9.3 Maxima and Minima

We look at *relative* minima and maxima points on two–dimensional surfaces. For all points in a circular region containing (a, b), there is a

- *relative minimum* at (a, b) if $f(a, b) \leq f(x, y)$
- *relative maximum* at (a, b) if $f(a, b) \geq f(x, y)$

More than this, for function $z = f(x, y)$, a relative minimum or relative maximum are located at *critical point* (a, b) where, as shown in figure,

$$f_x(a, b) = 0, \quad f_y(a, b) = 0.$$

If, for example, maximum in x-axis direction, but minimum in y-axis direction (or vis-versa) as shown in right figure, then critical point (a, b) is a *saddlepoint*, neither a maximum nor a minimum.

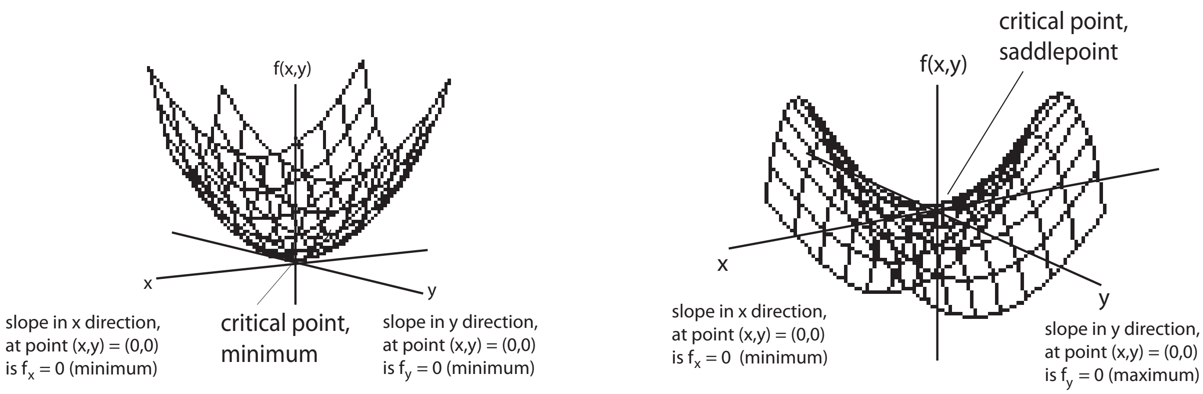

Figure 9.12 (critical point, extremum and saddlepoint)

Relative minima and maxima points are identified using the *discriminant test*:

- find f_x, f_y, f_{xx}, f_{yy}, f_{xy}
- find (a, b) such that $f_x(a, b) = 0$ and $f_y(a, b) = 0$
- find discriminant $D = f_{xx}(a, b) \cdot f_{yy}(a, b) - [f_{xy}(a, b)]^2$
- then
  - f (relative) maximum at (a, b) if $D > 0$ and $f_{xx}(a, b) < 0$
  - f (relative) minimum at (a, b) if $D > 0$ and $f_{xx}(a, b) > 0$

- f saddlepoint at (a, b) if $D < 0$
- test not applicable, gives no information, if $D = 0$

Notice discriminant test only considers the sign of $f_{xx}(a, b)$ alone, does not consider the sign of $f_{yy}(a, b)$–this is because if $D > 0$ then *both* $f_{xx}(a, b)$ and $f_{yy}(a, b)$ *must* either positive or both must be negative; in other words, both must be the same sign, and so only one of the two need be checked. *Boundary* relative extrema are *not* considered. Functions are *assumed* to be *differentiable*. Finally, *absolute* extrema are *not* considered in this course.

Exercise 9.3 (Maxima and Minima)

1. *Maxima–Minima of $z = 3x + 4y$?*

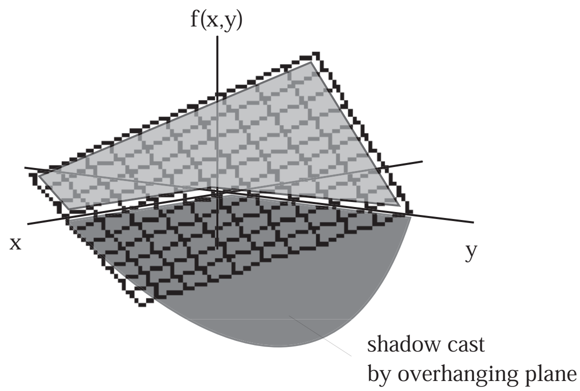

Figure 9.13 (Maxima–Minima of $z = 3x + 4y$?)

(a) *Possible extrema: locating critical points.*

Recall, since

$$f_x(x, y) = 3 \cdot 1x^{1-1} + 0 =$$

(i) **0** (ii) **3** (iii) **4**

and

$$f_y(x, y) = 0 + 4 \cdot 1y^{1-1} =$$

(i) **0** (ii) **3** (iii) **4**

plane $z = 3x + 4y$ (i) **does have** (ii) **does not have any** critical points and so, consequently, no minima or maxima.

there are no critical points because $f_x(x, y) = 3 \neq 0$ and $f_y(x, y) = 4 \neq 0$

(b) *Identifying which critical points are extrema: discriminant test.*

Since no critical points, discriminant test (i) **is** (ii) **is not** applicable here.

(c) *Related questions.*

If *domain* of plane $z = 3x + 4y$ was constrained in some x–y region, say $-5 \leq x \leq 5$ and $-5 \leq y \leq 5$, minima or maxima point(s) must necessarily appear (i) **along the edge** (ii) **in the interior** of this bounded plane.

2. *Maxima–Minima of $z = x^2$.*

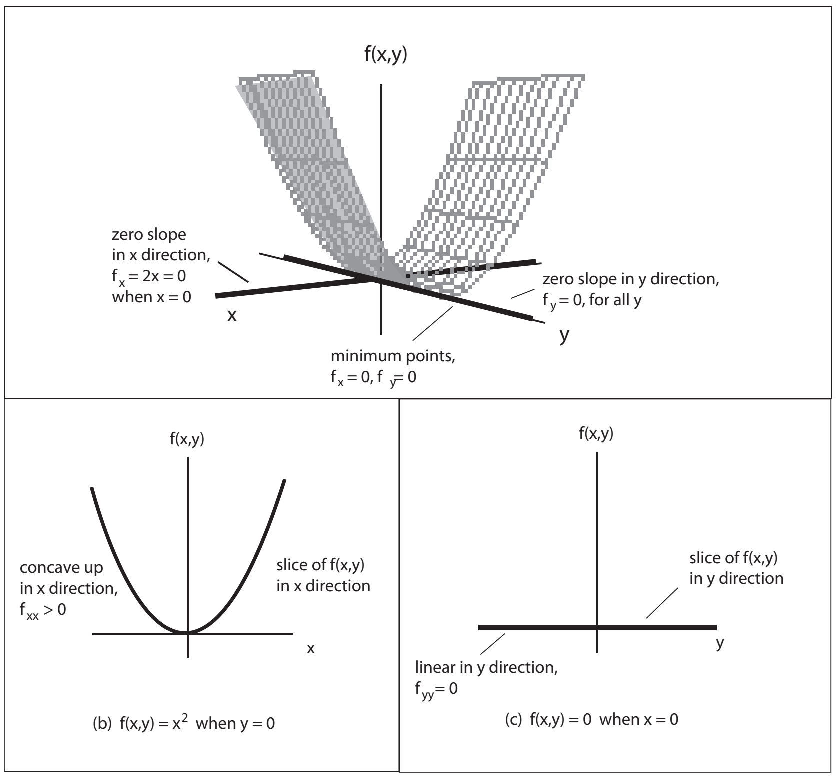

Figure 9.14 (Maxima–Minima of $z = x^2$)

(a) *Possible extrema: locating critical points.*

Recall, since

$$f_x(x, y) = 2x^{2-1} =$$

(i) **2** (ii) $\boldsymbol{2x}$ (iii) $\boldsymbol{2x^2}$

and

$$f_y(x, y) = \frac{\partial}{\partial y}\left(x^2\right) =$$

(i) **0** (ii) **3** (iii) **4**

so *all* critical points which satisfy

$$f_x(x,y) = 2x = 0, \quad f_y(x,y) = 0$$

are $(a,b) =$ (i) $\mathbf{(0,0)}$ (ii) $\boldsymbol{(0,y), y \geq 0}$ (iii) $\boldsymbol{(0,y), -\infty < y < \infty}$

y can be anything, $-\infty < y < \infty$, because slope in y-axis direction is always zero, $f_y(x,y) = 0$

(b) *Identifying which critical points are extrema: discriminant test.*

Since

$$f_{xx}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}(2x) = 2 \cdot 1x^{1-1} =$$

(i) **2** (ii) $\boldsymbol{2x}$ (iii) $\boldsymbol{2x^2}$

and

$$f_{xy}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2x) =$$

(i) **0** (ii) **3** (iii) **4**

$f_{xy} = 0$ because $2x$ is a *constant* with respect to y

and

$$f_{yy}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}(0) =$$

(i) **0** (ii) **3** (iii) **4**

so, summarizing,

$$f_{xx}(x,y) = 2, \quad f_{yy}(x,y) = 0, \quad f_{xy}(x,y) = 0$$

so, at critical points $(a,b) = (0,y)$,

$$f_{xx}(0,y) =$$

(i) **0** (ii) **2** (iii) **8**

$f_{xx}(0,0) = 2$ because $f_{xx}(x,y) = 2$ for any (x,y) including $(x,y) = (0,y)$

and

$$f_{yy}(0,y) =$$

(i) **0** (ii) **2** (iii) **8**

$f_{yy}(0,0) = 0$ because $f_{yy}(x,y) = 0$ for any (x,y) including $(x,y) = (0,y)$

and

$$f_{xy}(0, y) =$$

(i) **0** (ii) **2** (iii) **8**

$f_{xy}(0,0) = 0$ because $f_{xy}(x,y) = 0$ for any (x,y) including $(x,y) = (0,y)$

so, at critical points $(a,b) = (0,y)$, $-\infty < y < \infty$,

$$\begin{aligned} D &= f_{xx}(a,b) \cdot f_{yy}(a,b) - [f_{xy}(a,b)]^2 \\ &= f_{xx}(0,y) \cdot f_{yy}(0,y) - [f_{xy}(0,y)]^2 \\ &= 2 \cdot 0 - [0]^2 = \end{aligned}$$

(i) **0** (ii) **1** (iii) **2**

and so, since $D = 0$, discriminant test here
(i) **says critical points** $(a,b) = (0,y)$ **are minima**
(ii) **says critical points** $(a,b) = (0,y)$ **are maxima**
(iii) **says critical point** $(a,b) = (0,y)$ **is a saddlepoint**
(iv) **unable to tell if critical point is a minimum/maximum**

Although discriminant test unable to tell if minima or maxima, critical points $(a,b) = (0,y)$ clearly are minima because $f_{xx}(0,y) = 2 > 0$ and $f_{yy}(0,y) = 0$ which means function $f(x,y) = x^2$
(i) **concave up in both the** x**–axis direction and** y**–axis direction**
(ii) **concave up in** x**–axis direction, linear in** y**–axis direction**
(iii) **concave up in** y**–axis direction, linear in** x**–axis direction**
(iv) **concave down in both** x**–axis direction and** y**–axis direction**

3. *Maxima–Minima of* $z = x^2 + y^2$.

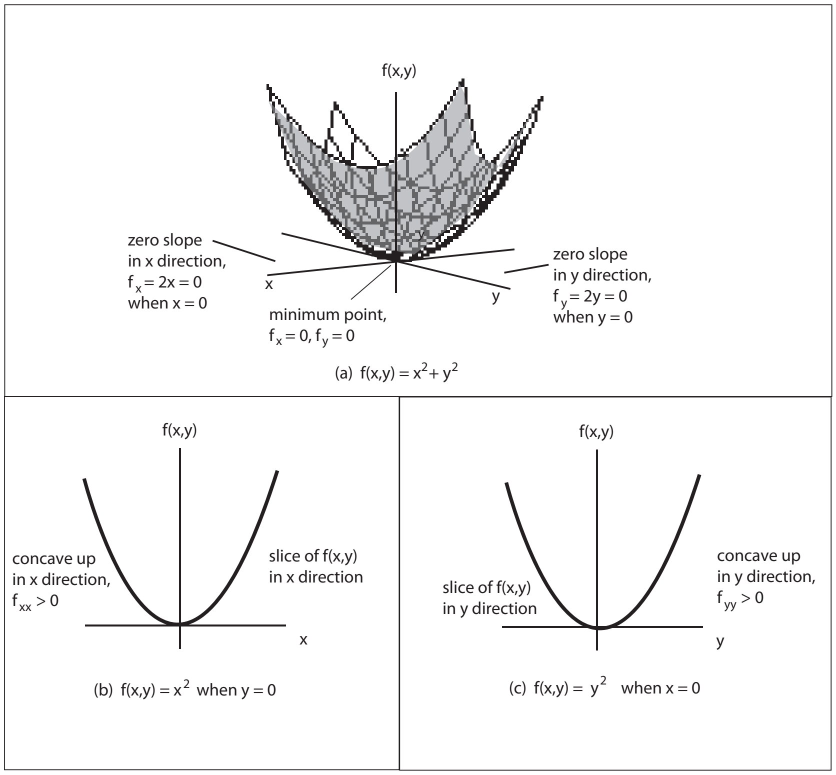

Figure 9.15 (Maxima–Minima of $z = x^2 + y^2$)

(a) *Possible extrema: locating critical points.*

Recall, since

$$f_x(x, y) = 2x^{2-1} + 0 =$$

(i) $\boldsymbol{2x}$ (ii) $\boldsymbol{2y}$ (iii) $\boldsymbol{2x + 2y}$

derivative of x^2 with respect to x is $2x$, but derivative of y^2 is *zero* because y^2 *constant* with respect to x

and

$$f_y(x, y) = 0 + 2y^{2-1} =$$

(i) $\boldsymbol{2x}$ (ii) $\boldsymbol{2y}$ (iii) $\boldsymbol{2x + 2y}$

derivative of y^2 with respect to y is $2y$, but derivative of x^2 is zero because x^2 *constant* with respect to y

so critical point(s) which satisfy

$$f_x(x, y) = 2x = 0, \quad f_y(x, y) = 2y = 0$$

is/are $(a, b) =$ (i) $\boldsymbol{(0, 0)}$ (ii) $\boldsymbol{(0, y), y \geq 0}$ (iii) $\boldsymbol{(0, y), -\infty < y < \infty}$

(b) *Identifying which critical point(s) is/are extrema: discriminant test.*

Since

$$f_{xx}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(2x) = 2 \cdot 1x^{1-1} =$$

(i) **0** (ii) **2** (iii) **4**

and

$$f_{xy}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(2x) =$$

(i) **0** (ii) **2** (iii) **4**

$f_{xy} = 0$ because $2x$ is a *constant* with respect to y

and

$$f_{yy}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(2y) = 2 \cdot 1y^{1-1} =$$

(i) **0** (ii) **2** (iii) **4**

so, summarizing,

$$f_{xx}(x,y) = 2, \quad f_{yy}(x,y) = 2, \quad f_{xy}(x,y) = 0$$

so, at critical point $(a,b) = (0,0)$,

$$f_{xx}(0,0) =$$

(i) **0** (ii) **2** (iii) **8**

$f_{xx}(0,0) = 2$ because $f_{xx}(x,y) = 2$ for any (x,y) including $(x,y) = (0,0)$

and

$$f_{yy}(0,0) =$$

(i) **0** (ii) **2** (iii) **8**

$f_{yy}(0,0) = 2$ because $f_{yy}(x,y) = 2$ for any (x,y) including $(x,y) = (0,0)$

and

$$f_{xy}(0,0) =$$

(i) **0** (ii) **2** (iii) **8**

$f_{xy}(0,0) = 0$ because $f_{xy}(x,y) = 0$ for any (x,y) including $(x,y) = (0,0)$

so, at critical point $(a, b) = (0, 0)$,

$$\begin{aligned} D &= f_{xx}(a,b) \cdot f_{yy}(a,b) - [f_{xy}(a,b)]^2 \\ &= f_{xx}(0,0) \cdot f_{yy}(0,0) - [f_{xy}(0,0)]^2 \\ &= 2 \cdot 2 - [0]^2 = \end{aligned}$$

(i) **0** (ii) **2** (iii) **4**

since $D = 4 > 0$ and $f_{xx}(0,0) = 2 > 0$, discriminant test
(i) **says critical point** $(a, b) = (0, 0)$ **is a minimum**
(ii) **says critical point** $(a, b) = (0, 0)$ **is a maximum**
(iii) **says critical point** $(a, b) = (0, 0)$ **is a saddlepoint**
(iv) **unable to tell if critical point is a minimum/maximum**

furthermore, at minimum point $(a, b) = (0, 0)$,
function $f(0, 0) = (0)^2 + (0)^2 =$ (i) **0** (ii) **2** (iii) **4**

4. *Maxima–Minima of* $z = \frac{-4}{1+x^2+y^2}$.

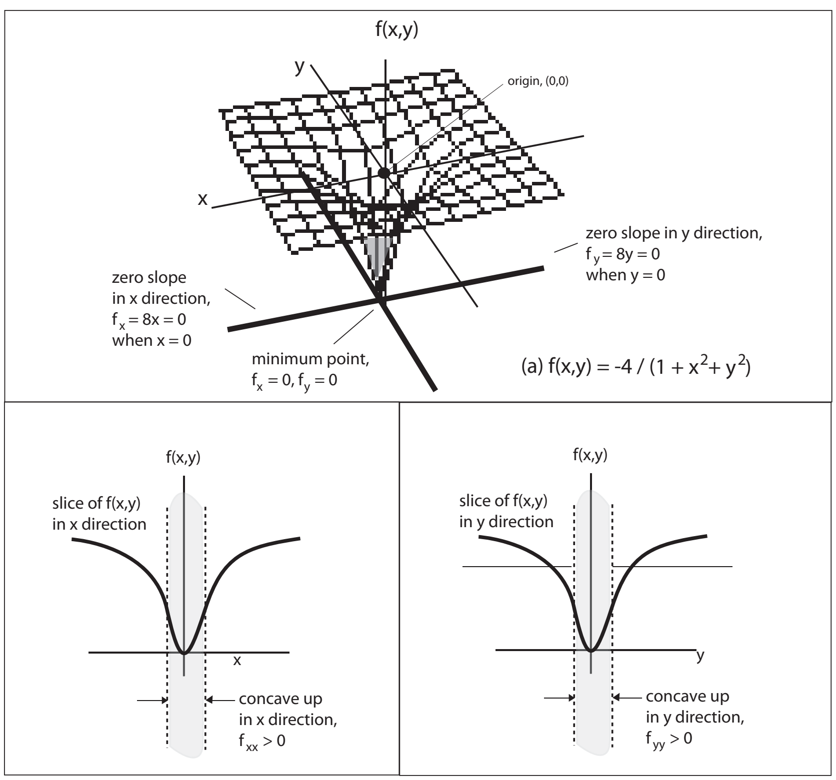

Figure 9.16 (Maxima–Minima of $z = \frac{-4}{1+x^2+y^2}$)

(a) *Possible extrema: locating critical points.*

Recall, since

$$f_x(x,y) = \frac{8x}{(1+x^2+y^2)^2}$$

and

$$f_y(x,y) = \frac{8y}{(1+x^2+y^2)^2}$$

so critical point(s) which satisfy

$$f_x(x,y) = \frac{8x}{(1+x^2+y^2)^2} = 0, \quad f_y(x,y) = \frac{8y}{(1+x^2+y^2)^2} = 0$$

or, since $(1+x^2+y^2)^2 > 0$, equivalently

$$f_x(x,y) = 8x = 0, \quad f_y(x,y) = 8y = 0$$

is/are $(a,b) =$ (i) $\mathbf{(0,0)}$ (ii) $\boldsymbol{(0,y), y \geq 0}$ (iii) $\boldsymbol{(x,0), x \geq 0}$

(b) *Identifying which critical point(s) is/are extrema: discriminant test.*

Recall,

$$f_{xx}(x,y) = \frac{8-32x^2}{(1+x^2+y^2)^3}, \quad f_{yy}(x,y) = \frac{8-32y^2}{(1+x^2+y^2)^2}, \quad f_{xy}(x,y) = \frac{8xy}{(1+x^2+y^2)^2}$$

so, at critical point $(a,b) = (0,0)$,

$$f_{xx}(0,0) = \frac{8-32(0)^2}{(1+(0)^2+(0)^2)^3} =$$

(i) **0** (ii) **2** (iii) **8**

and

$$f_{yy}(0,0) = \frac{8-32(0)^2}{(1+(0)^2+(0)^2)^2} =$$

(i) **0** (ii) **2** (iii) **8**

and

$$f_{xy}(0,0) = \frac{8(0)(0)}{(1+(0)^2+(0)^2)^2} =$$

(i) **0** (ii) **2** (iii) **8**

and, finally,

$$\begin{aligned} D &= f_{xx}(a,b) \cdot f_{yy}(a,b) - [f_{xy}(a,b)]^2 \\ &= f_{xx}(0,0) \cdot f_{yy}(0,0) - [f_{xy}(0,0)]^2 \\ &= 8 \cdot 8 - [0]^2 = \end{aligned}$$

(i) **0** (ii) **8** (iii) **64**

since $D = 64 > 0$ and $f_{xx}(0,0) = 8 > 0$, discriminant test
(i) **says critical point** $(a,b) = (0,0)$ **is a minimum**
(ii) **says critical point** $(a,b) = (0,0)$ **is a maximum**
(iii) **says critical point** $(a,b) = (0,0)$ **is a saddlepoint**
(iv) **unable to tell if critical point is a minimum/maximum**

furthermore, at minimum point $(a,b) = (0,0)$,
function $f(0,0) = \frac{-4}{1+(0)^2+(0)^2}$ = (i) $\mathbf{-8}$ (ii) $\mathbf{-4}$ (iii) **4**

5. *Maxima–Minima of* $z = 5y + 4x - x^2 - y^2$.

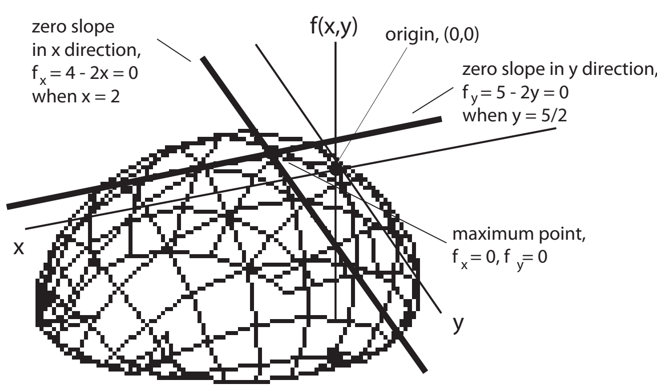

Figure 9.17 (Maxima–Minima of $z = 5y + 4x - x^2 - y^2$)

(a) *Possible extrema: locating critical points.*

Since

$$f_x(x,y) = 0 + 4 \cdot 1x^{1-1} - 2x^{2-1} - 0 =$$

(i) $\mathbf{4 - 2x}$ (ii) $\mathbf{5 - 2y}$ (iii) $\mathbf{2x + 2y}$

and

$$f_y(x,y) = 5 \cdot 1y^{1-1} + 0 - 0 - 2y^{2-1} =$$

(i) $\mathbf{4 - 2x}$ (ii) $\mathbf{5 - 2y}$ (iii) $\mathbf{2x + 2y}$

so critical point which satisfies

$$f_x(x,y) = 4 - 2x = 0, \quad f_y(x,y) = 5 - 2y = 0$$

is $(a,b) =$ (i) $\mathbf{(0,0)}$ (ii) $\left(\mathbf{2, \frac{5}{2}}\right)$ (iii) $\left(\mathbf{\frac{5}{2}, 2}\right)$

if $4 - 2x = 0$, then $4 = 2x$, or $x = 2$; also, if $5 - 2y = 0$, then $5 = 2y$ or $y = \frac{5}{2}$

(b) *Identifying which critical point(s) is/are extrema: discriminant test.*

Since

$$f_{xx}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(4 - 2x) = 0 - 2 \cdot 1x^{1-1} =$$

(i) $\mathbf{0}$ (ii) $\mathbf{-2}$ (iii) $\mathbf{-4}$

and

$$f_{xy}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(4 - 2x) = 0 + 0 =$$

(i) $\mathbf{0}$ (ii) $\mathbf{-2}$ (iii) $\mathbf{-4}$

and

$$f_{yy}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(5 - 2y) = 0 - 2 \cdot 1y^{1-1} =$$

(i) $\mathbf{0}$ (ii) $\mathbf{-2}$ (iii) $\mathbf{-4}$

so, summarizing,

$$f_{xx}(x,y) = -2, \quad f_{yy}(x,y) = -2, \quad f_{xy}(x,y) = 0$$

so, at critical point $(a,b) = \left(2, \frac{5}{2}\right)$,

$$f_{xx}\left(2, \frac{5}{2}\right) =$$

(i) $\mathbf{0}$ (ii) $\mathbf{-2}$ (iii) $\mathbf{-8}$

$f_{xx}\left(2, \frac{5}{2}\right) = -2$ because $f_{xx}(x,y) = 2$ for any (x,y) including $(x,y) = \left(2, \frac{5}{2}\right)$

and

$$f_{yy}\left(2, \frac{5}{2}\right) =$$

(i) **0** (ii) **−2** (iii) **−8**

$f_{yy}\left(2, \frac{5}{2}\right) = -2$ because $f_{yy}(x,y) = 2$ for any (x,y) including $(x,y) = \left(2, \frac{5}{2}\right)$

and

$$f_{xy}\left(2, \frac{5}{2}\right) =$$

(i) **0** (ii) **−2** (iii) **−8**

$f_{xy}\left(2, \frac{5}{2}\right) = 0$ because $f_{xy}(x,y) = 0$ for any (x,y) including $(x,y) = \left(2, \frac{5}{2}\right)$

so, at critical point $(a,b) = \left(2, \frac{5}{2}\right)$,

$$\begin{aligned} D &= f_{xx}(a,b) \cdot f_{yy}(a,b) - [f_{xy}(a,b)]^2 \\ &= f_{xx}\left(2, \frac{5}{2}\right) \cdot f_{yy}\left(2, \frac{5}{2}\right) - \left[f_{xy}\left(2, \frac{5}{2}\right)\right]^2 \\ &= (-2)(-2) - [0]^2 = \end{aligned}$$

(i) **0** (ii) **−4** (iii) **4**

since $D = 4 > 0$ and $f_{xx}(0,0) = -2 < 0$, discriminant test
(i) **says critical point** $(a,b) = \left(2, \frac{5}{2}\right)$ **is a minimum**
(ii) **says critical point** $(a,b) = \left(2, \frac{5}{2}\right)$ **is a maximum**
(iii) **says critical point** $(a,b) = \left(2, \frac{5}{2}\right)$ **is a saddlepoint**
(iv) **unable to tell if critical point is a minimum/maximum**

furthermore, at maximum point $(a,b) = \left(2, \frac{5}{2}\right)$,
$f\left(2, \frac{5}{2}\right) = 5\left(\frac{5}{2}\right) + 4(2) - (2)^2 - \left(\frac{5}{2}\right)^2 =$ (i) **8.25** (ii) **9.25** (iii) **10.25**

6. *Maxima–Minima of* $z = 2x^2 - 2y^2$.

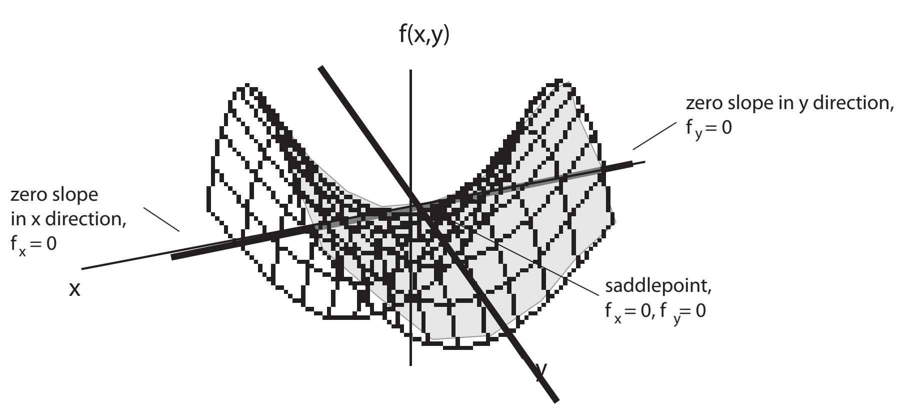

Figure 9.18 (Maxima–Minima of $z = 2x^2 - 2y^2$)

(a) *Possible extrema: locating critical points.*

Recall, since

$$f_x(x,y) = 2 \cdot 2x^{2-1} + 0 =$$

(i) $\boldsymbol{4x}$ (ii) $\boldsymbol{-4y}$ (iii) $\boldsymbol{4x - 4y}$

and

$$f_y(x,y) = 0 - 2 \cdot 2y^{2-1} =$$

(i) $\boldsymbol{4x}$ (ii) $\boldsymbol{-4y}$ (iii) $\boldsymbol{4x - 4y}$

so critical point(s) which satisfy

$$f_x(x,y) = 4x = 0, \quad f_y(x,y) = -4y = 0$$

is/are $(a,b) =$ (i) $\mathbf{(0,0)}$ (ii) $\mathbf{(4,-4)}$ (iii) $\mathbf{(0,-4)}$

(b) *Identifying which critical point(s) is/are extrema: discriminant test.*

Since

$$f_{xx}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(4x) = 4 \cdot 1x^{1-1} =$$

(i) **0** (ii) **4** (iii) **−4**

and

$$f_{xy}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(4x) =$$

(i) **0** (ii) **4** (iii) **−4**

$f_{xy} = 0$ because $4x$ is a *constant* with respect to y

and

$$f_{yy}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(-4y) = -4 \cdot 1y^{1-1} =$$

(i) **0** (ii) **4** (iii) **−4**

so, summarizing,

$$f_{xx}(x,y) = 4, \quad f_{yy}(x,y) = -4, \quad f_{xy}(x,y) = 0$$

so, at critical point $(a,b) = (0,0)$,

$$f_{xx}(0,0) =$$

(i) **0** (ii) **4** (iii) **−4**

$f_{xx}(0,0) = 4$ because $f_{xx}(x,y) = 4$ for any (x,y) including $(x,y) = (0,0)$

and

$$f_{yy}(0,0) =$$

(i) **0** (ii) **4** (iii) **−4**

$f_{yy}(0,0) = -4$ because $f_{yy}(x,y) = -4$ for any (x,y) including $(x,y) = (0,0)$

and

$$f_{xy}(0,0) =$$

(i) **0** (ii) **4** (iii) **−4**

$f_{xy}(0,0) = 0$ because $f_{xy}(x,y) = 0$ for any (x,y) including $(x,y) = (0,0)$

so, at critical point $(a,b) = (0,0)$,

$$\begin{aligned} D &= f_{xx}(a,b) \cdot f_{yy}(a,b) - [f_{xy}(a,b)]^2 \\ &= f_{xx}(0,0) \cdot f_{yy}(0,0) - [f_{xy}(0,0)]^2 \\ &= (4)(-4) - [0]^2 = \end{aligned}$$

(i) **−16** (ii) **16** (iii) **4**

since $D = -16 < 0$, discriminant test
(i) **says critical point** $(a,b) = (0,0)$ **is a minimum**
(ii) **says critical point** $(a,b) = (0,0)$ **is a maximum**
(iii) **says critical point** $(a,b) = (0,0)$ **is a saddlepoint**
(iv) **unable to tell if critical point is a minimum/maximum**

furthermore, at saddlepoint $(a,b) = (0,0)$,
function $f(0,0) = 2(0)^2 - 2(0)^2 =$ (i) **0** (ii) **2** (iii) **4**

7. *Application: Maximizing Profit.*

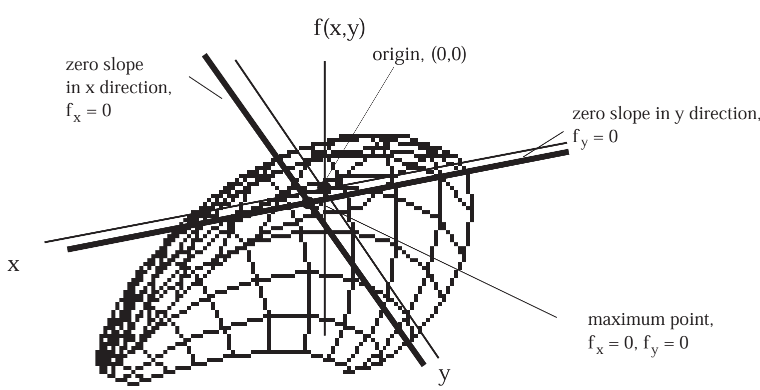

Figure 9.19 (Maximizing Profit)

Revenue of selling x scarfs, at \$5 each, and y pairs of gloves, at \$22 each, is $R(x,y) = 5x+22y$. Cost of production is $C(x,y) = 3x^2-3xy+3y^2-10x+22y-5$. How many scarfs and gloves must be sold to maximize profit?

(a) *Profit.* Since

$$R(x,y) = 5x + 22y, \quad C(x,y) = 3x^2 - 3xy + 3y^2 - 10x + 22y - 5$$

then profit $P(x,y) = R(x,y) - C(x,y) =$

(i) $\mathbf{-3x^2 + 3xy - 3y^2 + 15x + 5}$
(ii) $\mathbf{3x^2 + 3xy - 3y^2 + 15x + 5}$
(iii) $\mathbf{6x^2 + 3xy - 3y^2 + 15x + 5}$

(b) *Possible extrema: locating critical points.*

Since $P(x,y) = -3x^2 + 3xy - 3y^2 + 15x + 5$

$$f_x(x,y) = -3 \cdot 2x^{2-1} + 3x^{1-1}y - 0 + 15x^{1-1} + 0 =$$

(i) $\mathbf{-6x + 3y + 15}$ (ii) $\mathbf{3x - 6y}$ (iii) $\mathbf{-6}$

and

$$f_y(x,y) = 0 + 3xy^{1-1} - 3 \cdot 2y^{2-1} + 0 + 0 =$$

(i) $\mathbf{-6x + 3y + 15}$ (ii) $\mathbf{3x - 6y}$ (iii) $\mathbf{-6}$

so critical point which satisfies

$$f_x(x,y) = -6x + 3y + 15 = 0, \quad f_y(x,y) = 3x - 6y = 0$$

is $(a,b) =$ (i) $\mathbf{(0,0)}$ (ii) $\mathbf{\left(\frac{30}{9}, \frac{15}{9}\right)}$ (iii) $\mathbf{\left(\frac{15}{9}, \frac{30}{9}\right)}$

since $3x - 6y = 0$, then $3x = 6y$, or $x = 2y$, so $-6x + 3y + 15 = -6(2y) + 3y + 15 = 0$, or $-9y + 15 = 0$, so $y = \frac{15}{9}$ and $x = 2\left(\frac{15}{9}\right) = \frac{30}{9}$

(c) *Identifying which critical point(s) is/are extrema: discriminant test.*

Since

$$f_{xx}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(-6x + 3y + 15) = -6 \cdot 1x^{1-1} + 0 + 0 =$$

(i) **0** (ii) **−6** (iii) **−4**

and

$$f_{xy}(x, y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(-6x + 3y + 15) = 0 + 3y^{1-1} + 0 =$$

(i) **0** (ii) **−6** (iii) **3**

and

$$f_{yy}(x, y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(3x - 6y) = 0 - 6y^{1-1} =$$

(i) **0** (ii) **−6** (iii) **3**

so, summarizing,

$$f_{xx}(x, y) = -6, \quad f_{yy}(x, y) = -6, \quad f_{xy}(x, y) = 3$$

so, at critical point $(a, b) = \left(\frac{30}{9}, \frac{15}{9}\right)$,

$$f_{xx}\left(\frac{30}{9}, \frac{15}{9}\right) =$$

(i) **−6** (ii) **0** (iii) **3**

and

$$f_{yy}\left(\frac{30}{9}, \frac{15}{9}\right) =$$

(i) **−6** (ii) **0** (iii) **3**

and

$$f_{xy}\left(\frac{30}{9}, \frac{15}{9}\right) =$$

(i) **−6** (ii) **0** (iii) **3**

so, at critical point $(a, b) = \left(\frac{30}{9}, \frac{15}{9}\right)$,

$$\begin{aligned} D &= f_{xx}(a, b) \cdot f_{yy}(a, b) - [f_{xy}(a, b)]^2 \\ &= f_{xx}\left(\frac{30}{9}, \frac{15}{9}\right) \cdot f_{yy}\left(\frac{30}{9}, \frac{15}{9}\right) - \left[f_{xy}\left(\frac{30}{9}, \frac{15}{9}\right)\right]^2 \\ &= (-6)(-6) - [3]^2 = \end{aligned}$$

(i) **0** (ii) **27** (iii) **36**

since $D = 27 > 0$ and $f_{xx}(0,0) = -6 < 0$, discriminant test
(i) **says critical point** $(a,b) = \left(\frac{30}{9}, \frac{15}{9}\right)$ **is a minimum**
(ii) **says critical point** $(a,b) = \left(\frac{30}{9}, \frac{15}{9}\right)$ **is a maximum**
(iii) **says critical point** $(a,b) = \left(\frac{30}{9}, \frac{15}{9}\right)$ **is a saddlepoint**
(iv) **unable to tell if critical point is a minimum/maximum**

furthermore, at maximum point $(a,b) = \left(\frac{30}{9}, \frac{15}{9}\right)$,
$f\left(\frac{30}{9}, \frac{15}{9}\right) = -3\left(\frac{30}{9}\right)^2 + 3\left(\frac{30}{9}\right)\left(\frac{15}{9}\right) - 3\left(\frac{15}{9}\right)^2 + 15\left(\frac{30}{9}\right) + 5 =$
(i) **8** (ii) **10** (iii) **30**

9.4 Lagrange Multipliers

The method of *Lagrange multipliers* seeks to solve *constrained optimization problems*:

$$\text{optimize } f(x,y)$$
$$\text{subject to } g(x,y) = 0,$$

where $f(x,y)$ is called the *objective function* and $g(x,y) = 0$ the (equality) *constraint*. The steps in this method include:

- create Lagrange function

  $$F(x,y,\lambda) = f(x,y) - \lambda \cdot g(x,y)$$

  a constraint such as $r(x,y) = c$ must be rewritten as $g(x,y) = r(x,y) - c = 0$

- determine partial derivatives $F_x(x,y,\lambda)$, $F_y(x,y,\lambda)$, $F_\lambda(x,y,\lambda)$

- create and solve system of equations

  $$F_x(x,y,\lambda) = 0, \quad F_y(x,y,\lambda) = 0, \quad F_\lambda(x,y,\lambda) = 0$$

  for critical points (which may be minima, maxima or saddlepoints)

Notice the method of Lagrange multipliers is used to determine extrema for *two* equations; specifically, it identifies *relative* critical points (minima, maxima or saddlepoints) of one (differentiable) function $f(x,y)$ subject to (constrained by) another *equality constraint* equation, $g(x,y) = 0$. Although not considered here, Lagrange multipliers method can also deal with *multiple* equality constraints. A generalization of the Lagrange multipliers method, which involves *linear inequalities*, is the *Kuhn-Tucker method*, but this generalization is not considered here. As shown previously, although typically more difficult to undertake, sometimes it is possible to combine $f(x,y)$ and $g(x,y) = 0$ by, for example, solving for x in one equation, substituting into the other equation and then determining extrema from the one combined equation of one variable. The method of Lagrange multipliers is compared to this method in one example below.

Exercise 9.4 (Lagrange Multipliers)

1. *Maximizing rectangular garden area with restricted fence length.*
   Darlene has 200 feet of rabbit fence to enclose a garden she wishes to put in her backyard. What should be the length and width, (x, y), of this rectangular garden to maximize the area, $A = xy$?

   (a) *Lagrange function*

   Since there are four sides, two lengths and two widths, of the 200 foot fence around the garden, the constrained optimization problem is

   $$\text{maximize } A(x, y) = xy$$
   $$\text{subject to } 2x + 2y = 200$$

   related Lagrange function is

   $$F(x, y, \lambda) = A(x, y) - \lambda \cdot g(x, y) =$$

   (i) $\boldsymbol{xy - \lambda(200)}$ (ii) $\boldsymbol{xy - \lambda(2x + 2y)}$ (iii) $\boldsymbol{xy - \lambda(2x + 2y - 200)}$
   notice how $2x + 2y = 200$ has been rewritten $g(x, y) = 2x - 2y - 200 = 0$

   (b) *partial derivatives*

   since $F(x, y, \lambda) = xy - \lambda(2x + 2y - 200) = xy - 2x\lambda - 2y\lambda + 200\lambda$,

   $$F_x(x, y, \lambda) = 1x^{1-1}y - 2x^{1-1}\lambda - 0 + 0 =$$

   (i) $\boldsymbol{y - 2\lambda}$ (ii) $\boldsymbol{x - 2\lambda}$ (iii) $\boldsymbol{-2x - 2y + 200}$

   and

   $$F_y(x, y, \lambda) = xy^{1-1} - 0 - 2y^{1-1}\lambda + 0 =$$

   (i) $\boldsymbol{y - 2\lambda}$ (ii) $\boldsymbol{x - 2\lambda}$ (iii) $\boldsymbol{-2x - 2y + 200}$

   and

   $$F_\lambda(x, y, \lambda) = 0 - \lambda^{1-1}(2x + 2y - 200) =$$

   (i) $\boldsymbol{y - 2\lambda}$ (ii) $\boldsymbol{x - 2\lambda}$ (iii) $\boldsymbol{-2x - 2y + 200}$

   (c) *system of equations*

   $$F_x(x, y, \lambda) = 0, \quad F_y(x, y, \lambda) = 0, \quad F_\lambda(x, y, \lambda) = 0$$

   gives

   $$\begin{aligned} F_x(x, y, \lambda) &= y - 2\lambda = 0 \\ F_y(x, y, \lambda) &= x - 2\lambda = 0 \\ F_\lambda(x, y, \lambda) &= -2x - 2y + 200 = 0 \end{aligned}$$

and so from first two equations,

$$\lambda = \frac{x}{2}, \quad \lambda = \frac{y}{2}$$

which implies $\frac{x}{2} = \frac{y}{2}$ or $x = y$ and so

$$-2x - 2y + 200 = -2(y) - 2y + 200 = 0$$

or $y =$ (i) **40** (ii) **50** (iii) **60**
since $-4y + 200 = 0$, then $y = \frac{200}{4} = 50$

and so $x = y =$ (i) **40** (ii) **50** (iii) **60**

so $(x, y) =$ (i) $\mathbf{(40, 60)}$ (ii) $\mathbf{(50, 50)}$ (iii) $\mathbf{(60, 40)}$

also area $A(x, y) = xy = 50(50) =$ (i) **800** (ii) **1000** (iii) **2500**

(d) *(previous) solution method 2, NOT using Lagrange multipliers*

Since

$$A(x, y) = xy, \quad 2x + 2y = 200$$

then $2y = 200 - 2x$ or $y = 100 - x$, so

$$A = xy = x(100 - x) = 100x - x^2$$

and since length and area cannot be negative;
in other words, $x \geq 0$ and $A = x(100 - x) \geq 0$, or

(i) $\boldsymbol{x \geq 0, x \leq 100}$ or $\mathbf{[0, 100]}$
(ii) $\boldsymbol{x \leq 0, x \leq 100}$ or $\mathbf{(-\infty, 100]}$
(iii) $\boldsymbol{x \geq 0, x \geq 100}$ or $\mathbf{[0, \infty)}$

and since

$$A_x = 100(1)x^{2-1} - 2x^{1-1} = 100 - 2x,$$

there is *one* critical number at
$x = \frac{100}{2} =$ (i) **50** (ii) **2** (iii) **100**
and since $A(50) = 100(50) - (50)^2 = 2500$,
there is a critical *point* $(c, A(c)) = (50, 2500)$.

and so *lower endpoint* at $x = 0$,
$A(0) = 100(0) - (0)^2 =$ (i) **0** (ii) **100** (iii) **50**

and *upper endpoint* at $x = 100$,
$A(2) = 100(100) - (100)^2 =$ (i) **0** (ii) **50** (iii) $\boldsymbol{\infty}$

so, summarizing,

length candidates, x	area, A
0	0
50	2500
100	0

maximum area is
(i) **0** (ii) **2500** (iii) **50** (iv) **none**
which is what we got using method of Lagrange multipliers

and occurs when length x is
(i) **0** (ii) **2500** (iii) **50** (iv) **none**
which, again, is what we got using method of Lagrange multipliers

also, width $y = 100 - x = 100 - 50 = 50$

2. *Minimize* $f(x, y) = x^2 + y^2$ *with constraint* $xy = 1$.

   (a) *Lagrange function*

   Since constrained optimization problem is

$$\text{minimize } f(x, y) = x^2 + y^2$$
$$\text{subject to } xy = 1$$

   related Lagrange function is

$$F(x, y, \lambda) = f(x, y) - \lambda \cdot g(x, y) =$$

   (i) $\boldsymbol{x^2 + y^2 - \lambda(xy - 1)}$ (ii) $\boldsymbol{x^2 + y^2 - \lambda}$ (iii) $\boldsymbol{x^2 + y^2 - \lambda(xy)}$
   notice how $xy = 1$ has been rewritten $g(x, y) = xy - 1 = 0$

   (b) *partial derivatives*

   since $F(x, y, \lambda) = x^2 + y^2 - \lambda(xy - 1) = x^2 + y^2 - xy\lambda + \lambda$,

$$F_x(x, y, \lambda) = 2x^{2-1} + 0 - x^{1-1}y\lambda + 0 =$$

   (i) $\boldsymbol{2y - x\lambda}$ (ii) $\boldsymbol{2x - y\lambda}$ (iii) $\boldsymbol{-xy + 1}$

and

$$F_y(x, y, \lambda) = 0 + 2y^{2-1} - xy^{1-1}\lambda + 0 =$$

(i) $\boldsymbol{2y - x\lambda}$ (ii) $\boldsymbol{2x - y\lambda}$ (iii) $\boldsymbol{-xy + 1}$

and

$$F_\lambda(x, y, \lambda) = 0 + 0 - xy\lambda^{1-1} + \lambda^{1-1} =$$

(i) $\boldsymbol{2y - x\lambda}$ (ii) $\boldsymbol{2x - y\lambda}$ (iii) $\boldsymbol{-xy + 1}$

(c) *system of equations*

$$F_x(x, y, \lambda) = 0, \quad F_y(x, y, \lambda) = 0, \quad F_\lambda(x, y, \lambda) = 0$$

gives

$$\begin{aligned} F_x(x, y, \lambda) &= 2x - y\lambda = 0 \\ F_y(x, y, \lambda) &= 2y - x\lambda = 0 \\ F_\lambda(x, y, \lambda) &= -xy + 1 = 0 \end{aligned}$$

and so from first two equations,

$$\lambda = \frac{2x}{y}, \quad \lambda = \frac{2y}{x}$$

which implies $\frac{2x}{y} = \frac{2y}{x}$ or $x^2 = y^2$ and so from third equation

$$\begin{aligned} -xy + 1 &= 0 \\ xy &= 1 \\ x^2y^2 &= xy \quad \text{multiplying both sides by } xy \\ x^2(x^2) &= 1 \quad \text{since } y^2 = x^2 \text{ and } xy = 1 \\ x^4 &= 1 \end{aligned}$$

so $x =$ (i) $\mathbf{-1}$ (ii) $\mathbf{1}$ (iii) $\mathbf{\pm 1}$
since $x^4 = 1$, then $x = \pm\sqrt[4]{1} = \pm 1$

and, in a similar way, $y =$ (i) $\mathbf{-1}$ (ii) $\mathbf{1}$ (iii) $\mathbf{\pm 1}$

and since $xy = 1$ (implying x and y have the same sign)
$(x, y) =$ (choose one or more) (i) $\mathbf{(-1, -1)}$ (ii) $\mathbf{(-1, 1)}$ (iii) $\mathbf{(1, 1)}$

also $f(x, y) = x^2 + y^2 = (-1)^2 + (-1)^2 = (1)^2 + (1)^2 =$ (i) $\mathbf{0}$ (ii) $\mathbf{1}$ (iii) $\mathbf{2}$

(d) *minima, maxima or saddlepoint?*

Choose (x, y) which satisfies $xy = 1$; for example, $(x, y) = \left(\frac{1}{3}, 3\right)$,

$$xy = \left(\frac{1}{3}\right)(3) = 1$$

and notice

$$f(x,y) = x^2+y^2 = \left(\frac{1}{3}\right)^2+(3)^2 = \frac{82}{9} > f(x,y) = x^2+y^2 = (-1)^2+(-1)^2 = (1)^2+(1)^2 = 2$$

in other words, since $f(x, y)$ at choice $(x, y) = \left(\frac{1}{3}\right)(3)$ is larger than at either critical point, $(x, y) = (-1, -1)$, $(x, y) = (1, 1)$, this indicates $f(x, y)$ at critical points are
(i) **minima** (ii) **maxima** (iii) **saddlepoints**

3. *Minimize* $f(x, y) = 3x^2 + y^2 - 2xy$ *with constraint* $xy = 1$.

(a) *Lagrange function*

Since constrained optimization problem is

$$\text{minimize } f(x, y) = 3x^2 + y^2 - 2xy$$
$$\text{subject to } xy = 1$$

related Lagrange function is

$$F(x, y, \lambda) = f(x, y) - \lambda \cdot g(x, y) =$$

(i) $\boldsymbol{3x^2 + y^2 - 2xy - \lambda(xy - 1)}$
(ii) $\boldsymbol{3x^2 + y^2 - 2xy - \lambda}$
(iii) $\boldsymbol{3x^2 + y^2 - 2xy - \lambda(xy)}$
notice how $xy = 1$ has been rewritten $g(x, y) = xy - 1 = 0$

(b) *partial derivatives*

since $F(x, y, \lambda) = 3x^2 + y^2 - 2xy - \lambda(xy - 1) = 3x^2 + y^2 - 2xy - xy\lambda + \lambda$,

$$F_x(x, y, \lambda) = 3 \cdot 2x^{2-1} + 0 - 2x^{1-1}y - x^{1-1}y\lambda + 0 =$$

(i) $\boldsymbol{6x - 2y - y\lambda}$ (ii) $\boldsymbol{2y - 2x - x\lambda}$ (iii) $\boldsymbol{-xy + 1}$

and

$$F_y(x, y, \lambda) = 0 + 2y^{2-1} - 2xy^{1-1} - xy^{1-1}\lambda + 0 =$$

(i) $\boldsymbol{6x - 2y - y\lambda}$ (ii) $\boldsymbol{2y - 2x - x\lambda}$ (iii) $\boldsymbol{-xy + 1}$

and

$$F_\lambda(x, y, \lambda) = 0 + 0 - xy\lambda^{1-1} + \lambda^{1-1} =$$

(i) $\boldsymbol{6x - 2y - y\lambda}$ (ii) $\boldsymbol{2y - 2x - x\lambda}$ (iii) $\boldsymbol{-xy + 1}$

(c) *system of equations*

$$F_x(x, y, \lambda) = 0, \quad F_y(x, y, \lambda) = 0, \quad F_\lambda(x, y, \lambda) = 0$$

gives

$$\begin{aligned} F_x(x, y, \lambda) &= 6x - 2y - y\lambda = 0 \\ F_y(x, y, \lambda) &= 2y - 2x - x\lambda = 0 \\ F_\lambda(x, y, \lambda) &= -xy + 1 = 0 \end{aligned}$$

and so from first two equations,

$$\lambda = \frac{6x - 2y}{y}, \quad \lambda = \frac{2y - 2x}{x}$$

so

$$\begin{aligned} \frac{6x - 2y}{y} &= \frac{2y - 2x}{x} \\ \frac{6x - 2y}{y} \cdot \frac{x}{x} &= \frac{2y - 2x}{x} \cdot \frac{y}{y} \\ \frac{6x^2 - 2xy}{xy} &= \frac{2y^2 - 2xy}{xy} \\ 6x^2 - 2xy &= 2y^2 - 2xy \quad \text{multiplying both sides by } xy \\ 6x^2 &= 2y^2 \quad \text{adding } 2xy \text{ to both sides} \\ y^2 &= 3x^2 \quad \text{dividing by 3, flipping sides} \end{aligned}$$

so from the third equation

$$\begin{aligned} -xy + 1 &= 0 \\ xy &= 1 \\ x^2y^2 &= xy \quad \text{multiplying both sides by } xy \\ x^2(3x^2) &= 1 \quad \text{since } y^2 = 3x^2 \text{ and } xy = 1 \\ 3x^4 &= 1 \end{aligned}$$

so $x =$ (i) $\boldsymbol{-\sqrt[4]{\frac{1}{3}}}$ (ii) $\boldsymbol{\sqrt[4]{\frac{1}{3}}}$ (iii) $\boldsymbol{\pm\sqrt[4]{\frac{1}{3}}}$
since $3x^4 = 1$ or $x^4 = \frac{1}{3}$, then $x = \pm\sqrt[4]{\frac{1}{3}}$

and, in a similar way, $y =$ (i) $\mathbf{-\sqrt[4]{3}}$ (ii) $\mathbf{\sqrt[4]{3}}$ (iii) $\mathbf{\pm\sqrt[4]{3}}$
since $x^2y^2 = xy$ and $x^2 = \frac{y^2}{3}$ then $\frac{y^2}{3} \cdot y^2 = 1$, and so $y = \pm\sqrt[4]{3}$

and since $xy = 1$ (implying x and y have the same sign)
$(x, y) =$ (i) $\left(\mathbf{-\frac{1}{\sqrt[4]{3}}, -\sqrt[4]{3}}\right)$ (ii) $\left(\mathbf{-\frac{1}{\sqrt[4]{3}}, \sqrt[4]{3}}\right)$ (iii) $\left(\mathbf{\frac{1}{\sqrt[4]{3}}, \sqrt[4]{3}}\right)$

also

$$f(x, y) = 3x^2 + y^2 - 2xy = 3\left(\frac{1}{\sqrt[4]{3}}\right)^2 + \left(\sqrt[4]{3}\right)^2 - 2\left(\frac{1}{\sqrt[4]{3}}\right)\left(\sqrt[4]{3}\right) =$$

(i) $\mathbf{\frac{3}{\sqrt{3}} + \sqrt{3} - 2}$ (ii) $\mathbf{\frac{3}{\sqrt{3}}}$ (iii) $\mathbf{\frac{3}{\sqrt{3}} + \sqrt{3}}$
because of symmetry, $f(x, y)$ is the same whether $\left(-\frac{1}{\sqrt[4]{3}}, -\sqrt[4]{3}\right)$ or $\left(\frac{1}{\sqrt[4]{3}}, \sqrt[4]{3}\right)$ used
also notice $\left(\sqrt[4]{3}\right)^2 = \left(3^{\frac{1}{4}}\right)^2 = 3^{\frac{1}{2}} = \sqrt{3}$

(d) *minima, maxima or saddlepoint?*

Choose (x, y) which satisfies $xy = 1$; for example, $(x, y) = \left(\frac{1}{3}, 3\right)$,

$$xy = \left(\frac{1}{3}\right)(3) = 1$$

and notice

$$\begin{aligned} f(x, y) &= 3x^2 + y^2 - 2xy = 3\left(\frac{1}{3}\right)^2 + (3)^2 - 2\left(\frac{1}{3}\right)(3) = \frac{22}{3} \approx 7.33 \\ &> f(x, y) = 3\left(\frac{1}{\sqrt[4]{3}}\right)^2 + \left(\sqrt[4]{3}\right)^2 - 2\left(\frac{1}{\sqrt[4]{3}}\right)\left(\sqrt[4]{3}\right) = \frac{3}{\sqrt{3}} + \sqrt{3} - 2 \approx 1.46 \end{aligned}$$

in other words, since $f(x, y)$ at choice $(x, y) = \left(\frac{1}{3}\right)(3)$ is larger than at either critical point, $(x, y) = \left(-\frac{1}{\sqrt[4]{3}}, -\sqrt[4]{3}\right)$ and $(x, y) = \left(\frac{1}{\sqrt[4]{3}}, \sqrt[4]{3}\right)$, this indicates $f(x, y)$ at critical points are
(i) **minima** (ii) **maxima** (iii) **saddlepoints**

4. *Minimize* $f(x, y) = x^2 + y^2 + 2z^2$ *with constraint* $x + y + z = 1$.

(a) *Lagrange function*

Since constrained optimization problem is

$$\begin{gathered} \text{minimize } f(x, y) = x^2 + y^2 + 2z^2 \\ \text{subject to } x + y + z = 1 \end{gathered}$$

related Lagrange function is

$$F(x,y,z,\lambda) = f(x,y,z) - \lambda \cdot g(x,y,z) =$$

(i) $\boldsymbol{x^2+y^2+2z^2-\lambda(x+y+z)}$
(ii) $\boldsymbol{x^2+y^2-\lambda(x+y+z-1)}$
(iii) $\boldsymbol{x^2+y^2+2z^2-\lambda(x+y+z-1)}$
notice how $x+y+z=1$ has been rewritten $g(x,y)=x+y+z-1=0$

(b) *partial derivatives*

since

$$F(x,y,z,\lambda) = x^2+y^2+2z^2-\lambda(x+y+z-1) = x^2+y^2+2z^2-x\lambda-y\lambda-z\lambda+\lambda$$

then

$$F_x(x,y,z,\lambda) = 2x^{2-1}+0+0-x^{1-1}\lambda-0-0+0 =$$

(i) $\boldsymbol{2x-\lambda}$ (ii) $\boldsymbol{2y-\lambda}$ (iii) $\boldsymbol{4z-\lambda}$ (iv) $\boldsymbol{-x-y-z+1}$

and

$$F_y(x,y,z,\lambda) = 0+2y^{2-1}+0-0-y^{1-1}\lambda+0+0 =$$

(i) $\boldsymbol{2x-\lambda}$ (ii) $\boldsymbol{2y-\lambda}$ (iii) $\boldsymbol{4z-\lambda}$ (iv) $\boldsymbol{-x-y-z+1}$

and

$$F_z(x,y,z,\lambda) = 0+0+2\cdot 2z^{2-1}-0-0-z^{1-1}\lambda+0 =$$

(i) $\boldsymbol{2x-\lambda}$ (ii) $\boldsymbol{2y-\lambda}$ (iii) $\boldsymbol{4z-\lambda}$ (iv) $\boldsymbol{-x-y-z+1}$

and

$$F_\lambda(x,y,z,\lambda) = 0+0+0-\lambda^{1-1}(x+y+z-1) =$$

(i) $\boldsymbol{2x-\lambda}$ (ii) $\boldsymbol{2y-\lambda}$ (iii) $\boldsymbol{4z-\lambda}$ (iv) $\boldsymbol{-x-y-z+1}$

(c) *system of equations*

$$F_x(x,y,z,\lambda)=0, \quad F_y(x,y,z,\lambda)=0, \quad F_z(x,y,z,\lambda)=0, \quad F_\lambda(x,y,z,\lambda)=0$$

gives

$$\begin{aligned}
F_x(x,y,\lambda) &= 2x-\lambda = 0 \\
F_y(x,y,\lambda) &= 2y-\lambda = 0 \\
F_y(x,y,\lambda) &= 4z-\lambda = 0 \\
F_\lambda(x,y,\lambda) &= -x-y-z+1 = 0
\end{aligned}$$

and so from first three equations,

$$\lambda = 2x = 2y = 4z$$

or $z = \frac{x}{2}$, $y = x$, so from the last equation

$$\begin{aligned} -x - y - z + 1 &= 0 \\ -x - (x) - \left(\frac{x}{2}\right) &= -1 \\ -\frac{5x}{2} &= -1 \end{aligned}$$

so $x =$ (i) $-\mathbf{\frac{2}{5}}$ (ii) $\mathbf{\frac{2}{5}}$ (iii) $\pm\mathbf{\frac{2}{5}}$

and so, since $y = x$, $y =$ (i) $-\mathbf{\frac{2}{5}}$ (ii) $\mathbf{\frac{2}{5}}$ (iii) $\pm\mathbf{\frac{2}{5}}$

and also, since $z = \frac{x}{2}$, $z =$ (i) $-\mathbf{\frac{1}{5}}$ (ii) $\mathbf{\frac{1}{5}}$ (iii) $\pm\mathbf{\frac{1}{5}}$

and so
$(x, y, z) =$ (i) $\left(\mathbf{\frac{2}{5}, \frac{2}{5}, \frac{1}{5}}\right)$ (ii) $\left(\mathbf{\frac{2}{5}, -\frac{2}{5}, \frac{1}{5}}\right)$ (iii) $\left(\mathbf{\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}}\right)$

also

$$f(x, y, z) = x^2 + y^2 + 2z^2 = \left(\frac{2}{5}\right)^2 + \left(\frac{2}{5}\right)^2 + 2\left(\frac{1}{5}\right)^2 =$$

(i) $\mathbf{\frac{8}{25}}$ (ii) $\mathbf{\frac{9}{25}}$ (iii) $\mathbf{\frac{10}{25}}$

(d) *minima, maxima or saddlepoint?*

Choose (x, y, z) which satisfies $x + y + z = 1$; say, $(x, y, z) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$,

$$x + y + z = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$$

and notice

$$\begin{aligned} f(x, y, z) &= x^2 + y^2 + 2z^2 = \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + 2\left(\frac{1}{3}\right)^2 = \frac{4}{9} \\ &> f(x, y, z) = x^2 + y^2 + 2z^2 = \left(\frac{2}{5}\right)^2 + \left(\frac{2}{5}\right)^2 + 2\left(\frac{1}{5}\right)^2 = \frac{10}{25} \end{aligned}$$

so, since $f(x, y, z)$ at choice $(x, y, z) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is larger than at critical point, $(x, y, z) = \left(\frac{2}{5}, \frac{2}{5}, \frac{1}{5}\right)$, this indicates $f(x, y, z)$ at the critical point is (i) **minimum** (ii) **maximum** (iii) **saddlepoint**

5. *Minimize* $f(x,y) = 2x^2 + y^2 + 2xy + 4x$ *with constraint* $y^2 = x + 1$.

   (a) *Lagrange function*

   Since constrained optimization problem is

$$\text{minimize } f(x,y) = 2x^2 + y^2 + 2xy + 4x$$
$$\text{subject to } y^2 = x + 1$$

   related Lagrange function is

$$F(x,y,\lambda) = f(x,y) - \lambda \cdot g(x,y) =$$

   (i) $\mathbf{2x^2 + y^2 + 2xy + 4x - \lambda(y^2 - x - 1)}$
   (ii) $\mathbf{2x^2 + y^2 + 2xy + 4x}$
   (iii) $\mathbf{-\lambda(y^2 - x - 1)}$
   notice how $y^2 = x + 1$ has been rewritten $g(x,y) = y^2 - x - 1 = 0$

   (b) *partial derivatives*

   since $F(x,y,\lambda) = 2x^2 + y^2 + 2xy + 4x - \lambda(y^2 - x - 1) = 2x^2 + y^2 + 2xy + 4x - \lambda y^2 + \lambda x + \lambda)$,

$$F_x(x,y,\lambda) = 2 \cdot 2x^{2-1} + 0 + 2x^{1-1}y + 4x^{1-1} - 0 + \lambda x^{1-1} + 0 =$$

   (i) $\mathbf{4x + 2y + 4 + \lambda}$ (ii) $\mathbf{2y + 2x - 2y\lambda}$ (iii) $\mathbf{-y^2 + x + 1}$

   and

$$F_y(x,y,\lambda) = 0 + 2y^{2-1} + 2xy^{1-1} + 0 - 2y^{2-1}\lambda + 0 + 0 =$$

   (i) $\mathbf{4x + 2y + 4 + \lambda}$ (ii) $\mathbf{2y + 2x - 2y\lambda}$ (iii) $\mathbf{-y^2 + x + 1}$

   and

$$F_\lambda(x,y,\lambda) = 0 + 0 + 0 + 0 - y^2 + x + 1 =$$

   (i) $\mathbf{4x + 2y + 4 + \lambda}$ (ii) $\mathbf{2y + 2x - 2y\lambda}$ (iii) $\mathbf{-y^2 + x + 1}$

   (c) *system of equations*

$$F_x(x,y,\lambda) = 0, \quad F_y(x,y,\lambda) = 0, \quad F_\lambda(x,y,\lambda) = 0$$

   gives

$$\begin{aligned} F_x(x,y,\lambda) &= 4x + 2y + 4 + \lambda = 0 \\ F_y(x,y,\lambda) &= 2y + 2x - 2y\lambda = 0 \\ F_\lambda(x,y,\lambda) &= -y^2 + x + 1 = 0 \end{aligned}$$

and so from first two equations,

$$\lambda = 4x + 2y + 4, \quad \lambda = \frac{2y + 2x}{2y}$$

so

$$\begin{aligned}
4x + 2y + 4 &= \frac{2y + 2x}{2y} \\
4x + 2y + 4 &= \frac{x + y}{y} \\
4xy + 2y^2 + 4y &= x + y \\
4xy + 2y^2 + 3y - x &= 0 \\
x(4y - 1) + 2y^2 + 3y &= 0 \\
(y^2 - 1)(4y - 1) + 2y^2 + 3y &= 0 \quad \text{since } y^2 = x + 1, \text{ then } x = y^2 - 1 \\
4y^3 + y^2 - y + 1 &= 0
\end{aligned}$$

so $y \approx$ (i) $\mathbf{-0.8689}$ (ii) $\mathbf{-0.2450}$ (iii) $\mathbf{-0.4565}$

let $\text{Y}_1 = 4y^3 + y^2 - y + 1$, then MATH Solver 0 ENTER, then ALPHA ENTER, to give $X = -.8688...$

check there is only one real root (the other two are complex): WINDOW -2 2 1 -0.5 2 1 1, then GRAPH

and so $x = y^2 - 1 \approx$ (i) $\mathbf{-0.8689}$ (ii) $\mathbf{-0.2450}$ (iii) $\mathbf{-0.4565}$

$x \approx (-0.8689)^2 - 1 \approx -0.2450$

and $(x, y) =$
(i) $\mathbf{(-0.2450, -0.8689)}$
(ii) $\mathbf{(0.2450, -0.8689)}$
(iii) $\mathbf{(0.2450, 0.8689)}$

also

$f(x, y) = 2x^2 + y^2 + 2xy + 4x \approx 2(-0.2450)^2 + (-0.8689)^2 + 2(-0.2450)(-0.8689) + 4(-0.8689) \approx$

(i) $\mathbf{-2.006}$ (ii) $\mathbf{-2.113}$ (iii) $\mathbf{-2.175}$

(d) *minima, maxima or saddlepoint?*

Choose (x, y) which satisfies $y^2 = x + 1$; for example, $(x, y) = (0, 1)$,

$$y^2 = 1^2 = x + 1 = 0 + 1$$

and notice

$$\begin{aligned}
f(x, y) &= 2x^2 + y^2 + 2xy + 4x = 2(0)^2 + (1)^2 + 2(0)(1) + 4(0) = 1 \\
&> f(x, y) \approx 2(-0.245)^2 + (-0.8689)^2 + 2(-0.245)(-0.8689) + 4(-0.8689) \approx -2.175
\end{aligned}$$

so, since $f(x, y)$ at choice $(x, y) = (0, 1)$ is larger than at critical point, $(x, y) = (-0.2450, -0.8689)$, this indicates $f(x, y)$ at critical point is
(i) **minimum** (ii) **maximum** (iii) **saddlepoint**

6. *Application: profit and electric fans.*

Atomic rotation from x proton charge, y muon charge and z quark charge is

$$R(x, y, z) = -x^2 - y^2 + z^2.$$

What (x, y, z) charges minimize atomic rotation if there is a total charge of 10?

(a) *Lagrange function*

Since constrained optimization problem is

$$\begin{gathered}\text{minimize } f(x, y) = -x^2 - 2y^2 + z^2 \\ \text{subject to } x + y + z = 10\end{gathered}$$

related Lagrange function is

$$F(x, y, z, \lambda) = f(x, y, z) - \lambda \cdot g(x, y, z) =$$

(i) $\boldsymbol{x^2 + y^2 + 2z^2 - \lambda(x + y + z)}$
(ii) $\boldsymbol{x^2 + y^2 - \lambda(x + y + z - 1)}$
(iii) $\boldsymbol{-x^2 - 2y^2 + z^2 - \lambda(x + y + z - 10)}$
notice how $x + y + z = 10$ has been rewritten $g(x, y) = x + y + z - 10 = 0$

(b) *partial derivatives*

since

$$F(x, y, z, \lambda) = -x^2 - 2y^2 + z^2 - \lambda(x + y + z - 10) = -x^2 - 2y^2 + z^2 - x\lambda - y\lambda - z\lambda + 10\lambda$$

then

$$F_x(x, y, z, \lambda) = -2x^{2-1} - 0 + 0 - x^{1-1}\lambda - 0 - 0 + 0 =$$

(i) $\boldsymbol{-2x - \lambda}$ (ii) $\boldsymbol{-4y - \lambda}$ (iii) $\boldsymbol{2z - \lambda}$ (iv) $\boldsymbol{-x - y - z + 10}$

and

$$F_y(x, y, z, \lambda) = -0 - 2 \cdot 2y^{2-1} + 0 - 0 - y^{1-1}\lambda + 0 + 0 =$$

(i) $\boldsymbol{-2x - \lambda}$ (ii) $\boldsymbol{-4y - \lambda}$ (iii) $\boldsymbol{2z - \lambda}$ (iv) $\boldsymbol{-x - y - z + 10}$

and

$$F_z(x, y, z, \lambda) = -0 - 0 + 2z^{2-1} - 0 - 0 - z^{1-1}\lambda + 0 =$$

(i) $\boldsymbol{-2x - \lambda}$ (ii) $\boldsymbol{-4y - \lambda}$ (iii) $\boldsymbol{2z - \lambda}$ (iv) $\boldsymbol{-x - y - z + 10}$

and

$$F_\lambda(x, y, z, \lambda) = -0 - 0 + 0 - \lambda^{1-1}(x + y + z - 10) =$$

(i) $\boldsymbol{-2x - \lambda}$ (ii) $\boldsymbol{-4y - \lambda}$ (iii) $\boldsymbol{2z - \lambda}$ (iv) $\boldsymbol{-x - y - z + 10}$

(c) *system of equations*

$$F_x(x, y, z, \lambda) = 0, \quad F_y(x, y, z, \lambda) = 0, \quad F_z(x, y, z, \lambda) = 0, \quad F_\lambda(x, y, z, \lambda) = 0$$

gives

$$\begin{aligned} F_x(x, y, \lambda) &= -2x - \lambda = 0 \\ F_y(x, y, \lambda) &= -4y - \lambda = 0 \\ F_y(x, y, \lambda) &= 2z - \lambda = 0 \\ F_\lambda(x, y, \lambda) &= -x - y - z + 10 = 0 \end{aligned}$$

and so from first three equations,

$$\lambda = -2x = -4y = 2z$$

or $z = -x$, $y = \frac{x}{2}$, so from the last equation

$$\begin{aligned} -x - y - z + 10 &= 0 \\ -x - \left(\frac{x}{2}\right) - (-x) &= -10 \\ -\frac{x}{2} &= -10 \end{aligned}$$

so $x =$ (i) **18** (ii) **19** (iii) **20**

and so, since $y = \frac{x}{2}$, $y =$ (i) **9** (ii) **9.5** (iii) **10**

and also, since $z = -x$, $z =$ (i) $\boldsymbol{-18}$ (ii) $\boldsymbol{-19}$ (iii) $\boldsymbol{-20}$

so $(x, y, z) =$ (i) $\boldsymbol{(20, 10, -10)}$ (ii) $\boldsymbol{(20, 10, -20)}$ (iii) $\boldsymbol{(20, -10, -20)}$

also

$$f(x, y, z) = -x^2 - 2y^2 + z^2 = -(20)^2 - 2(10)^2 + (-20)^2 =$$

(i) $\boldsymbol{-100}$ (ii) $\boldsymbol{-200}$ (iii) $\boldsymbol{-300}$

(d) *minima, maxima or saddlepoint?*

Choose (x, y, z) which satisfies $x + y + z = 10$; say, $(x, y, z) = (0, 0, 10)$,

$$x + y + z = 0 + 0 + 10 = 10$$

and notice

$$\begin{aligned} f(x, y, z) &= -x^2 - 2y^2 + z^2 = -(0)^2 - 2(0)^2 + (10)^2 = 100 \\ &> f(x, y, z) = -x^2 - 2y^2 + z^2 - (20)^2 - 2(10)^2 + (-20)^2 = -200 \end{aligned}$$

so, since $f(x, y, z)$ at choice $(x, y, z) = (0, 0, 10)$ is smaller than at critical point, $(x, y, z) = (20, 10, -20)$, this indicates $f(x, y, z)$ at critical point is (i) **minimum** (ii) **maximum** (iii) **saddlepoint**

7. *Understanding method of Lagrange multipliers.*

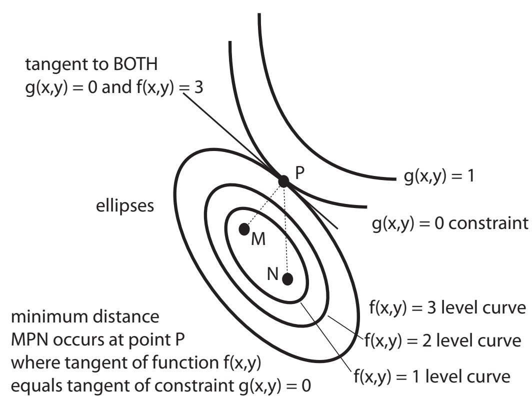

Figure 9.20 (Understanding Lagrange Multipliers)

(i) **True** (ii) **False** Roughly speaking, minimum distance of path MPN, a possible function $f(x, y)$, that must touch, is constrained by, $g(x, y) = 0$ occurs at point P which has both a tangent to $f(x, y)$, the *gradient* ("derivative") $\nabla f(x, y)$, and also a tangent to $g(x, y)$, the gradient $\nabla g(x, y)$; that is, both tangents are parallel to one another, although not necessarily of the same length, so

$$\nabla f(x, y) = \lambda \nabla g(x, y)$$

where λ indicates the difference in length. But this equality is equivalent to creating the Lagrange function

$$F(x, y, \lambda) = f(x, y) - \lambda \cdot g(x, y)$$

then solving the system of equations

$$F_x(x, y, \lambda) = 0, \quad F_y(x, y, \lambda) = 0, \quad F_\lambda(x, y, \lambda) = 0$$

for critical points (which may be minima, maxima or saddlepoints).