Chapter 7

Process and Measurement System Capability Analysis

7.1 Introduction

Our main goal in this chapter is to look at different ways to measure the capability of a process. We also look at related issues of calculating measurement error.

7.2 Process Capability Analysis Using a Histogram

This section is more or less a review of previous material on the histogram and on normal probability plots. Two new statistics, $\hat{\beta}_1$ and $\hat{\beta}_2$, which measure *skewness* (how asymmetric a probability distribution is) and *kurtosis* (how peaked/flat a probability distribution is), respectively, are discussed, where

$$\sqrt{\hat{\beta}_1} = \frac{M_3}{(M_2)^{3/2}}, \quad \hat{\beta}_2 = \frac{M_4}{M_2^2}$$

where the moments are calculated by

$$M_j = \frac{1}{n}\sum_{j=1}^{n}(x_i - \bar{x})^j, \quad j = 1, 2, 3, 4$$

It is shown how different values of skewness and kurtosis can be used to identify probability distributions (not necessarily the normal) which best describes sampled data.

7.3 Process Capability Ratios

SAS program: att6-7-3-icecream-capability

We look at point estimates and confidence intervals of various measures of the capability of a process, including

- Potential capability, $\hat{C}_p$
- Actual capability, $\hat{C}_{pk}$
- Capability which accounts for off–centering, $\hat{C}_{pm}$
- Third generation capability, $\hat{C}_{pkm}$

Exercise 7.1 (Process Capability Ratios)

1. *Ice Cream Treat Weight, Raw Data*
   Ice cream treats are weighed (in ounces), four at a time ($n = 4$).

sample	x_1	x_2	x_3	x_4
1	7	9	12	14
2	9	5	5	12
3	7	10	11	7
4	8	9	5	14
5	7	8	11	12
6	12	11	9	10
7	16	9	10	7
8	6	5	11	3
9	8	7	10	14
10	15	17	9	14
11	7	12	13	17
12	5	11	8	12
13	16	10	13	14
14	8	13	9	13
15	11	8	11	16
16	15	11	10	14
17	10	9	11	10
18	16	8	11	12
19	8	7	10	13
20	15	17	10	15

   If specifications are 10.05 ± 3.5, calculate the point estimates and confidence intervals of the various capability measures.

(a) *Potential capability,* $\hat{C}_p$
From SAS,
$\bar{\bar{x}}$ = (choose one) **10.45** / **10.55** / **10.65**
$\bar{R}$ = (choose one) **6.6** / **7.6** / **8.6**
$\hat{\sigma}$ = (choose one) **3.18** / **4.18** / **5.18**
(which also, in addition to the SAS output, equals $\bar{R}/d_2 = 6.6/2.059$)
Also, the specification limits are,
LSL = (choose one) **6.55** / **7.55** / **8.55**
USL = (choose one) **11.55** / **12.55** / **13.55**
and so

$$\hat{C}_p = \frac{USL - LSL}{6\hat{\sigma}} =$$

(choose one) **0.25** / **0.37** / **0.55**
(which can also be found on the SAS output)

If a value of 1.33 is considered to be a *minimally* acceptable value of potential capability, then the ice cream process given here
(choose one) **is** / **is not** capable.

Related to this, since

$$P = \frac{1}{\hat{C}_p} = \frac{1}{0.37} = 2.70,$$

the *product variability* is approximately 270% of the total specification band (which is terrible).

(b) *Actual capability,* $\hat{C}_{pk}$
From SAS,
lower $\hat{C}_{pl}$ = (choose one) **0.25** / **0.42** / **0.55**
(which equals $\frac{\hat{\mu}-LSL}{3\hat{\sigma}}) = \frac{10.55-6.55}{3(3.18)})$
upper $\hat{C}_{pu}$ = (choose one) **0.25** / **0.31** / **0.55**
(which equals $\frac{USL-\hat{\mu}}{3\hat{\sigma}}) = \frac{13.55-10.55}{3(3.18)}$
then the minimum of the two, $\hat{C}_{pk}$ = (choose one) **0.25** / **0.31** / **0.42**
(which equals $\min(\hat{C}_{pl}, \hat{C}_{pu})) = \min(0.42, 0.31))$

Since $\hat{C}_{pk} = 0.31 < \hat{C}_p = 0.37$, the process mean is off–center, and is, in fact, *above* the specified (nominal) mean since $\hat{C}_{pk} = \hat{C}_{pu} = 0.31$, the upper capability value.

(c) *Comparing* $\hat{C}_p$, $\hat{C}_{pk}$

The $\hat{C}_{pk}$ is said to be the *actual* capability since it accounts for where the process is relative to the specification limits. Since $\hat{C}_p$ does not do this, it said to be the *potential* capability; the capability that *would* occur *if* the process mean happened to match the specification or nominal mean.

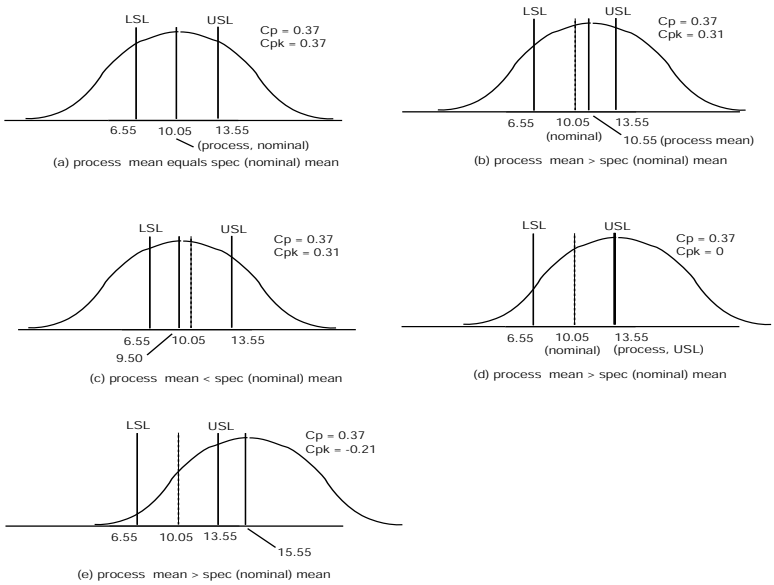

Figure 7.1 (Comparing $\hat{C}_p$, $\hat{C}_{pk}$)

Of the five graphs given in the figure above, the one figure which best describes the ice cream data here is graph
(choose one) **(a)** / **(b)** / **(c)** / **(d)** / **(e)**
Also, from the figure above, $\hat{C}_{pk} = 0.31$ when (choose best one)

i. the process is off–center *above* the specification (nominal) mean
ii. the process is off–center *below* the specification (nominal) mean
iii. the process is off–center *either* above or below the specification mean

(d) *Capability which accounts for off–centering,* $\hat{C}_{pm}$
Since the *target* (or, equivalently, specification (nominal) mean) is
$T =$ (choose one) **10.05** / **10.55** / **13.55**
then $V = \frac{\bar{\bar{x}}-T}{\hat{\sigma}} = \frac{10.55-10.05}{3.18} =$ (choose one) **0.16** / **0.31** / **0.42**
and[1]

$$\hat{C}_{pm} = \frac{\hat{C}_p}{\sqrt{1+V^2}} = \frac{0.37}{\sqrt{1+(0.16)^2}} =$$

(choose one) **0.36** / **0.39** / **0.42**
which, like $\hat{C}_{pk}$ indicates the process is off–center because $\hat{C}_{pm} = 0.36 < \hat{C}_p = 0.37$, but, more than this, $\hat{C}_{pm}$ can tell us if process mean is in the specification band; specifically, since
$\frac{6\hat{C}_{pm}}{2} =$ (choose one) **0.57** / **1.08** / **1.42**
this indicates the process mean is *in* (as opposed to outside)
of the specification band.

(e) *More on* $\hat{C}_{pm}$
If $\frac{6\hat{C}_{pm}}{2} = 2$ this would have indicated that the process mean was in the *middle half* of the specification band.
If $\frac{6\hat{C}_{pm}}{2} = 5$ this would have indicated that the process mean was in the *middle fifth* of the specification band.
If $0 \leq \frac{6\hat{C}_{pm}}{2} \leq 1$ this would have indicated that the process mean was inside (as opposed to outside) the specification band.
If $\frac{6\hat{C}_{pm}}{2} \leq 0$ this would have indicated that the process mean was outside the specification band.
In other words, the larger $\frac{6\hat{C}_{pm}}{2}$ is, the (choose one) **less** / **more** precisely known where the process mean is in the specification band.

(f) *Understanding* $\hat{C}_{pm}$
Using the ice cream data,

$$\begin{aligned} |\mu - T| &< \frac{1}{6\hat{C}_{pm}}(USL - LSL) \\ |\mu - T| &< \frac{1}{6(0.36)}(USL - LSL) \\ |\mu - T| &< \frac{1}{2.16}(USL - LSL) \end{aligned}$$

which says (choose one)

[1]Notice that if $V = 0$ (centered, $\bar{\bar{x}} = T$), then

$$\hat{C}_{pm} = \frac{\hat{C}_p}{\sqrt{1+0^2}} = \hat{C}_p$$

i. that since the distance between the process mean (μ) and the specification mean (or target, T) is less than one *half* of the specification band, then the process mean must be in either half of the specification band; in other words, in the specification band.

ii. that since the distance between the process mean (μ) and the specification mean (or target, T) is less than one *quarter* of the specification band, then the process mean must be in either quarter above or below the target mean in (in other words, in the middle half of) the specification band.

(g) *Third generation capability, $\hat{C}_{pkm}$*
Since $\hat{\xi} = V = \frac{\hat{\mu}-T}{\hat{\sigma}} = \frac{10.55-10.05}{3.18} = 0.16$, then

$$\hat{C}_{pkm} = \frac{\hat{C}_{pk}}{\sqrt{1+\xi^2}} = \frac{0.31}{\sqrt{1+0.16^2}} =$$

(choose one) **0.31** / **1.08** / **1.42**
The $\hat{C}_{pkm}$ measure of capability is a *third* generation measure because it is calculated from the two *second* generation measure of capability, $\hat{C}_{pk}$. Notice that the $\hat{C}_{pkm}$ is the same as $\hat{C}_{pm}$, only $\hat{C}_p$ is used in the first case and $\hat{C}_{pk}$ is used in the second case.

2. *Another example, summary statistics*
Let $n = 20$, $\bar{\bar{x}} = 70$, $\hat{\sigma} = S = 0.75$, $LSL = 67$, $USL = 73$

(a) *Potential capability, $\hat{C}_p$, point estimate*
Since $n = 20$, $\bar{\bar{x}} = 70$, $\hat{\sigma} = S = 0.75$, $LSL = 67$, $USL = 73$,
and so $\hat{C}_p = \frac{USL-LSL}{6\hat{\sigma}} = \frac{73-67}{6(0.75)} =$
(choose one) **0.31** / **1.08** / **1.33**

(b) *Potential capability, $\hat{C}_p$, 95% confidence interval*[2]
Since $n = 20$, $\bar{\bar{x}} = 70$, $\hat{\sigma} = S = 0.75$, $LSL = 67$, $USL = 73$,

$$\begin{aligned}
\hat{C}_p\sqrt{\frac{\chi^2_{1-\alpha/2,n-1}}{n-1}} &\leq C_p \leq \hat{C}_p\sqrt{\frac{\chi^2_{\alpha/2,n-1}}{n-1}} \\
\left(\frac{4}{3}\right)\sqrt{\frac{\chi^2_{1-0.05/2,20-1}}{20-1}} &\leq C_p \leq \left(\frac{4}{3}\right)\sqrt{\frac{\chi^2_{0.05/2,20-1}}{20-1}} \\
\left(\frac{4}{3}\right)\sqrt{\frac{8.91}{20-1}} &\leq C_p \leq \left(\frac{4}{3}\right)\sqrt{\frac{32.85}{20-1}}
\end{aligned}$$

[2]where $\chi^2_{0.05/2,25-1}$, for example, is calculated using
PRGM INVCHI2 24 ENTER 0.025 ENTER

(choose one)
$\mathbf{0.80 \le C_p \le 1.42}$ / $\mathbf{0.91 \le C_p \le 1.75}$ / $\mathbf{1.03 \le C_p \le 1.82}$
The interval is too wide to be useful. It stretches from capable (1.75) to incapable (0.91); using table 7-3, page 360, the parts per million (ppm) defective ranges from 6,934 to around 0.34.

(c) *Actual capability, $\hat{C}_{pk}$, point estimate*
Since $n = 20$, $\bar{\bar{x}} = 70$, $\hat{\sigma} = S = 0.75$, $LSL = 67$, $USL = 73$,
then $\hat{C}_{pl} = \frac{\hat{\mu}-LSL}{3\hat{\sigma}} = \frac{70-67}{3(0.75)} = \frac{4}{3}$
and $\hat{C}_{pu} = \frac{USL-\hat{\mu}}{3\hat{\sigma}} = \frac{73-70}{3(0.75)} = \frac{4}{3}$
and $\hat{C}_{pk} = \min(\hat{C}_{pl}, \hat{C}_{pu}) = \min\left(\frac{4}{3}, \frac{4}{3}\right) =$
(choose one) $\mathbf{\frac{3}{3}}$ / $\mathbf{\frac{4}{3}}$ / $\mathbf{\frac{5}{3}}$

(d) *Actual capability, $\hat{C}_{pk}$, 95% confidence interval*[3]
Since $n = 20$, $\bar{\bar{x}} = 70$, $\hat{\sigma} = S = 0.75$, $LSL = 67$, $USL = 73$,

$$\begin{aligned}
\hat{C}_{pk}\left(1 - z_{\alpha/2}\sqrt{\frac{1}{9n\hat{C}_{pk}^2} + \frac{1}{2(n-1)}}\right) &\le C_{pk} \le \hat{C}_{pk}\left(1 + z_{\alpha/2}\sqrt{\frac{1}{9n\hat{C}_{pk}^2} + \frac{1}{2(n-1)}}\right) \\
\left(\frac{4}{3}\right)\left(1 - z_{0.05/2}\sqrt{\frac{1}{9(20)(4/3)} + \frac{1}{2(20-1)}}\right) &\le C_{pk} \le \left(\frac{4}{3}\right)\left(1 + z_{0.05/2}\sqrt{\frac{1}{9(20)(4/3)} + \frac{1}{2(20-1)}}\right) \\
\left(\frac{4}{3}\right)\left(1 - 1.96\sqrt{\frac{1}{9(20)(4/3)} + \frac{1}{2(20-1)}}\right) &\le C_{pk} \le \left(\frac{4}{3}\right)\left(1 + 1.96\sqrt{\frac{1}{9(20)(4/3)} + \frac{1}{2(20-1)}}\right)
\end{aligned}$$

(choose one)
$\mathbf{0.88 \le C_{pk} \le 1.79}$ / $\mathbf{0.91 \le C_{pk} \le 1.75}$ / $\mathbf{1.03 \le C_{pk} \le 1.82}$

(e) *Probability of type I error*
The probability of making a type I error (of accidently saying the process is out of control) is,
since $n = 20$, $\bar{\bar{x}} = 70$, $\hat{\sigma} = S = 0.75$, $LSL = 67$, $USL = 73$,

$$\begin{aligned}
& P\{\text{accidently saying out of control}|\hat{\mu} = 70\} \\
= \; & 1 - P\{\text{correctly saying in control}|\hat{\mu} = 70\} \\
= \; & 1 - P\{LSL \le \bar{\bar{x}} \le USL|\hat{\mu} = 70\} \\
= \; & 1 - [P\{\bar{\bar{x}} \le USL|\hat{\mu} = 70\} - P\{\bar{\bar{x}} < LSL|\hat{\mu} = 70\}] \\
= \; & 1 - P\{\bar{\bar{x}} \le USL|\hat{\mu} = 70\} + P\{\bar{\bar{x}} < LSL|\hat{\mu} = 70\} \\
= \; & 1 - P\{Z \le \frac{USL - \hat{\mu}}{\hat{\sigma}}|\hat{\mu} = 70\} + P\{Z < \frac{LSL - \hat{\mu}}{\hat{\sigma}}|\hat{\mu} = 70\} \\
= \; & 1 - P\{Z \le \frac{73 - 70}{\sqrt{0.75}}\} + P\{Z < \frac{67 - 70}{0.75}\} \\
= \; & 1 - P\{Z \le 4\} + P\{Z < -4\} \; =
\end{aligned}$$

[3]where $z_{0.05/2}$, for example, is calculated using 2nd DISTR InvNorm(ENTER 0.975 ENTER

(choose one) **0** / **0.99** / **1**

7.4 Process Capability Analysis Using a Control Chart

We covered this material in the previous section.

7.5 Process Capability Analysis Using Designed Experiments

The variability in a fixed single–factor ANOVA design is given by

$$\sigma^2_{\text{total}} = \sigma^2_{\text{treatment effect}} + \sigma^2_{\text{error}}$$

The variability for other experimental designs can also be specified.

7.6 Gage and Measurement System Capability Studies

SAS program: att6-7-6-carparts-gage

A *gage* is an instrument used to measure the length (weight, volume, . . .) of objects. It is very important that a gage is accurate and precise (capable) because, otherwise, the objects would be measured incorrectly. Assuming

$$\begin{aligned}
\sigma^2_{\text{total}} &= \sigma^2_{\text{product}} + \sigma^2_{\text{gage}} \\
&= \sigma^2_{\text{product}} + \sigma^2_{\text{measurement error}} \\
&= \sigma^2_{\text{product}} + \sigma^2_{\text{repeatability}} + \sigma^2_{\text{reproducibility}}
\end{aligned}$$

would like σ^2_{gage} to be as small as possible.

Exercise 7.2 (Gage and Measurement System Capability Studies)

1. *P/T ratio and other stuff*
   Different car parts are each weighed four times by a weight gage.

part	m_1	m_2	m_3	m_4
1	7	9	12	14
2	9	5	5	12
3	7	10	11	7
4	8	9	5	14
5	7	8	11	12
6	12	11	9	10
7	16	9	10	7
8	6	5	11	3
9	8	7	10	14
10	15	17	9	14
11	7	12	13	17
12	5	11	8	12
13	16	10	13	14
14	8	13	9	13
15	11	8	11	16
16	15	11	10	14
17	10	9	11	10
18	16	8	11	12
19	8	7	10	13
20	15	17	10	15

   (a) *Determine σ^2_{gage}*
   Using the SAS output and the appendix VI table,

$$\hat{\sigma}_{\text{gage}} = \frac{\bar{R}}{d_2} = \frac{6.55}{2.059} =$$

   (choose one) **1.98** / **2.99** / **3.18**
   This measure does not have much meaning until it is compared to a required (specified) variability, such as $USL - LSL$.

   (b) *Determine P/T ratio*
   If $USL = 13.55$ and $LSL = 6.55$, using the SAS output,

$$P/T = \frac{6\hat{\sigma}_{\text{gage}}}{USL - LSL} = \frac{6(3.18)}{13.55 - 6.55} =$$

   (choose one) **1.98** / **2.73** / **3.18**
   Since a P/T ratio of *less* than 0.1 is often taken to mean an adequate gage capability, the gage, in this case, is clearly a
   (choose one) **poor** / **high** quality gage.

(c) *Determine* σ^2_{total}
From SAS,

$$\sigma^2_{\text{total}} \approx S^2 =$$

(choose one) **6.98** / **9.73** / **10.73**

(d) *Determine* $\sigma^2_{product}$
Since we are assuming

$$\sigma^2_{\text{total}} = \sigma^2_{\text{product}} + \sigma^2_{\text{gage}}$$

then

$$\begin{aligned} \sigma^2_{\text{product}} &= \sigma^2_{\text{total}} - \sigma^2_{\text{gage}} \\ &\approx \hat{\sigma}^2_{\text{total}} - \hat{\sigma}^2_{\text{gage}} \\ &\approx 10.73 - 3.18^2 \end{aligned}$$

(choose one) **0.6178** / **0.7333** / **0.8773**

or, in other words, the gage variability is

$$\frac{\sigma_{\text{gage}}}{\sigma_{\text{product}}} \times 100 \approx \frac{3.18}{\sqrt{0.6178}} \times 100 = 404.6\%$$

of (four times bigger than) the product variability. Again, this indicates a very poor quality gage.

2. *Variability of different operators: repeatability and reproducibility*
   Different car parts are each weighed four times by a weight gage.

	Operator 1		Operator 2	
part	m_1	m_2	m_1	m_2
1	7	9	12	14
2	9	5	5	12
3	7	10	11	7
4	8	9	5	14
5	7	8	11	12
6	12	11	9	10
7	16	9	10	7
8	6	5	11	3
9	8	7	10	14
10	15	17	9	14
11	7	12	13	17
12	5	11	8	12
13	16	10	13	14
14	8	13	9	13
15	11	8	11	16
16	15	11	10	14
17	10	9	11	10
18	16	8	11	12
19	8	7	10	13
20	15	17	10	15

(a) *Determine* $\sigma^2_{repeatability}$
From SAS, $\bar{\bar{x}}_1 = 10.5$, $\bar{R}_1 = 3.2$, and $\bar{\bar{x}}_2 = 11.1$, $\bar{R}_2 = 3.8$,
so $\bar{\bar{R}} = \frac{1}{2}(\bar{R}_1 + \bar{R}_2) = \frac{1}{2}(3.2 + 3.8) = 3.5$
and so

$$\hat{\sigma}_{\text{repeatability}} = \bar{\bar{R}}/d_2 = 3.5/1.128 =$$

(choose one) **2.80** / **3.10** / **4.87**
(where d_2 is based on $n = 2$ measurements)

(b) *Determine* $\sigma^2_{reproducibility}$
Since $\bar{\bar{x}}_{\max} = \max(\bar{\bar{x}}_1, \bar{\bar{x}}_2) = \max(10.05, 11.1) = 11.1$
$\bar{\bar{x}}_{\min} = \min(\bar{\bar{x}}_1, \bar{\bar{x}}_2) = \min(10.05, 11.1) = 10.05$
and so $R_{\bar{\bar{x}}} = \bar{\bar{x}}_{\max} - \bar{\bar{x}}_{\min} = 11.1 - 10.05 = 1.05$
and so

$$\hat{\sigma}_{\text{reproducibility}} = R_{\bar{\bar{x}}}/d_2 = 1.05/1.128 =$$

(choose one) **0.93** / **1.10** / **2.87**
(where d_2 is based on $n = 2$ operators)

(c) *Comparing $\sigma^2_{repeatability}$ and $\sigma^2_{reproducibility}$*
The $\sigma^2_{\text{repeatability}}$ measures the inherent precision of the gage, whereas $\sigma^2_{\text{reproducibility}}$ measures the variability due to the different operators. Since $\sigma^2_{\text{repeatability}} = 3.10 > \sigma^2_{\text{reproducibility}} = 0.93$, most of the variability is due to the (choose one) **gage** / **operators**.

(d) *Measurement error*

$$\begin{aligned}\hat{\sigma}^2_{\text{gage}} &= \hat{\sigma}^2_{\text{measurement error}} \\ &= \hat{\sigma}^2_{\text{repeatability}} + \hat{\sigma}^2_{\text{reproducibility}} \\ &= 3.1^2 + 0.93^2 =\end{aligned}$$

(choose one) **10.47** / **11.10** / **12.87**

(e) *Determine P/T ratio*
If $USL = 13.55$ and $LSL = 6.55$,

$$P/T = \frac{6\hat{\sigma}_{\text{gage}}}{USL - LSL} = \frac{6\sqrt{10.47}}{13.55 - 6.55} =$$

(choose one) **1.98** / **2.77** / **3.18**

(f) *Comparing Measurement errors*
The variance in measurement error, here, $\hat{\sigma}^2_{\text{gage}} = 10.47$, which takes into account the variability due to repeatability and reproducibility (which uses different operators) is (choose one) **smaller** / **larger** than the variance in measurement error above, $\hat{\sigma}^2_{\text{gage}} = 10.11$, which does not take into account the variance due to different operators.

7.7 Setting Specification Limits on Discrete Components

In this section, we discuss how to add up the variability of items in both a linear and non–linear way.

Exercise 7.3 (Setting Specification Limits on Discrete Components)
The lengths (in cm), x_1, x_2, x_3 and x_4, of four car parts are found to follow a normal distribution where
$\mu_1 = 1.5$, $\mu_2 = 2.5$, $\mu_3 = 2.0$, $\mu_4 = 2.0$
$\sigma_1 = 0.5$, $\sigma_2 = 0.5$, $\sigma_3 = 0.9$, $\sigma_4 = 0.4$

1. *Addition of linear variability: tolerance stack–up*
   These car parts are assembled in such a way that there four lengths *add* together. What is the chance that the length of the assembled car parts exceeds 8.5 cm?

   Since $\mu_1 + \mu_2 + \mu_3 + \mu_4 = 2.5 + 1.5 + 2.0 + 2.0 = 8.0$
   and $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2} = \sqrt{0.5^2 + 0.5^2 + 0.9^2 + 0.4^2} = 1.21$

   $$P\{X > 8.5\} =$$

   (choose one) **0.21** / **0.27** / **0.34**
   (use normcdf(8.5, E99, 8.0, 1.21))

2. *Addition of nonlinear variability*
   These car parts are assembled in such a way that there four lengths *multiply* together. What is the mean and variance of the assembled car parts?

   $$\begin{aligned} A &= g(x_1, x_2, x_3, x_4) = x_1 \times x_2 \times x_3 \times x_4 \\ &\approx g(x_1, x_2, x_3, x_4) + \sum_{i=1}^{4} (x_i - \mu) \frac{\partial g}{\partial x_i}|_{\mu_{x_1}, \mu_{x_2}, \mu_{x_3}, \mu_{x_4}} \\ &= \mu_{x_1}\mu_{x_2}\mu_{x_3}\mu_{x_4} + (x_1 - \mu)\mu_{x_2}\mu_{x_3}\mu_{x_4} \\ &\quad +(x_2 - \mu)\mu_{x_1}\mu_{x_3}\mu_{x_4} + (x_3 - \mu)\mu_{x_1}\mu_{x_2}\mu_{x_4} + (x_4 - \mu)\mu_{x_1}\mu_{x_2}\mu_{x_3} \end{aligned}$$

   and so
   $$\mu_A \approx \mu_{x_1}\mu_{x_2}\mu_{x_3}\mu_{x_4} = (2.5)(1.5)(2.0)(2.0) =$$

   (choose one) **13** / **14** / **15**

and

$$
\begin{aligned}
\sigma_A^2 &\approx \sum_{i=1}^{3} \left(\frac{\partial g}{\partial x_i}|_{\mu_L,\mu_H,\mu_W} \right)^2 \sigma_i^2 \\
&= (\mu_{x_2}\mu_{x_3}\mu_{x_4})^2\sigma_{x_1}^2 + (\mu_{x_1}\mu_{x_3}\mu_{x_4})^2\sigma_{x_2}^2 \\
&\quad +(\mu_{x_1}\mu_{x_2}\mu_{x_4})^2\sigma_{x_3}^2 + (\mu_{x_1}\mu_{x_2}\mu_{x_3})^2\sigma_{x_4}^2 \\
&= ((1.5)(2.0)(2.0))^2(0.5)^2 + ((2.5)(2.0)(2.0))^2(0.5)^2 \\
&\quad +((2.5)(1.5)(2.0))^2(0.9)^2 + ((2.5)(1.5)(2.0))^2(0.4)^2 \\
&= 88.56
\end{aligned}
$$

7.8 Estimating the Natural Tolerance Limits of a Process

Not covered.