# Chapter 4

# Calculating the Derivative

Recall, the derivative is the slope of the tangent to a curve. We found using the definition directly to calculate a derivative is complicated. Consequently, we look at techniques that allow us to algebraically calculate the derivative quickly.

## 4.1 Techniques for Finding Derivatives

Various notations for the derivative of $y = f(x)$ include

$$f'(x), \quad \frac{dy}{dx}, \quad \frac{d}{dx}\left[f(x)\right], \quad D_x[f(x)].$$

Some rules for differentiation include:

- *Constant rule.* Derivative of a constant function, $f(x) = k$, $k$ real, is zero:

$$f'(x) = 0.$$

- *Power rule.* Derivative of $f(x) = x^n$, $n$ real, is

$$f'(x) = nx^{n-1}.$$

- *Constant times function rule.* Derivative of $f(x) = k \cdot g(x)$, $k$ real, $g'(x)$ exists:

$$f'(x) = kg'(x).$$

- *Sum or difference rule.* Derivative of $f(x) = u(x) \pm v(x)$, and $u'(x), v'(x)$ exist:

$$f'(x) = u'(x) \pm v'(x).$$

**Exercise 4.1 (Techniques for Finding Derivatives)**

1. *Constant rule.*
   (a) If $f(x) = 3$ (equivalently, $y = 3$), then derivative:
      i. $f'(x) =$ (i) **0** (ii) **1** (iii) **3**
      ii. $\frac{dy}{dx} =$ (i) **0** (ii) **1** (iii) **3**
      iii. $\frac{d}{dx}[f(x)] =$ (i) **0** (ii) **1** (iii) **3**
      iv. $D_x[f(x)] =$ (i) **0** (ii) **1** (iii) **3**
   (b) If $f(x) = 3$ and $x = 3$, derivative $f'(3) =$ (i) **0** (ii) **1** (iii) **3**
   Function $f(x) = 3$ for all $x$, including when $x = 3$.
   (c) If $y = 3$ and $x = \pi$, derivative $f'(\pi) =$ (i) **0** (ii) **1** (iii) **3**
   Function $f(x) = 3$ for all $x$, including when $x = \pi$.
   (d) If $f(x) = 3^2$, derivative $f'(x) =$ (i) **0** (ii) **1** (iii) **3**
   (e) If $y = \pi$, derivative $\frac{dy}{dx} =$ (i) **0** (ii) **1** (iii) **3**
   (f) If $f(x) = \pi^2$, derivative $\frac{d}{dx}[f(x)] =$ (i) **0** (ii) **1** (iii) **3**
   (g) If $f(x) = \pi^2$, derivative $D_x[f(x)] =$ (i) **0** (ii) **1** (iii) **3**
   (h) If $f(x) = 3$ and $x = 3$, derivative $\frac{dy}{dx} =$ (i) **0** (ii) **1** (iii) **3**
   (i) Function $f(x) = 3$ is a (i) **horizontal** (ii) **vertical** line and so the slope (derivative) $f'(x)$ must be (i) **0** (ii) **1** (iii) **undefined**
   (j) (i) **True** (ii) **False** If $f(x) = k$, $k$ real, by definition $f'(x)$ is

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{k - k}{h} = \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} 0 = 0 \end{aligned}$$

2. *Power Rule.*
   (a) If $f(x) = x^2$ (equivalently, $y = x^2$), then derivative:
      i. $f'(x) = 2x^{2-1} =$ (i) **2** (ii) $\boldsymbol{2x}$ (iii) $\boldsymbol{2x^3}$
      ii. $\frac{dy}{dx} =$ (i) $\boldsymbol{x}$ (ii) $\boldsymbol{2x}$ (iii) $\boldsymbol{2x^3}$
      iii. $\frac{d}{dx}[f(x)] =$ (i) $\boldsymbol{x}$ (ii) $\boldsymbol{2x}$ (iii) $\boldsymbol{2x^3}$
      iv. $D_x[f(x)] =$ (i) $\boldsymbol{x}$ (ii) $\boldsymbol{2x}$ (iii) $\boldsymbol{2x^3}$
   (b) If $f(x) = x^3$ then $f'(x) =$ (i) **3** (ii) $\boldsymbol{3x}$ (iii) $\boldsymbol{3x^2}$
   (c) If $y = x^4$, $\frac{dy}{dx} =$ (i) $\boldsymbol{4x^3}$ (ii) $\boldsymbol{4x^2}$ (iii) $\boldsymbol{4x}$
   (d) If $f(t) = t^4$ then $\frac{d}{dt}[f(t)] =$ (i) $\boldsymbol{4x^3}$ (ii) $\boldsymbol{4t^3}$ (iii) $\boldsymbol{3t^4}$
   (e) If $f(x) = x = x^1$ then $D_x[f(x)] =$ (i) **0** (ii) $\boldsymbol{1x^0 = 1}$ (iii) $\boldsymbol{x}$
   (f) If $f(x) = x^{-3}$ then $f'(x) =$ (i) $\boldsymbol{-3x^{-4}}$ (ii) $\boldsymbol{3x^{-4}}$ (iii) $\boldsymbol{-3x^{-2}}$

(g) If $y = x^{-7}$ then $\frac{dy}{dx} =$ (i) $\boldsymbol{-7x^{-8}}$ (ii) $\boldsymbol{-8x^{-8}}$ (iii) $\boldsymbol{-7x^{-6}}$

(h) If $y = z^{-7}$ then $\frac{dy}{dz} =$ (i) $\boldsymbol{-\frac{7}{z^8}}$ (ii) $\boldsymbol{-\frac{7}{x^8}}$ (iii) $\boldsymbol{-\frac{7}{z^6}}$

(i) If $f(x) = x^{-7.2}$, $\frac{d}{dx}[f(t)] =$ (i) $\boldsymbol{-7.2x^{-8.2}}$ (ii) $\boldsymbol{-8.2x^{-6.2}}$ (iii) $\boldsymbol{-6.2x^{-8.2}}$

(j) If $f(x) = \sqrt{x} = x^{\frac{1}{2}}$ then $D_x[f(x)] =$ (i) $\boldsymbol{\frac{3}{2}x^{-\frac{1}{2}}}$ (ii) $\boldsymbol{-\frac{1}{2}x^{-\frac{1}{2}}}$ (iii) $\boldsymbol{\frac{1}{2}x^{-\frac{1}{2}}}$

(k) If $f(x) = \sqrt{x}$ then $D_x[f(x)] =$ (i) $\boldsymbol{\frac{3}{2\sqrt{x}}}$ (ii) $\boldsymbol{-\frac{1}{2\sqrt{x}}}$ (iii) $\boldsymbol{\frac{1}{2\sqrt{x}}}$

(l) If $f(x) = x^{\frac{5}{3}}$ then $f'(x) =$ (i) $\boldsymbol{-\frac{5}{3}x^{-\frac{2}{3}}}$ (ii) $\boldsymbol{-\frac{1}{3}x^{-\frac{1}{3}}}$ (iii) $\boldsymbol{\frac{5}{3}x^{\frac{2}{3}}}$

(m) If $f(x) = x^2$ (or $y = x^2$), so $f'(x) = 2x^{2-1} = 2x$, then at $x = 7$,

i. $f'(7) = 2(7) =$ (i) **14** (ii) $\boldsymbol{2x}$ (iii) **7**

ii. $\frac{dy}{dx}|_{x=7} =$ (i) **14** (ii) $\boldsymbol{2x}$ (iii) **7**

iii. $\frac{d}{dx}[f(7)] =$ (i) **14** (ii) $\boldsymbol{2x}$ (iii) **7**

iv. $D_x[f(7)] =$ (i) **14** (ii) $\boldsymbol{2x}$ (iii) **7**

(n) If $f(x) = x^{-3}$ then $f'(x) = -3x^{-4}$
and so $f'(3) = -3(3)^{-4} \approx$ (i) $\boldsymbol{-3x^{-4}}$ (ii) $\boldsymbol{-0.037}$ (iii) $\boldsymbol{-0.333}$
and $f'(4) = -3(4)^{-4} \approx$ (i) $\boldsymbol{-0.012}$ (ii) $\boldsymbol{-0.037}$ (iii) $\boldsymbol{-0.333}$
and $f'(-4) = -3(-4)^{-4} \approx$ (i) $\boldsymbol{-0.012}$ (ii) $\boldsymbol{-0.037}$ (iii) $\boldsymbol{-0.333}$

3. *Constant times a function rule.*

(a) If $f(x) = 10x^5$ (equivalently, $y = 10x^5$), then derivative:

i. $f'(x) = 10\left(5x^{5-1}\right) =$ (i) $\boldsymbol{5x^4}$ (ii) $\boldsymbol{10x^4}$ (iii) $\boldsymbol{50x^4}$

ii. $\frac{dy}{dx} =$ (i) $\boldsymbol{5x^4}$ (ii) $\boldsymbol{10x^4}$ (iii) $\boldsymbol{50x^4}$

iii. $\frac{d}{dx}[f(x)] =$ (i) $\boldsymbol{5x^4}$ (ii) $\boldsymbol{10x^4}$ (iii) $\boldsymbol{50x^4}$

iv. $D_x[f(x)] =$ (i) $\boldsymbol{5x^4}$ (ii) $\boldsymbol{10x^4}$ (iii) $\boldsymbol{50x^4}$

(b) If $f(x) = -\frac{2}{3}x^3$ then $f'(x) = -\frac{2}{3}\left(3x^{3-1}\right) =$ (i) $\boldsymbol{-2x^2}$ (ii) $\boldsymbol{3x^2}$ (iii) $\boldsymbol{-3x^3}$

(c) If $y = 8p^4$ then $\frac{dy}{dp} = 8\left(4p^{4-1}\right) =$ (i) $\boldsymbol{32x^3}$ (ii) $\boldsymbol{8p^3}$ (iii) $\boldsymbol{32p^3}$

(d) If $f(x) = 5x^{\frac{5}{7}}$ then $\frac{d}{dx}[f(x)] =$ (i) $\boldsymbol{\frac{5}{7}x^{-\frac{2}{7}}}$ (ii) $\boldsymbol{\frac{25}{7}x^{-\frac{2}{7}}}$ (iii) $\boldsymbol{5x^{-\frac{2}{7}}}$

(e) If $f(x) = \frac{7}{x} = 7x^{-1}$ then $D_x[f(x)] =$ (i) $\boldsymbol{7x^{-1}}$ (ii) $\boldsymbol{-7x^{-2}}$ (iii) $\boldsymbol{-2x^{-7}}$
and so $D_x[f(2)] = -7(2)^{-2} =$ (i) $\boldsymbol{-2.25}$ (ii) $\boldsymbol{-1.75}$ (iii) $\boldsymbol{-2.75}$
and $D_x[f(8)] = -7(8)^{-2} \approx$ (i) $\boldsymbol{-0.23}$ (ii) $\boldsymbol{-0.45}$ (iii) $\boldsymbol{-0.11}$

(f) (i) **True** (ii) **False** If $f(x) = k \cdot g(x)$, $k$ real, by definition $f'(x)$ is

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{kg(x+h) - kg(x)}{h} \\ &= \lim_{h\to 0} k\frac{g(x+h) - g(x)}{h} \\ &= k\lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\ &= k \cdot g'(x) \end{aligned}$$

4. *Sum or difference rule.*

   (a) If $f(x) = 2x^3 - 4x$ then $f'(x) = 2\left(3x^{3-1}\right) - 4\left(1x^{1-1}\right) =$
   (i) $\boldsymbol{3x^2 - 4}$ (ii) $\boldsymbol{6x^2 - 4}$ (iii) $\boldsymbol{6x^2 - 4x}$

   (b) If $y = 8p^4 + 4\sqrt{p} - 5 = 8p^4 + 4p^{\frac{1}{2}} - 5$ then $\frac{dy}{dp} = 8\left(4p^{4-1}\right) + 4\left(\frac{1}{2}p^{\frac{1}{2}-1}\right) + 0 =$
   (i) $\boldsymbol{32p^3 + \frac{1}{p}}$ (ii) $\boldsymbol{32p^3 + \frac{1}{\sqrt{p}} + 5}$ (iii) $\boldsymbol{32p^3 + \frac{2}{\sqrt{p}}}$

   (c) If $f(x) = 5x^{\frac{5}{7}} + \frac{7}{x} = 5x^{\frac{5}{7}} + 7x^{-1}$ then $\frac{d}{dx}\left[f(x)\right] =$
   (i) $\boldsymbol{\frac{25}{7}x^{-\frac{2}{7}} - 7x^{-2}}$ (ii) $\boldsymbol{5x^{-\frac{2}{7}} - 7x^{-2}}$ (iii) $\boldsymbol{\frac{5}{7}x^{-\frac{2}{7}} + 7x^{-2}}$

   (d) If $f(x) = \frac{7+4\sqrt{x}}{x} = \frac{7}{x} + \frac{4x^{\frac{1}{2}}}{x} = 7x^{-1} + 4x^{\frac{1}{2}}$
   then $D_x[f(x)] = 7\left(-x^{-1-1}\right) + 4\left(\frac{1}{2}x^{\frac{1}{2}-1}\right) =$
   (i) $\boldsymbol{\frac{7}{x^2} + \frac{2}{\sqrt{x}}}$ (ii) $\boldsymbol{-\frac{7}{x^2} + \frac{2}{\sqrt{x}}}$ (iii) $\boldsymbol{-\frac{7}{x^2} - \frac{2}{\sqrt{x}}}$

   (e) If $f(x) = \left(2x^3 - 4x\right)^2 = 4x^6 - 16x^4 + 16x^2$
   then $f'(x) = 4\left(6x^{6-1}\right) - 16\left(4x^{4-1}\right) + 16\left(2x^{2-1}\right) =$
   (i) $\boldsymbol{24x^5 - 64x^3 + 32x}$ (ii) $\boldsymbol{24x^5 + 64x^3 + 32}$ (iii) $\boldsymbol{24x^5 - 64x^3 - 32}$
   and so $f'(2) = 24(2)^5 - 64(2)^3 + 32(2) =$ (i) **321** (ii) **322** (iii) **320**
   and $f'(0) = 24(0)^5 - 64(0)^3 + 32(0) =$ (i) **0** (ii) $\boldsymbol{-8}$ (iii) **undefined**

   (f) (i) **True** (ii) **False** If $f(x) = u(x) + v(x)$, then

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{[u(x+h) + v(x+h)] - [u(x) + v(x)]}{h} \\ &= \lim_{h\to 0} \frac{[u(x+h) - u(x)] + [v(x+h) - v(x)]}{h} \\ &= \lim_{h\to 0} \left[\frac{u(x+h) - u(x)}{h} + \frac{v(x+h) - v(x)}{h}\right] \\ &= \lim_{h\to 0} \frac{u(x+h) - u(x)}{h} + \lim_{h\to 0} \frac{v(x+h) - v(x)}{h} \\ &= u'(x) + v'(x) \end{aligned}$$

5. *Tangent lines.*

   (a) *Locating point(s) of $f(x) = 2x^2 + 5x$ where tangent line(s) horizontal.*
   since $f'(x) = 2\left(2x^{2-1}\right) + 5\left(x^{1-1}\right) =$ (i) $\boldsymbol{2x + 5}$ (ii) $\boldsymbol{4x + 5}$ (iii) $\boldsymbol{4x - 5}$
   and *horizontal* tangent line occurs when $f'(x) = 0$, then

$$f'(x) = 4x + 5 = 0$$

   when $x_1 =$ (i) $\boldsymbol{-1.25}$ (ii) **1.25** (iii) **0**
   and since $f(x_1) = f(-1.25) = 2(-1.25)^2 + 5(-1.25) =$
   (i) $\boldsymbol{-3.125}$ (ii) $\boldsymbol{-3.225}$ (iii) **3.325**
   so point on $f(x)$ when tangent line is horizontal is $(x_1, f(x_1)) =$

(i) $\mathbf{(-1.25, 3.125)}$ (ii) $\mathbf{(1.25, -3.125)}$ (iii) $\mathbf{(-1.25, -3.125)}$
and also equation of tangent line at $(-1.25, -3.125)$ is

$$\begin{aligned} y - f(x_1) &= f'(x_1)(x - x_1) \\ y - f(-1.25) &= f'(-1.25)(x - (-1.25)) \\ y - (-3.125) &= 0(x + 1.25) \end{aligned}$$

or (i) $\boldsymbol{y = -3.125}$ (ii) $\boldsymbol{y = 3.125}$ (iii) $\boldsymbol{y = 0}$

(b) *Locating all $x$ where tangent lines to $f(x) = x^3 + x^2$ are horizontal.*
since $f'(x) = 3x^{3-1} + x^{2-1} =$ (i) $\boldsymbol{3x^2 + x}$ (ii) $\boldsymbol{3x + x}$ (iii) $\boldsymbol{2x^3 + x}$
and *horizontal* tangent lines occur when $f'(x) = 0$, then

$$f'(x) = 3x^2 + x = x(3x + 1) = 0$$

when $x_1 =$ (choose *two*!) (i) $\mathbf{-\frac{1}{3}}$ (ii) $\mathbf{\frac{1}{3}}$ (iii) $\mathbf{0}$

(c) *Points where slopes to tangent lines to $f(x) = \frac{1}{3}x^3 + \frac{5}{2}x^2 + 3x$ equal -3.*
since $f'(x) = \frac{1}{3}(3x^{3-1}) + \frac{5}{2}(2x^{2-1}) + 3(x^{1-1}) =$
(i) $\boldsymbol{x^2 - 5x + 3}$ (ii) $\boldsymbol{x^2 + 5x + 3}$ (iii) $\boldsymbol{x^2 + 5x - 3}$
and tangent lines with slope -3 occur when $f'(x) = -3$,

$$f'(x) = x^2 + 5x + 3 = -3$$

so $x^2 + 5x + 6 = 0$ or $(x + 3)(x + 2) = 0$
when $x_1 =$ (choose *two*!) (i) $\mathbf{-2}$ (ii) $\mathbf{-3}$ (iii) $\mathbf{0}$
and since $f(x_1) = f(-2) = \frac{1}{3}(-2)^3 + \frac{5}{2}(-2)^2 + 3(-2) =$
(i) $\mathbf{\frac{2}{3}}$ (ii) $\mathbf{\frac{1}{3}}$ (iii) $\mathbf{\frac{4}{3}}$
and also $f(x_1) = f(-3) = \frac{1}{3}(-3)^3 + \frac{5}{2}(-3)^2 + 3(-3) =$
(i) **4.5** (ii) **4.4** (iii) **4.3**
so *two* points on $f(x)$ when slope of tangent line -3 is $(x_1, f(x_1)) =$
(i) $\mathbf{\left(-2, \frac{4}{3}\right)}$ (ii) $\mathbf{(-3, 4.5)}$ (iii) $\mathbf{(-2, -3)}$

6. *Application: life sciences.* Circumference, $C$ (in cm), of a healing wound is

$$C(r) = 2\pi r$$

(a) Circumference of a wound with radius $r = 3$ cm, is
$C(3) =$ (i) $\boldsymbol{3\pi}$ (ii) $\boldsymbol{6\pi}$ (iii) $\boldsymbol{8\pi}$

(b) Rate of change of circumference with respect to radius is
$C'(r) =$ (i) $\boldsymbol{2\pi}$ (ii) $\boldsymbol{3\pi}$ (iii) $\boldsymbol{4\pi}$

(c) At $r = 5$ cm, $C'(5) =$ (i) $\boldsymbol{2\pi}$ (ii) $\boldsymbol{3\pi}$ (iii) $\boldsymbol{4\pi}$

7. *Application: social sciences.* Population of a city is given by

$$P(t) = 50,000 + 1500t^{1.5}$$

(a) Rate of change of population with respect to time is
$P'(t) =$ (i) $\mathbf{1500t}$ (ii) $\mathbf{2250t^{0.5}}$ (iii) $\mathbf{t^{0.5}}$

(b) At $t = 2$, growth rate is $P'(2) =$ (i) **3182** (ii) **4574** (iii) **4834**

8. *Application: physics.* Function which relates distance, $s$, to time, $t$:

$$s(t) = 3t^3 + 2t$$

(a) At $t = 0$, distance $s(0) =$ (i) **0** (ii) **1** (iii) **2**
and at $t = 2$, $s(2) =$ (i) **3** (ii) **14** (iii) **28**

(b) Velocity (rate of change of distance with respect to time) is
$v(t) = s'(t) =$ (i) $\mathbf{3t^2 + 2}$ (ii) $\mathbf{9t^2 + 2}$ (iii) $\mathbf{9t^2}$
and at $t = 2$, $v(2) = 9(2)^2 + 2 =$ (i) **3** (ii) **14** (iii) **38**

(c) Acceleration (rate of change of *velocity* with respect to time) is
$a(t) = v'(t) =$ (i) $\mathbf{18t}$ (ii) $\mathbf{9t^2 + 2}$ (iii) $\mathbf{9t^2}$
and at $t = 2$, $a(2) = 18(2) =$ (i) **3** (ii) **14** (iii) **36**

9. *Application: business.* Revenue, $R(x)$, and cost, $C(x)$, functions are

$$\begin{aligned} R(x) &= 3x, \\ C(x) &= 0.01x^2 + 2.4x + 45, \end{aligned}$$

where $x$ is quantity of items produced.

(a) Marginal revenue $R'(x) =$
(i) $\mathbf{3x - (0.01x^2 + 2.4x)}$ (ii) $\mathbf{3x - (0.01x^2 + 45)}$ (iii) **3**
so marginal revenue of 50 items $R'(50) =$ (i) **3** (ii) **15** (iii) **20**

(b) Marginal cost $C'(x) =$
(i) $\mathbf{3x - (0.01x^2 + 2.4x)}$ (ii) $\mathbf{3x - (0.01x^2 + 45)}$ (iii) $\mathbf{0.02x + 2.4}$
so marginal cost of 50 items $C'(50) =$ (i) **1.40** (ii) **2.40** (iii) **3.40**

(c) Profit $P(x) = R(x) - C(x) =$
(i) $\mathbf{3x - (0.01x^2 + 2.4x)}$
(ii) $\mathbf{3x - (0.01x^2 + 45)}$
(iii) $\mathbf{-0.01x^2 + 0.6x - 45}$
so marginal profit $P'(x) =$
(i) $\mathbf{3x - (0.01x^2 + 2.4x)}$ (ii) $\mathbf{3x - (0.01x^2 + 45)}$ (iii) $\mathbf{-0.02x + 0.6}$
marginal profit of 50 items $P'(50) =$ (i) $\mathbf{-0.40}$ (ii) $\mathbf{-0.60}$ (iii) $\mathbf{-0.80}$

(d) Quantity $x$ when marginal profit zero, $P'(x) = 0$, so

$$P'(x) = -0.02x + 0.6 = 0$$

when $x = \frac{-0.6}{-0.02} =$ (i) **30** (ii) **40** (iii) **50**
where profit is $P(30) = -0.01(30)^2 + 0.6(30) - 45 =$
(i) $\mathbf{-30}$ (ii) $\mathbf{-36}$ (iii) $\mathbf{-42}$

## 4.2 Derivatives of Products and Quotients

*Product rule:* If $f(x) = u(x) \cdot v(x)$, $u'(x)$ and $v'(x)$ exist, then

$$f'(x) = v(x) \cdot u'(x) + u(x) \cdot v'(x).$$

*Quotient rule:* If $y = \frac{u(x)}{v(x)}$, $u'(x)$ and $v'(x)$ exist, and $v(x) \neq 0$, then

$$f'(x) = \frac{v(x) \cdot u'(x) - u(x) \cdot v'(x)}{[v(x)]^2}.$$

**Exercise 4.2 (Derivatives of Products and Quotients)**

1. *Product Rule*

   (a) Consider $f(x) = (4x - 1)(5x + 4)$.
   let $u(x) = 4x - 1$ and $v(x) = 5x + 4$.

   Then, $u'(x) =$
   (i) $\boldsymbol{4x^2}$
   (ii) $\boldsymbol{4}$
   (iii) $\boldsymbol{-1}$

   and $v'(x) =$
   (i) $\boldsymbol{5x^2}$
   (ii) $\boldsymbol{5}$
   (iii) $\boldsymbol{3 + 4x}$

   and so $v(x)u'(x) =$
   (i) $\boldsymbol{(5x + 4)(4)}$
   (ii) $\boldsymbol{(5x + 4)(4x^2)}$
   (iii) $\boldsymbol{(5x^2 + 4)(4)}$

   and $u(x)v'(x) =$
   (i) $\boldsymbol{(4x^2 - 1)(5)}$
   (ii) $\boldsymbol{(4x - 1)(5)}$
   (iii) $\boldsymbol{(4x - 1)(5x)}$

   and so $f'(x) = v(x) \cdot u'(x) + u(x) \cdot v'(x) =$
   (i) $\boldsymbol{(5x + 4)(4) + (4x - 1)(5)}$
   (ii) $\boldsymbol{(5x - 4)(4) + (4x - 1)(5)}$
   (iii) $\boldsymbol{(5x + 4)(4) - (4x - 1)(5)}$

which equals
(i) $\mathbf{40x - 11}$
(ii) $\mathbf{40x + 11}$
(iii) $\mathbf{39 + 11}$

(b) Consider $f(x) = (\sqrt{x} + x)(x^3 + x^2)$.
let $u(x) = \sqrt{x} + x = x^{1/2} + x$ and $v(x) = x^3 + x^2$.

Then, $u'(x) =$
(i) $\mathbf{3x^2 - 7}$
(ii) $\mathbf{3x^2 + 3}$
(iii) $\mathbf{\frac{1}{2}x^{-1/2} + 1}$

and $v'(x) =$
(i) $\mathbf{3x^2 - 7}$
(ii) $\mathbf{3x^2 + 3}$
(iii) $\mathbf{3x^2 + 2x}$

and so $v(x)u'(x) =$
(i) $\mathbf{(2x^3 + 2x^2 - 4x)(3x^2 + 3)}$
(ii) $\mathbf{(x^3 + x^2)(\frac{1}{2}x^{-1/2} + 1)}$
(iii) $\mathbf{6x^2 + 4x - 4}$

and $u(x)v'(x) =$
(i) $\mathbf{(2x^3 + 2x^2 - 4x)(3x^2 + 3)}$
(ii) $\mathbf{3x^2 + 3}$
(iii) $\mathbf{(x^{1/2} + x)(3x^2 + 2x)}$

and so $\frac{dy}{dx} = v(x) \cdot u'(x) + u(x) \cdot v'(x) =$
(i) $\mathbf{(2x^3 + 2x^2 - 4x)(3x^2 + 3)}$
(ii) $\mathbf{(x^3 + x^2)(\frac{1}{2}x^{-1/2} + 1) + (x^{1/2} + x)(3x^2 + 2x)}$
(iii) $\mathbf{(2x^3 + 2x^2 - 4x)(3x^2 + 3) + (x^3 + 3x)(6x^2 + 4x - 4)}$
or
$\frac{1}{2}x^{\frac{5}{2}} + \frac{1}{2}x^{\frac{3}{2}} + x^3 + x^2 + 3x^{\frac{5}{2}} + 3x^3 + 2x^{\frac{3}{2}} + 2x^2$

which equals
(i) $\mathbf{4x^3 + \frac{5}{2}x^{\frac{5}{2}} + 3x^2 + \frac{5}{2}x^{\frac{3}{2}}}$
(ii) $\mathbf{4x^3 + \frac{7}{2}x^{\frac{5}{2}} + 3x^2 + \frac{5}{2}x^{\frac{3}{2}}}$
(iii) $\mathbf{4x^3 + \frac{7}{2}x^{\frac{3}{2}} + 3x^2 + \frac{5}{2}x^{\frac{1}{2}}}$

(c) Consider $f(x) = (x^3 + 3x)(2x^3 + 2x^2 - 4x)$.

let $u(x) = x^3 + 3x$ and $v(x) = 2x^3 + 2x^2 - 4x$.

Then, $u'(x) =$
(i) $\mathbf{3x^2 - 7}$
(ii) $\mathbf{3x^2 + 3}$
(iii) $\mathbf{4x - \frac{7}{3}}$

and $v'(x) =$
(i) $\mathbf{3x^2 - 7}$
(ii) $\mathbf{3x^2 + 3}$
(iii) $\mathbf{6x^2 + 4x - 4}$

so $v(x)u'(x) =$
(i) $\mathbf{(2x^3 + 2x^2 - 4x)(3x^2 + 3)}$
(ii) $\mathbf{3x^2 + 3}$
(iii) $\mathbf{6x^2 + 4x - 4}$

and $u(x)v'(x) =$
(i) $\mathbf{(2x^3 + 2x^2 - 4x)(3x^2 + 3)}$
(ii) $\mathbf{3x^2 + 3}$
(iii) $\mathbf{(x^3 + 3x)(6x^2 + 4x - 4)}$

and so $f'(x) = v(x) \cdot u'(x) + u(x) \cdot v'(x) =$
(i) $\mathbf{(2x^3 + 2x^2 - 4x)(3x^2 + 3)}$
(ii) $\mathbf{3x^2 + 3}$
(iii) $\mathbf{(2x^3 + 2x^2 - 4x)(3x^2 + 3) + (x^3 + 3x)(6x^2 + 4x - 4)}$
or
$6x^5 + 6x^4 - 12x^3 + 6x^3 + 6x^2 - 12x + 6x^5 + 18x^3 + 4x^4 + 12x^2 - 4x^3 - 12x$

which equals
(i) $\mathbf{12x^5 + 10x^4 + 8x^3 + 18x^2 - 24x}$
(ii) $\mathbf{6x^5 + 12x^4 - 16x^3 + 14x^2 - 24x}$
(iii) $\mathbf{6x^5 + 12x^4 + 8x^3 + 14x^2 - 24x}$

(d) Consider $y = \left(\frac{2}{x^2} + 3\right)\left(\frac{1}{x^3} + 2x - 4\right)$.
Let $u(x) = \frac{2}{x^2} + 3 = 2x^{-2} + 3$ and $v(x) = \frac{1}{x^3} + 2x - 4 = x^{-3} + 2x - 4$.

Then $u'(x) =$
(i) $\mathbf{-4x^{-3}}$
(ii) $\mathbf{3x^2 + 3}$
(iii) $\mathbf{6x^2 + 4x - 4}$

and $v'(x) =$
(i) $\boldsymbol{3x^2 + 3}$
(ii) $\boldsymbol{6x^2 + 4x - 4}$
(iii) $\boldsymbol{-3x^{-4} + 2}$

and so $v(x)u'(x) =$
(i) $\boldsymbol{3x^2 + 3}$
(ii) $\boldsymbol{(x^{-3} + 2x - 4)\,(-4x^{-3})}$
(iii) $\boldsymbol{6x^2 + 4x - 4}$

and $u(x)v'(x) =$
(i) $\boldsymbol{3x^2 + 3}$
(ii) $\boldsymbol{(2x^{-2} + 3)\,(-3x^{-4} + 2)}$
(iii) $\boldsymbol{6x^2 + 4x - 4}$

and so $\frac{d}{dx}\,[f(x)] = v(x) \cdot u'(x) + u(x) \cdot v'(x) =$
(i) $\boldsymbol{(x^{-3} + 2x - 4)\,(-4x^{-3}) + (2x^{-2} + 3)\,(-3x^{-4} + 2)}$
(ii) $\boldsymbol{(x^{-2} + 2x - 4)\,(-4x^{-3}) + (2x^{-2} + 3)\,(-3x^{-4} + 2)}$
(iii) $\boldsymbol{(x^{-3} + 2x - 4)\,(-4x^{-3}) + (2x^{-2} + 3)\,(-3x^{-4} - 2)}$
or
$-4x^{-6} - 8x^{-2} + 16x^{-3} - 6x^{-6} - 9x^{-4} + 4x^{-2} + 6$

which equals
(i) $\boldsymbol{4x^{-2} + 16x^{-3} - 9x^{-4} - 10x^{-6} + 6}$
(ii) $\boldsymbol{-4x^{-2} - 16x^{-3} - 9x^{-4} - 10x^{-6} + 6}$
(iii) $\boldsymbol{-4x^{-2} + 16x^{-3} - 9x^{-4} - 10x^{-6} + 6}$

(e) *Using product rule definition.*
If $u(4) = 5$, $u'(4) = 6$, $v(4) = 9$ and $v'(4) = -4$, then
$D_x[f(4)] = v(4) \cdot u'(4) + u(4) \cdot v'(4) =$ (i) **34** (ii) **35** (iii) **36**

2. *Quotient rule*

(a) Consider $f(x) = \frac{4x-1}{5x+4}$.
let $u(x) = 4x - 1$ and $v(x) = 5x + 4$.

Then, $u'(x) =$
(i) $\boldsymbol{4x^2}$
(ii) $\boldsymbol{4}$
(iii) $\boldsymbol{-1}$

and $v'(x) =$
(i) $\boldsymbol{5x^2}$

(ii) $\mathbf{5}$
(iii) $\boldsymbol{3 + 4x}$

and so $v(x)u'(x) =$
(i) $\boldsymbol{(5x + 4)(4)}$
(ii) $\boldsymbol{(5x + 4)(4x^2)}$
(iii) $\boldsymbol{(5x^2 + 4)(4)}$

and $u(x)v'(x) =$
(i) $\boldsymbol{(4x^2 - 1)(5)}$
(ii) $\boldsymbol{(4x - 1)(5)}$
(iii) $\boldsymbol{(4x - 1)(5x)}$

and so $f'(x) = \frac{v(x)\cdot u'(x) - u(x)\cdot v'(x)}{[v(x)]^2} =$
(i) $\boldsymbol{\frac{(5x+4)(4)-(4x-1)(5)}{(5x+4)^2}}$ (ii) $\boldsymbol{\frac{(5x+4)(4)+(4x-1)(5)}{(5x+4)^2}}$ (iii) $\boldsymbol{\frac{(5x+4)(4)+(4x-1)(5)}{(4x-1)^2}}$

which equals
(i) $\boldsymbol{\frac{21}{(5x+4)^2}}$ (ii) $\boldsymbol{\frac{40x+11}{(5x+4)^2}}$ (iii) $\boldsymbol{\frac{20}{(5x+4)^2}}$

(b) Consider $y = \frac{x^{1/2}+x}{x^3+x^2}$.
let $u(x) = x^{1/2} + x$ and $v(x) = x^3 + x^2$.

Then, $u'(x) =$
(i) $\boldsymbol{3x^2 - 7}$
(ii) $\boldsymbol{3x^2 + 3}$
(iii) $\boldsymbol{\frac{1}{2}x^{-1/2} + 1}$

and $v'(x) =$
(i) $\boldsymbol{3x^2 - 7}$
(ii) $\boldsymbol{3x^2 + 3}$
(iii) $\boldsymbol{3x^2 + 2x}$

and so $v(x)u'(x) =$
(i) $\boldsymbol{(2x^3 + 2x^2 - 4x)(3x^2 + 3)}$
(ii) $\boldsymbol{(x^3 + x^2)(\frac{1}{2}x^{-1/2} + 1)}$
(iii) $\boldsymbol{6x^2 + 4x - 4}$

and $u(x)v'(x) =$
(i) $\boldsymbol{(2x^3 + 2x^2 - 4x)(3x^2 + 3)}$
(ii) $\boldsymbol{3x^2 + 3}$
(iii) $\boldsymbol{(x^{1/2} + x)(3x^2 + 2x)}$

and so $\frac{dy}{dx} = \frac{v(x)\cdot u'(x) - u(x)\cdot v'(x)}{[v(x)]^2} =$
(i) $\boldsymbol{\frac{(x^3+x^2)(\frac{1}{2}x^{-1/2}+1)+(x^{1/2}+x)(3x^2+2x)}{(x^3+x^2)^2}}$
(ii) $\boldsymbol{\frac{(x^3+x^2)(\frac{1}{2}x^{-1/2}+1)-(x^{1/2}+x)(3x^2+2x)}{(x^3+x^2)^2}}$
(iii) $\boldsymbol{\frac{(3x+2x)(\frac{1}{2}x^{-1/2}+1)-(x^{1/2}+x)(3x^2+2x)}{(x^3+x^2)^2}}$
or
$\frac{\frac{1}{2}x^{\frac{5}{2}}+\frac{1}{2}x^{\frac{3}{2}}+x^3+x^2-3x^{\frac{5}{2}}-3x^3-2x^{\frac{3}{2}}-2x^2}{(x^3+x^2)^2}$

which equals
(i) $\boldsymbol{\frac{-2x^3-\frac{5}{2}x^{\frac{5}{2}}-x^2-\frac{3}{2}x^{\frac{3}{2}}}{(x^3+x^2)^2}}$
(ii) $\boldsymbol{\frac{4x^3+\frac{5}{2}x^{\frac{5}{2}}+3x^2+\frac{5}{2}x^{\frac{3}{2}}}{(x^3+x^2)^2}}$
(iii) $\boldsymbol{\frac{-4x^3-\frac{5}{2}x^{\frac{5}{2}}-3x^2-\frac{5}{2}x^{\frac{3}{2}}}{(x^3+x^2)^2}}$

(c) Consider $f(t) = \frac{t}{t^2+3}$.
let $u(t) = t$ and $v(t) = t^2 + 3$.

Then, $u'(t) =$
(i) $\mathbf{1}$
(ii) $\boldsymbol{t}$
(iii) $\boldsymbol{t^2}$

and $v'(t) =$
(i) $\boldsymbol{2t}$
(ii) $\boldsymbol{2t^2}$
(iii) $\boldsymbol{4t}$

and so $v(t)u'(t) =$
(i) $\boldsymbol{(t^2+3)(t)}$
(ii) $\boldsymbol{(t+3)(1)}$
(iii) $\boldsymbol{(t^2+3)(1)}$

and $u(t)v'(t) =$
(i) $\boldsymbol{(t^2)(t)}$
(ii) $\boldsymbol{(t^2+1)(1)}$
(iii) $\boldsymbol{(t)(2t)}$

and so $f'(t) = \frac{v(t)\cdot u'(t) - u(t)\cdot v'(t)}{[v(t)]^2} =$
(i) $\boldsymbol{\frac{(t^2+3)(1)-(t)(2t)}{(t^2+3)^2}}$ (ii) $\boldsymbol{\frac{(t^2+3)(1)+(t)(2t)}{(t^2+3)^2}}$ (iii) $\boldsymbol{\frac{(3t^2+3)(1)-(t)(2t)}{(t^2+3)^2}}$

which equals
(i) $\boldsymbol{\frac{-t^2}{(t^2+3)^2}}$ (ii) $\boldsymbol{\frac{-t^2+3}{(t^2-3)^2}}$ (iii) $\boldsymbol{\frac{-t^2+3}{(t^2+3)^2}}$

(d) *Equation of tangent line to* $y = f(t) = \frac{t}{t^2+3}$ *at point* $(t_1, f(t_1)) = \left(1, \frac{1}{4}\right)$.
Since $f'(t_1) = f'(1) = \frac{-(1)^2+3}{((1)^2+3)^2} =$ (i) $\boldsymbol{\frac{1}{8}}$ (ii) $\boldsymbol{\frac{1}{16}}$ (iii) $\boldsymbol{\frac{1}{32}}$ then

$$\begin{aligned} y - f(t_1) &= f'(t_1)(t - t_1) \\ y - f(1) &= f'(1)(t - 1) \\ y - \frac{1}{4} &= \frac{1}{8}(t - 1) \end{aligned}$$

or (i) $\boldsymbol{y = \frac{t+1}{8}}$ (ii) $\boldsymbol{y = \frac{t-1}{8}}$ (iii) $\boldsymbol{y = \frac{t+1}{4}}$

3. *Application: average cost.* Monthly fixed costs of using machine I are \$15,000 and marginal costs of manufacturing one widget using machine I is \$20. Suppose number of widgets produced restricted to $100 \le x \le 500$. Consequently, *average* costs are

$$\overline{C}(x) = \frac{20x + 15000}{x}, \quad 100 \le x \le 500.$$

let $u(x) = 20x + 15000$ and $v(x) = x$.

Then, $u'(x) =$
(i) $\mathbf{15000}$
(ii) $\boldsymbol{20x^2}$
(iii) $\mathbf{20}$

and $v'(x) =$
(i) $\boldsymbol{x}$
(ii) $\mathbf{1}$
(iii) $\boldsymbol{x^2}$

and so $v(x)u'(x) =$
(i) $\boldsymbol{(x)(20)}$
(ii) $\boldsymbol{(x^2)(20)}$
(iii) $\boldsymbol{(x)(20x)}$

and $u(x)v'(x) =$
(i) $\boldsymbol{(20x)(1)}$
(ii) $\boldsymbol{(20x + 15000)(1)}$
(iii) $\boldsymbol{(15000)(1)}$

and so $\overline{C}'(x) = \frac{v(x) \cdot u'(x) - u(x) \cdot v'(x)}{[v(x)]^2} =$
(i) $\boldsymbol{\frac{(x)(20)+(20x+15000)(1)}{(x)^2}}$ (ii) $\boldsymbol{\frac{(x)(20)-(20x+15000)(1)}{(x)^2}}$ (iii) $\boldsymbol{\frac{(x)(20)-(20x)(1)}{(x)^2}}$

which equals
(i) $\boldsymbol{\frac{15000}{x^2}}$ (ii) $\boldsymbol{-\frac{15000}{x^2}}$ (iii) $\boldsymbol{-\frac{15000}{x}}$