero.

24.6 ANOVA Models II and III with Unequal Sample Sizes

This material is not covered.