4.3 The Chain Rule

One possible *composition* of functions g and f is composed function $f[g(x)]$ whose values are given for all x in the domain of g such that $g(x)$ is in the domain of f. Roughly, composed function $f[g(x)]$ takes"output" of $g(x)$ and uses it as "input" of function $f(x)$, or that $g(x)$ is the inner layer and $f(x)$ is the outer layer of the function. The *chain rule* is used to find the derivative of the composed function $y = f[g(x)]$, where $y = f(u)$ and $u = g(x)$, and is given by,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = f'[g(x)] \cdot g'(x)$$

Exercise 4.3 (The Chain Rule)

1. *Composition of functions.*

   (a) Let $f(x) = x^2 + 1$ and $g(x) = x^2 - 1$.

   i. *Find* $f[g(4)]$

   since $g(4) = (4)^2 - 1 =$ (i) **16** (ii) **17** (iii) **15**
   $f[g(4)] = f[15] = (15)^2 + 1 =$ (i) **225** (ii) **224** (iii) **226**

   ii. *Find* $f[g(-1)]$

   since $g(-1) = (-1)^2 - 1 =$ (i) **0** (ii) **1** (iii) **2**
   $f[g(-1)] = f[0] = (0)^2 + 1 =$ (i) **2** (ii) **1** (iii) **3**

   iii. *Find* $f[g(x)]$ *in general*

   $$\begin{aligned} f[g(x)] &= f\left[x^2 - 1\right] \\ &= \left[x^2 - 1\right]^2 + 1 \\ &= x^4 - 2x^2 + 1 + 1 = \end{aligned}$$

   (i) $\boldsymbol{x^4 - 2x^2}$ (ii) $\boldsymbol{x^4 - 2x^2 + 2}$ (iii) $\boldsymbol{x^4 - 4x^2}$

   so $f[g(4)] = (4)^4 - 2(4)^2 + 2 =$ (i) **226** (ii) **225** (iii) **224**
   and $f[g(-1)] = (-1)^4 - 2(-1)^2 + 2 =$ (i) **1** (ii) **2** (iii) **3**

   iv. *Find* $g[f(4)]$ *(not* $f[g(4)]$*!)*

   since $f(4) = (4)^2 + 1 =$ (i) **15** (ii) **16** (iii) **17**
   $g[f(4)] = g[17] = (17)^2 - 1 =$ (i) **289** (ii) **290**
   (iii) **288**

v. *Find* $g[f(-1)]$

since $f(-1) = (-1)^2 + 1 =$ (i) **1** (ii) **2** (iii) **0**
$g[f(-1)] = g[2] = (2)^2 - 1 =$ (i) **3** (ii) **4** (iii) **5**

vi. *Find* $g[f(x)]$ *in general*

$$\begin{aligned} g[f(x)] &= g\left(x^2+1\right) \\ &= \left(x^2+1\right)^2 - 1 \end{aligned}$$

(i) $\boldsymbol{x^4 - 2x^2 + 2}$ (ii) $\boldsymbol{x^4 + 2x^2}$ (iii) $\boldsymbol{x^4 - 4x^2}$

so $g[f(4)] = (4)^4 + 2(4)^2 =$ (i) **288** (ii) **289** (iii) **290**
and $g[f(-1)] = (-1)^4 + 2(-1)^2 =$ (i) **3** (ii) **4** (iii) **5**

(b) Let $f(x) = 4x^2 + 4x$ and $g(x) = \frac{2}{x}$.

i. *Find f[g(x)]*

$$f[g(x)] = f\left(\frac{2}{x}\right) = 4\left(\frac{2}{x}\right)^2 + 4\left(\frac{2}{x}\right) = \frac{16}{x^2} + \frac{8}{x} =$$

(i) $\boldsymbol{x^4 - 2x^2 + 2}$ (ii) $\boldsymbol{x^4 - 2x^2}$ (iii) $\boldsymbol{\frac{16+8x}{x^2}}$

so $f[g(4)] = \frac{16+8(4)}{(4)^2} =$ (i) **3** (ii) **2** (iii) **1**

ii. *Find* $g[f(x)]$

$$g[f(x)] = g\left[4x^2 + 4x\right] = \frac{2}{4x^2 + 4x} =$$

(i) $\boldsymbol{x^4 - 2x^2 + 2}$ (ii) $\boldsymbol{\frac{1}{2x(x+1)}}$ (iii) $\boldsymbol{\frac{16+8x}{x^2}}$

so $g[f(4)] = \frac{1}{2(4)(4+1)} =$ (i) $\boldsymbol{\frac{1}{40}}$ (ii) $\boldsymbol{\frac{1}{8}}$ (iii) $\boldsymbol{\frac{1}{5}}$

(c) Let $f(x) = x^2 + 1$ and $g(x) = \sqrt{x^3 - 1}$.

i. *Find* $f[g(x)]$

$$f[g(x)] = f[\sqrt{x^3-1}] = \left(\sqrt{x^3-1}\right)^2 + 1 =$$

(i) $\boldsymbol{x^3}$ (ii) $\boldsymbol{(x^3 - 1)^2 + 1}$ (iii) $\boldsymbol{(x^2 + 1)^3 - 1}$

ii. *Find* $g[f(x)]$

$$g[f(x)] = g[x^2 + 1] = \sqrt{\left(x^2+1\right)^3 - 1} =$$

(i) $\boldsymbol{\sqrt{x^6 + 3x^4 + 3x^4}}$ (ii) $\boldsymbol{x^6 + 3x^4 + 3x^4}$ (iii) $\boldsymbol{(x^2 + 1)^3 - 1}$

iii. $f[g(x)] = x^3$
(i) **equals** (ii) **does not equal**
$g[f(x)] = \sqrt{x^6 + 3x^4 + 3x^4}$

(d) *Decomposing a function into two functions.*

i. Find $f(x)$ and $g(x)$ such that $f[g(x)] = \frac{1}{\sqrt{3x^2-2}}$,
if "inner" function $g(x) =$ (i) $\mathbf{3x^2}$ (ii) $\mathbf{3x}$ (iii) $\mathbf{3x^2 - 2}$
then "outer" function $f(x) =$ (i) $\mathbf{\frac{1}{x}}$ (ii) $\mathbf{\frac{1}{x^2}}$ (iii) $\mathbf{\frac{1}{\sqrt{x}}}$
To check decomposition, compose $f(x)$ and $g(x)$:

$$f[g(x)] = f[3x^2 - 2] = \frac{1}{\sqrt{3x^2 - 2}}$$

ii. Find $f(x)$ and $g(x)$ such that $f[g(x)] = \frac{1}{\sqrt{3x^2-2}}$,
if "inner" function $g(x) =$ (i) $\mathbf{3x^2}$ (ii) $\mathbf{3x^2 - 2}$ (iii) $\mathbf{3x}$
then "outer" function $f(x) =$ (i) $\mathbf{\frac{1}{x-2}}$ (ii) $\mathbf{\frac{1}{\sqrt{x-2}}}$ (iii) $\mathbf{\frac{1}{x^2}}$
To check decomposition, compose $f(x)$ and $g(x)$:

$$f[g(x)] = f[3x^2] = \frac{1}{\sqrt{3x^2 - 2}}$$

(The last two examples demonstrates there can be more than one way of decomposing a function.)

iii. Find $f(x)$ and $g(x)$ such that $f[g(x)] = (3x^2 - 2)^4$,
$g(x) =$ (i) $\mathbf{3x^2}$ (ii) $\mathbf{3x^2 - 2}$ (iii) $\mathbf{3x}$
$f(x) =$ (i) $\mathbf{3x^3}$ (ii) $\mathbf{x^4}$ (iii) $\mathbf{x^2}$
To check decomposition, compose $f(x)$ and $g(x)$:

$$f[g(x)] = f[3x^2 - 2] = \left(3x^2 - 2\right)^4$$

iv. Find $f(x)$ and $g(x)$ such that $f[g(x)] = e^{3x^2-2}$,
$g(x) =$ (i) $\mathbf{3x^2}$ (ii) $\mathbf{3x^2 - 2}$ (iii) $\mathbf{3x}$
$f(x) =$ (i) $\mathbf{e^x}$ (ii) $\mathbf{3x^3}$ (iii) $\mathbf{x^2}$
To check decomposition, compose $f(x)$ and $g(x)$:

$$f[g(x)] = f[3x^2 - 2] = e^{3x^2-2}$$

v. Find $f(x)$ and $g(x)$ such that $f[g(x)] = \ln(3x) + 3x + (3x)^2$,
$g(x) =$ (i) $\mathbf{3x}$ (ii) $\mathbf{3x^2}$ (iii) $\mathbf{3x^3}$
$f(x) =$ (i) $\mathbf{\ln x^2 + x + 2x}$ (ii) $\mathbf{e^x + x + x^2}$ (iii) $\mathbf{\ln x + x + x^2}$
To check decomposition, compose $f(x)$ and $g(x)$:

$$f[g(x)] = f[3x] = \ln(3x) + (3x) + (3x)^2$$

2. *Chain rule.* Find the derivative of following functions using the chain rule.

(a) $y = (3x + x^2)^2$

let $f[g(x)] = (3x + x^2)^2$
with "inner" function $g(x) =$ (i) $\boldsymbol{3x + x^2}$ (ii) $\boldsymbol{3x^2 - 2}$ (iii) $\boldsymbol{3x}$
and "outer" function $f(x) =$ (i) $\boldsymbol{x^3}$ (ii) $\boldsymbol{x^4}$ (iii) $\boldsymbol{x^2}$

with derivative $g'(x) =$ (i) $\boldsymbol{6x - 2}$ (ii) $\boldsymbol{3}$ (iii) $\boldsymbol{3 + 2x}$
and derivative $f'(x) =$ (i) $\boldsymbol{3x^2}$ (ii) $\boldsymbol{4x^3}$ (iii) $\boldsymbol{2x}$

and so by chain rule

$$\begin{aligned} f'[g(x)] \cdot g'(x) &= f'[3x + x^2] \cdot (3 + 2x) \\ &= 2\left[3x + x^2\right](3 + 2x) \\ &= 2\left(9x + 3x^2 + 6x^2 + 2x^3\right) \\ &= 2\left(9x + 9x^2 + 2x^3\right) = \end{aligned}$$

(i) $\boldsymbol{9x + 9x^2 + 2x^3}$
(ii) $\boldsymbol{18x + 18x^2 + 4x^3}$
(iii) $\boldsymbol{9 + 9x + 2x^2}$

(b) $y = \sqrt{3x + x^2}$

let $f[g(x)] = (3x + x^2)^{\frac{1}{2}}$
with "inner" function $g(x) =$ (i) $\boldsymbol{3x^2 - 2}$ (ii) $\boldsymbol{3x}$ (iii) $\boldsymbol{3x + x^2}$
and "outer" function $f(x) =$ (i) $\boldsymbol{x^{\frac{1}{2}}}$ (ii) $\boldsymbol{x^{\frac{3}{2}}}$ (iii) $\boldsymbol{x^{\frac{5}{2}}}$

with derivative $g'(x) =$ (i) $\boldsymbol{3 + 2x}$ (ii) $\boldsymbol{6x - 2}$ (iii) $\boldsymbol{3}$
and derivative $f'(x) =$ (i) $\boldsymbol{\frac{3}{2}x^{\frac{1}{2}}}$ (ii) $\boldsymbol{\frac{3}{2}x^{\frac{3}{2}}}$ (iii) $\boldsymbol{\frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}}$

and so by chain rule

$$\begin{aligned} f'[g(x)] \cdot g'(x) &= f'[3x + x^2] \cdot (3 + 2x) \\ &= \frac{1}{2\sqrt{3x + x^2}}(3 + 2x) = \end{aligned}$$

(i) $\boldsymbol{\frac{3+2x}{\sqrt{3x+x^2}}}$ (ii) $\boldsymbol{\frac{3+2x}{2\sqrt{3x+x^2}}}$ (iii) $\boldsymbol{\frac{3+2x}{3\sqrt{3x+x^2}}}$

(c) $y = (-2t^4 + 7t)^5$.

let $f[g(t)] = (-2t^4 + 7t)^5$

with "inner" function $g(t) =$ (i) $\boldsymbol{-2t^4}$ (ii) $\boldsymbol{-2t^4 + 7t}$ (iii) $\boldsymbol{t^5}$
and "outer" function $f(t) =$ (i) $\boldsymbol{-2t^4 + 7t}$ (ii) $\boldsymbol{t^5}$ (iii) $\boldsymbol{-2t^4 - 7t}$

with derivative $g'(t) =$ (i) $\boldsymbol{5t^4}$ (ii) $\boldsymbol{-2t^4 + 7t}$ (iii) $\boldsymbol{-8t^3 + 7}$
and derivative $f'(t) =$ (i) $\boldsymbol{-8t^3 + 7}$ (ii) $\boldsymbol{5t^4}$ (iii) $\boldsymbol{-8t^3}$

and so by chain rule

$$f'[g(t)] \cdot g'(t) \quad = \quad f'[-2t^4 + 7t] \cdot (-8t^3 + 7) =$$

(i) $\boldsymbol{5\left(-2t^4 + 7t\right)^4 \left(-8t^3 + 7\right)}$
(ii) $\boldsymbol{\left(-2t^4 + 7t\right)^4 \left(-8t^3 + 7\right)}$
(iii) $\boldsymbol{\left(-2t^4 + 7t\right) \left(-8t^3 + 7\right)}$

(d) $y = (3x^3 + 2x^2 - 4x)^{-4}$,

let $f[g(x)] = (3x^3 + 2x^2 - 4x)^{-4}$
"inner" function $g(x) =$ (i) $\boldsymbol{3x^3 + 2x^2}$ (ii) $\boldsymbol{3x^3 + 2x^2 - 4x}$ (iii) $\boldsymbol{x^{-4}}$
"outer" function $f(x) =$ (i) $\boldsymbol{-4x^{-5}}$ (ii) $\boldsymbol{3x^3 + 2x^2 - 4x}$ (iii) $\boldsymbol{x^{-4}}$

derivative $g'(x) =$ (i) $\boldsymbol{9x^2 + 4x - 4}$ (ii) $\boldsymbol{-4x^{-5}}$ (iii) $\boldsymbol{x^{-4}}$
derivative $f'(x) =$ (i) $\boldsymbol{9x^2 + 4x - 4}$ (ii) $\boldsymbol{-4x^{-5}}$ (iii) $\boldsymbol{-4x^{-5}}$

and so by chain rule

$$\begin{aligned} f'[g(x)] \cdot g'(x) &= f'[3x^3 + 2x^2 - 4x] \cdot (9x^2 + 4x - 4) \\ &= -4(3x^3 + 2x^2 - 4x)^{-5}(9x^2 + 4x - 4) = \end{aligned}$$

(i) $\boldsymbol{\frac{-4(9x^2+4x-4)}{(3x^3+2x^2-4x)^5}}$ (ii) $\boldsymbol{\frac{(9x^2+4x-4)}{(3x^3+2x^2-4x)^{-5}}}$ (iii) $\boldsymbol{\frac{-4(9x^2+4x-4)}{(3x^3+2x^2-4x)^{-5}}}$

3. *Chain rule and other rules.*

(a) $y = 34(3x + x^2)^2$

let $f[g(x)] = 34(3x + x^2)^2$
with "inner" function $g(x) =$ (i) $\boldsymbol{3x^2 - 2}$ (ii) $\boldsymbol{3x}$ (iii) $\boldsymbol{3x + x^2}$
and "outer" function $f(x) =$ (i) $\boldsymbol{34x^2}$ (ii) $\boldsymbol{x^2}$ (iii) $\boldsymbol{34x^3}$

with derivative $g'(x) =$ (i) $\boldsymbol{6x - 2}$ (ii) $\boldsymbol{3 + 2x}$ (iii) $\boldsymbol{3}$
and derivative $f'(x) =$ (i) $\boldsymbol{2x}$ (ii) $\boldsymbol{68x}$ (iii) $\boldsymbol{102x^3}$

and so by chain rule

$$f'[g(x)] \cdot g'(x) \quad = \quad f'[3x + x^2] \cdot (3 + 2x)$$

$$\begin{aligned} &= 68\left[3x + x^2\right](3 + 2x) \\ &= 68\left(9x + 3x^2 + 6x^2 + 2x^3\right) \\ &= 68\left(9x + 9x^2 + 2x^3\right) = \end{aligned}$$

(i) $\mathbf{612x + 612x^2 + 136x^3}$
(ii) $\mathbf{9x + 9x^2 + 2x^3}$
(iii) $\mathbf{612 + 612x + 136x^2}$

(b) *Chain rule and product rule.* $y = (4x^3 + 5)(3x^3 - x^2)^2$

Let $u(x) = 4x^3 + 5$ and $v(x) = (3x^3 - x^2)^2$

where derivative $u'(x) =$ (i) $\mathbf{12x^2}$ (ii) $\mathbf{4}$ (iii) $\mathbf{-1}$

and to find derivative $v'(x)$ let $v(x) = f[g(x)] = (3x^3 - x^2)^2$
with "inner" function $g(x) =$ (i) $\mathbf{3x^3 - x^2}$ (ii) $\mathbf{3x^2 - 2}$ (iii) $\mathbf{3x}$
and "outer" function $f(x) =$ (i) $\mathbf{x^3}$ (ii) $\mathbf{x^4}$ (iii) $\mathbf{x^2}$

with derivative $g'(x) =$ (i) $\mathbf{9x^2 - 2x}$ (ii) $\mathbf{9x - 2}$ (iii) $\mathbf{3}$
and derivative $f'(x) =$ (i) $\mathbf{2x}$ (ii) $\mathbf{3x^2}$ (iii) $\mathbf{4x^3}$

and so by chain rule

$$\begin{aligned} v'(x) = f'[g(x)] \cdot g'(x) &= f'[3x^3 - x^2] \cdot (9x^2 - 2x) \\ &= 2\left[3x^3 - x^2\right](9x^2 - 2x) \\ &= 2\left(27x^5 - 9x^4 - 6x^4 + 2x^3\right) \\ &= 2\left(27x^5 - 15x^4 + 2x^3\right) = \end{aligned}$$

(i) $\mathbf{9x + 9x^2 + 2x^3}$
(ii) $\mathbf{9 + 9x + 2x^2}$
(iii) $\mathbf{54x^5 - 30x^4 + 4x^3}$

and so $v(x)u'(x) =$
(i) $\mathbf{(3x^3 - x^2)(12x^2)}$
(ii) $\mathbf{(3x^3 - x^2)^2(12x^2)}$
(iii) $\mathbf{(3x^3 - x^2)^2(12x)}$

and $u(x)v'(x) =$
(i) $\mathbf{(4x^2)(18x + 18x^2 + 4x^3)}$
(ii) $\mathbf{(4x^3 + 5)(54x^5 - 30x^4 + 4x^3)}$

(iii) $\boldsymbol{(4x^3+5)(3x^3-x^2)^2}$

and so $f'(x) = v(x) \cdot u'(x) + u(x) \cdot v'(x) =$
(i) $\boldsymbol{(3x^3-x^2)^2 + (4x^3+5)(54x^5-30x^4+4x^3)}$
(ii) $\boldsymbol{(3x^3-x^2)^2(12x^2) + (4x^3+5)(54x^5-30x^4+4x^3)}$
(iii) $\boldsymbol{(4x^2) + (36x^5-24x^4+4x^3)}$

(c) *Chain rule and product rule.* $y = x(x^3+7x+4)^{\frac{1}{3}}$

Let $u(x) = x$ and $v(x) = (x^3+7x+4)^{\frac{1}{3}}$

where derivative $u'(x) =$ (i) $\mathbf{1}$ (ii) $\mathbf{2}$ (iii) $\mathbf{3}$

and to find derivative $v'(x)$ let $v(x) = f[g(x)] = (x^3+7x+4)^{\frac{1}{3}}$
with "inner" function $g(x) =$ (i) $\boldsymbol{3x^2+7}$ (ii) $\boldsymbol{x^3+7x+4}$ (iii) $\boldsymbol{3x}$
and "outer" function $f(x) =$ (i) $\boldsymbol{x^{\frac{1}{3}}}$ (ii) $\boldsymbol{x^{\frac{2}{3}}}$ (iii) $\boldsymbol{x^{\frac{4}{3}}}$

with derivative $g'(x) =$ (i) $\boldsymbol{6x-2}$ (ii) $\boldsymbol{\frac{1}{3}x^{-\frac{2}{3}}}$ (iii) $\boldsymbol{3x^2+7}$
and derivative $f'(x) =$ (i) $\boldsymbol{2x}$ (ii) $\boldsymbol{\frac{1}{3}x^{-\frac{2}{3}}}$ (iii) $\boldsymbol{3x^2+7}$

and so by chain rule

$$\begin{aligned} v'(x) = f'[g(x)] \cdot g'(x) &= f'[x^3+7x+4] \cdot (3x^2+7) \\ &= \frac{1}{3}\left[x^3+7x+4\right]^{-\frac{2}{3}}(3x^2+7) = \end{aligned}$$

(i) $\boldsymbol{\frac{2}{3[x^3+7x+4]^{\frac{2}{3}}}}$ (ii) $\boldsymbol{\frac{3x^2+7}{3[x^3+7x+4]^{-\frac{2}{3}}}}$ (iii) $\boldsymbol{\frac{3x^2+7}{3[x^3+7x+4]^{\frac{2}{3}}}}$

and so $v(x)u'(x) =$
(i) $\boldsymbol{(x^3+7x+4)^{-\frac{1}{3}}(1)}$
(ii) $\boldsymbol{(x^3+7x+4)^{\frac{1}{3}}(1)}$
(iii) $\boldsymbol{(x^3+7x+4)^{\frac{2}{3}}(1)}$

and $u(x)v'(x) =$
(i) $\boldsymbol{\left(\frac{3x^2+7}{3[x^3+7x+4]^{\frac{2}{3}}}\right)}$ (ii) $\boldsymbol{(x)\left(\frac{3x^2+7}{3[x^3+7x+4]^{\frac{2}{3}}}\right)}$ (iii) $\boldsymbol{(x)\left(\frac{3x^2+7}{[x^3+7x+4]^{\frac{2}{3}}}\right)}$

and so $f'(x) = v(x) \cdot u'(x) + u(x) \cdot v'(x) =$
(i) $\boldsymbol{(x^3+7x+4)^{\frac{1}{3}} + \frac{x(3x^2+7)}{3[x^3+7x+4]^{\frac{2}{3}}}}$
(ii) $\boldsymbol{(x^3+7x+4)^{-\frac{1}{3}} + \frac{x(3x^2+7)}{3[x^3+7x+4]^{\frac{2}{3}}}}$
(iii) $\boldsymbol{(x^3+7x+4)^{\frac{1}{3}} + \frac{3x^2+7}{3[x^3+7x+4]^{\frac{2}{3}}}}$

which equals

$$
\begin{aligned}
& (x^3+7x+4)^{\frac{1}{3}} + \frac{x(3x^2+7)}{3\left[x^3+7x+4\right]^{\frac{2}{3}}} \\
= \; & \frac{(x^3+7x+4)^{\frac{1}{3}} \cdot 3(x^3+7x+4)^{\frac{2}{3}}}{3(x^3+7x+4)^{\frac{2}{3}}} + \frac{x(3x^2+7)}{3\left[x^3+7x+4\right]^{\frac{2}{3}}} \\
= \; & \frac{3(x^3+7x+4)^1}{3(x^3+7x+4)^{\frac{2}{3}}} + \frac{x(3x^2+7)}{3\left[x^3+7x+4\right]^{\frac{2}{3}}} \\
= \; & \frac{3(x^3+7x+4)+x(3x^2+7)}{3(x^3+7x+4)^{\frac{2}{3}}} \\
= \; & \frac{6x^3+28x+12}{3(x^3+7x+4)^{\frac{2}{3}}}
\end{aligned}
$$

(d) *Chain rule and quotient rule.* $y = \frac{x}{(x^3+7x+4)^{\frac{1}{3}}}$.

Let $u(x) = x$ and $v(x) = (x^3+7x+4)^{\frac{1}{3}}$

where derivative $u'(x) =$ (i) $\mathbf{1}$ (ii) $\mathbf{2}$ (iii) $\mathbf{3}$

and to find derivative $v'(x)$ let $v(x) = f[g(x)] = (x^3+7x+4)^{\frac{1}{3}}$
with "inner" function $g(x) =$ (i) $\boldsymbol{3x^2+7}$ (ii) $\boldsymbol{3x}$ (iii) $\boldsymbol{x^3+7x+4}$
and "outer" function $f(x) =$ (i) $\boldsymbol{x^{\frac{2}{3}}}$ (ii) $\boldsymbol{x^{\frac{1}{3}}}$ (iii) $\boldsymbol{x^{\frac{4}{3}}}$

with derivative $g'(x) =$ (i) $\boldsymbol{6x-2}$ (ii) $\boldsymbol{3x^2+7}$ (iii) $\boldsymbol{\frac{1}{3}x^{-\frac{2}{3}}}$
and derivative $f'(x) =$ (i) $\boldsymbol{\frac{1}{3}x^{-\frac{2}{3}}}$ (ii) $\boldsymbol{2x}$ (iii) $\boldsymbol{3x^2+7}$

and so by chain rule

$$
\begin{aligned}
v'(x) = f'[g(x)] \cdot g'(x) &= f'[x^3+7x+4] \cdot (3x^2+7) \\
&= \frac{1}{3}\left[x^3+7x+4\right]^{-\frac{2}{3}}(3x^2+7) =
\end{aligned}
$$

(i) $\boldsymbol{\frac{3x^2+7}{3[x^3+7x+4]^{\frac{2}{3}}}}$ (ii) $\boldsymbol{\frac{2}{3[x^3+7x+4]^{\frac{2}{3}}}}$ (iii) $\boldsymbol{\frac{3x^2+7}{3[x^3+7x+4]^{-\frac{2}{3}}}}$

and so $v(x)u'(x) =$
(i) $\boldsymbol{(x^3+7x+4)^{\frac{1}{3}}(1)}$ (ii) $\boldsymbol{(x^3+7x+4)^{-\frac{1}{3}}(1)}$ (iii) $\boldsymbol{(x^3+7x+4)^{\frac{2}{3}}(1)}$

and $u(x)v'(x) =$
(i) $\boldsymbol{\left(\frac{3x^2+7}{3[x^3+7x+4]^{\frac{2}{3}}}\right)}$ (ii) $\boldsymbol{(x)\left(\frac{3x^2+7}{3[x^3+7x+4]^{\frac{2}{3}}}\right)}$ (iii) $\boldsymbol{(x)\left(\frac{3x^2+7}{[x^3+7x+4]^{\frac{2}{3}}}\right)}$

and so $f'(x) = \frac{v(x) \cdot u'(x) - u(x) \cdot v'(x)}{[v(x)]^2} =$

(i) $\boldsymbol{\frac{(x^3+7x+4)^{\frac{1}{3}} - \frac{x(3x^2+7)}{3[x^3+7x+4]^{\frac{2}{3}}}}{(x^3+7x+4)^{\frac{2}{3}}}}$
(ii) $\boldsymbol{\frac{(x^3+7x+4)^{\frac{1}{3}} + \frac{x(3x^2+7)}{3[x^3+7x+4]^{\frac{2}{3}}}}{(x^3+7x+4)^{\frac{2}{3}}}}$
(iii) $\boldsymbol{\frac{(x^3+7x+4)^{\frac{1}{3}} - \frac{x(3x^2+7)}{3[x^3+7x+4]^{\frac{2}{3}}}}{(x^3+7x+4)^{\frac{1}{3}}}}$
which equals

$$\begin{aligned}
& \frac{(x^3+7x+4)^{\frac{1}{3}} - \frac{x(3x^2+7)}{3[x^3+7x+4]^{\frac{2}{3}}}}{(x^3+7x+4)^{\frac{2}{3}}} \\
= & \frac{\frac{(x^3+7x+4)^{\frac{1}{3}} \cdot 3(x^3+7x+4)^{\frac{2}{3}}}{3(x^3+7x+4)^{\frac{2}{3}}} - \frac{x(3x^2+7)}{3(x^3+7x+4)^{\frac{2}{3}}}}{(x^3+7x+4)^{\frac{2}{3}}} \\
= & \frac{\frac{3(x^3+7x+4)^1}{3(x^3+7x+4)^{\frac{2}{3}}} - \frac{x(3x^2+7)}{3(x^3+7x+4)^{\frac{2}{3}}}}{(x^3+7x+4)^{\frac{2}{3}}} \\
= & \frac{\frac{3(x^3+7x+4)-x(3x^2+7)}{3(x^3+7x+4)^{\frac{2}{3}}}}{(x^3+7x+4)^{\frac{2}{3}}} \\
= & \frac{1}{3}\left[3(x^3+7x+4) - x(3x^2+7)\right] \\
= & \frac{14x+12}{3}
\end{aligned}$$

4. *Equation of tangent line to* $y = f(x) = \frac{x}{(x^3+7x+4)^{\frac{1}{3}}}$ *at* $x_1 = 0$.
   since $f(x_1) = f(1) = \frac{(0)}{((0)^3+7(0)+4)^{\frac{1}{3}}} =$ (i) $\mathbf{0}$ (ii) $\boldsymbol{\frac{1}{4^{\frac{1}{3}}}}$ (iii) $\boldsymbol{\frac{1}{16^{\frac{1}{3}}}}$
   and $f'(x_1) = f'(0) = \frac{14(0)+12}{3} =$ (i) $\boldsymbol{\frac{26}{3}}$ (ii) $\boldsymbol{\frac{23}{3}}$ (iii) $\mathbf{4}$ then

$$\begin{aligned}
y - f(x_1) &= f'(x_1)(x - x_1) \\
y - f(0) &= f'(0)(x - 0) \\
y - 0 &= 4(x - 0)
\end{aligned}$$

   or (i) $\boldsymbol{y = 4x}$ (ii) $\boldsymbol{y = 4x + 1}$ (iii) $\boldsymbol{y = 4x - 1}$

5. *Application: compound interest.*
   If \$700 is invested at an r interest compounded *monthly*, calculate the rate of change, A', of future amount, A, with respect to interest rate, r, after 8 years. What is A' when $r = 0.11$?

(a) Let $A = f[g(r)] = P\left(1 + \frac{r}{m}\right)^{mt} = 700\left(1 + \frac{r}{12}\right)^{(12)8}$

with "inner" function $g(r) =$ (i) $\mathbf{1 + \frac{r}{12}}$ (ii) $\mathbf{\frac{r}{12}}$ (iii) $\mathbf{700\left(1 + \frac{r}{12}\right)}$
and "outer" function $f(r) =$ (i) $\mathbf{700r^{96}}$ (ii) $\mathbf{r^{96}}$ (iii) $\mathbf{700r^{12}}$

with derivative $g'(r) =$ (i) $\mathbf{\frac{1}{12}}$ (ii) $\mathbf{67200r^{95}}$ (iii) $\mathbf{3 + 2x}$
and derivative $f'(r) =$ (i) $\mathbf{700r^{95}}$ (ii) $\mathbf{67200r^{95}}$ (iii) $\mathbf{\frac{1}{12}}$

and so by chain rule

$$\begin{aligned} A'(r) = f'[g(r)] \cdot g'(r) &= f'\left[1 + \frac{r}{12}\right] \cdot \left(\frac{1}{12}\right) \\ &= 67200\left(1 + \frac{r}{12}\right)^{95}\left(\frac{1}{12}\right) = \end{aligned}$$

(i) $\mathbf{5600\left(1 + \frac{r}{12}\right)^{95}}$ (ii) $\mathbf{\left(1 + \frac{r}{12}\right)^{95}}$ (iii) $\mathbf{5600\left(1 + \frac{r}{12}\right)^{96}}$

(b) $A'(0.11) = 5600\left(1 + \frac{0.11}{12}\right)^{95} \approx$ (i) **13325** (ii) **13425** (iii) **13525**

4.4 Derivatives of Exponential Functions

The *exponential* $e \approx 2.71828...$ is defined as

$$\lim_{m\to\infty}\left(1 + \frac{1}{m}\right).$$

The *(natural) exponential function* has the remarkable property it is its own derivative:

$$\frac{d}{dx}e^x = e^x,$$

whereas the derivative of the *exponential function* a^x, $a \neq 1$, is

$$\frac{d}{dx}a^x = (\ln a)a^x.$$

Exercise 4.4 (Derivatives of Exponential Functions)

1. *Natural exponential functions.*

   (a) If $f(x) = e^x$ then
   $f'(x) =$ (i) $\mathbf{0}$ (ii) $\boldsymbol{e}$ (iii) $\boldsymbol{e^x}$

(b) *Natural exponential and chain rule.* $y = 4e^{-7t}$.
Let $f[g(t)] = 4e^{-7t}$
with inner function $g(t) = -7t$ and outer function $f(t) = 4e^t$
and $g'(t) =$ (i) $\boldsymbol{-7}$ (ii) $\boldsymbol{-7e^t}$ (iii) $\boldsymbol{4e^{-7t}}$
and $f'(t) =$ (i) $\boldsymbol{4e^t}$ (ii) $\boldsymbol{e^t}$ (iii) $\boldsymbol{4}$
and so by chain rule

$$f'[g(t)] \cdot g'(t) = f'[-7t] \cdot (-7) = 4e^{-7t}(-7) =$$

(i) $\boldsymbol{-28e^{-7t}}$ (ii) $\boldsymbol{-14te^{2t}}$ (iii) $\boldsymbol{-14e^{2t}}$

(c) *Natural exponential and chain rule.* $y = 5e^{2x}$.
Let $f[g(x)] = 5e^{2x}$
with inner function $g(x) = 2x$ and outer function $f(x) = 5e^x$
and $g'(x) =$ (i) $\boldsymbol{2}$ (ii) $\boldsymbol{2x^2}$ (iii) $\boldsymbol{4x}$
and $f'(x) =$ (i) $\boldsymbol{x}$ (ii) $\boldsymbol{5e^x}$ (iii) $\boldsymbol{e^{2x}}$
and so by chain rule

$$f'[g(x)] \cdot g'(x) = f'[2x] \cdot (2) = 5e^{2x}(2) =$$

(i) $\boldsymbol{5xe^{2x}}$ (ii) $\boldsymbol{10e^{2x}}$ (iii) $\boldsymbol{2xe^{2x}}$

(d) *Natural exponential and chain rule.* Find $\frac{d}{dx}e^{-7\sqrt{x}}$.

Let $f[g(x)] = e^{-7\sqrt{x}} = e^{-7x^{\frac{1}{2}}}$ and $g(x) = -7x^{\frac{1}{2}}$ and $f(x) = e^x$

and $g'(x) =$ (i) $\boldsymbol{-7x^{\frac{1}{2}}}$ (ii) $\boldsymbol{-\frac{7}{2}x^{-\frac{1}{2}}}$ (iii) $\boldsymbol{-\frac{7}{2}x^{-\frac{3}{2}}}$
and $f'(x) =$ (i) $\boldsymbol{x}$ (ii) $\boldsymbol{e^x}$ (iii) $\boldsymbol{e^{2x}}$
and so by chain rule

$$f'[g(x)] \cdot g'(x) = f'\left[-7x^{\frac{1}{2}}\right] \cdot \left(-\frac{7}{2}x^{-\frac{1}{2}}\right) = e^{-7x^{\frac{1}{2}}}\left(-\frac{7}{2}x^{-\frac{1}{2}}\right) =$$

(i) $\boldsymbol{-\frac{7}{2}x^{-\frac{1}{2}}e^{-7\sqrt{x}}}$ (ii) $\boldsymbol{-\frac{7}{2}x^{-\frac{3}{2}}e^{-7\sqrt{x}}}$ (iii) $\boldsymbol{-7x^{\frac{1}{2}}e^{-7\sqrt{x}}}$

(e) $y = 5e^{2x} - e^{-7\sqrt{x}}$

Then $\frac{dy}{dx} =$
(i) $\boldsymbol{10e^{2x} + \frac{7}{2}x^{-1/2}e^{-7\sqrt{x}}}$
(ii) $\boldsymbol{10e^{2x} - \frac{7}{2}x^{-1/2}e^{-7\sqrt{x}}}$
(iii) $\boldsymbol{10e^{2x} + 7x^{1/2}e^{-7\sqrt{x}}}$
Hint: Use the answers from the previous two questions.

(f) *Natural exponential and product rule.* $y = x^3 e^x$.

Let $u(x) = x^3$ and $v(x) = e^x$
then $u'(x) =$ (i) $\boldsymbol{x^3}$ (ii) $\boldsymbol{3x^2}$ (iii) $\boldsymbol{e^{2x}}$
and $v'(x) =$ (i) $\boldsymbol{e^x}$ (ii) $\boldsymbol{3x^2}$ (iii) $\boldsymbol{x}$

and so $v(x)u'(x) =$
(i) $\boldsymbol{e^x\,(x^2)}$
(ii) $\boldsymbol{e^{3x}\,(3x^2)}$
(iii) $\boldsymbol{e^x\,(3x^2)}$

and $u(x)v'(x) =$
(i) $\boldsymbol{(3x^3)(e^x)}$
(ii) $\boldsymbol{(x^3)(e^x)}$
(iii) $\boldsymbol{(x^3)(e^3x)}$

and so $f'(x) = v(x) \cdot u'(x) + u(x) \cdot v'(x) =$
(i) $\boldsymbol{e^x\,(x^2) + (x^3)(e^x)}$
(ii) $\boldsymbol{xe^x + (x^3)(e^x)}$

(iii) $\boldsymbol{3x^2e^x + x^3e^x}$

(g) *Natural exponential and product rule.* $y = (2x+1)^2 e^{3x^5-5x^2}$.

Let $u(x) = (2x+1)^2$ and $v(x) = e^{3x^5-5x^2}$

then $u'(x) =$
(i) $\boldsymbol{4x + 4}$
(ii) $\boldsymbol{2(2x+1)^{2-1}(2) = 4(2x+1) = 8x + 4}$ (chain rule)
(iii) $\boldsymbol{x + 1}$

and $v'(x) =$
(i) $\boldsymbol{(3x^5 - 5x^2)e^{3x^5-5x^2}}$
(ii) $\boldsymbol{e^{3x^5-5x^2}(15x^4 - 10x) = (15x^4 - 10x)e^{3x^5-5x^2}}$ (chain rule)
(iii) $\boldsymbol{e^{3x^5-5x^2}}$

and so $v(x)u'(x) =$
(i) $\boldsymbol{e^{3x^5-5x^2}\,(8x+4)}$
(ii) $\boldsymbol{e^{3x^5-5x^2}\,(8x)}$
(iii) $\boldsymbol{e^{3x^5}\,(8x+4)}$

and $u(x)v'(x) =$
(i) $\boldsymbol{(2x+1)^2(15x^4 - 10x)e^{3x^5-5x^2}}$

(ii) $\boldsymbol{(2x)^2(15x^4 - 10x)e^{3x^5-5x^2}}$
(iii) $\boldsymbol{(2x+1)^2(15x^4)e^{3x^5-5x^2}}$

and so $f'(x) = v(x) \cdot u'(x) + u(x) \cdot v'(x) =$
(i) $\boldsymbol{e^{3x^5}\,(8x+4) + (2x+1)^2(15x^4 - 10x)e^{3x^5-5x^2}}$
(ii) $\boldsymbol{e^{3x^5-5x^2}\,(8x+4) + (2x+1)^2(15x^4 - 10x)e^{3x^5}}$
(iii) $\boldsymbol{e^{3x^5-5x^2}\,(8x+4) + (2x+1)^2(15x^4 - 10x)e^{3x^5-5x^2}}$
which simplifies to

$$\begin{aligned} & \left[(8x+4) + (2x+1)^2(15x^4 - 10x)\right] e^{3x^5-5x^2} \\ = & \left(60x^6 - 30x^5 + 15x^4 - 40x^3 + 20x^2 + 18x + 4\right) e^{3x^5-5x^2} \end{aligned}$$

(h) *Natural exponential and quotient rule.* $y = \frac{(2x+1)^2}{e^{3x^5-5x^2}}$.

Let $u(x) = (2x+1)^2$ and $v(x) = e^{3x^5-5x^2}$

then $u'(x) =$
(i) $\boldsymbol{4x+4}$
(ii) $\boldsymbol{2(2x+1)^{2-1}(2) = 4(2x+1) = 8x+4}$ (chain rule)
(iii) $\boldsymbol{x+1}$

and $v'(x) =$
(i) $\boldsymbol{(3x^5 - 5x^2)e^{3x^5-5x^2}}$
(ii) $\boldsymbol{e^{3x^5-5x^2}(15x^4 - 10x) = (15x^4 - 10x)e^{3x^5-5x^2}}$ (chain rule)
(iii) $\boldsymbol{e^{3x^5-5x^2}}$

and so $v(x)u'(x) =$
(i) $\boldsymbol{e^{3x^5-5x^2}\,(8x)}$
(ii) $\boldsymbol{e^{3x^5-5x^2}\,(8x+4)}$
(iii) $\boldsymbol{e^{3x^5}\,(8x+4)}$

and $u(x)v'(x) =$
(i) $\boldsymbol{(2x+1)^2(15x^4 - 10x)e^{3x^5-5x^2}}$
(ii) $\boldsymbol{(2x)^2(15x^4 - 10x)e^{3x^5-5x^2}}$
(iii) $\boldsymbol{(2x+1)^2(15x^4)e^{3x^5-5x^2}}$

and so $f'(x) = \frac{v(x) \cdot u'(x) - u(x) \cdot v'(x)}{[v(x)]^2} =$
(i) $\boldsymbol{e^{3x^5}\,(8x+4) + (2x+1)^2(15x^4 - 10x)e^{3x^5-5x^2}}$
(ii) $\boldsymbol{\frac{e^{3x^5-5x^2}(8x+4) - (2x+1)^2(15x^4-10x)e^{3x^5-5x^2}}{\left[e^{3x^5-5x^2}\right]^2}}$
(iii) $\boldsymbol{e^{3x^5-5x^2}\,(8x+4) + (2x+1)^2(15x^4 - 10x)e^{3x^5}}$

which simplifies to

$$\frac{(8x+4)-(2x+1)^2(15x^4-10x)}{e^{3x^5-5x^2}} = \frac{-60x^6-60x^5-15x^4+40x^3+40x^2+18x+4}{e^{3x^5-5x^2}}$$

(i) *A limit related to natural exponential function.*

$h \to$	1	0.1	0.01	0.001	0.00001
$f(h) = \frac{e^h-1}{h} \to$	1.718	1.052	1.005	1.001	1.000005

(Type 2nd TBLSET 1 1 Ask Auto, then 2nd TABLE 1 ENTER 0.1 ENTER and so on.)
so $\lim_{h\to 0} \frac{e^h-1}{h}$ = (i) **0** (ii) **1** (iii) **2**.

(j) (i) **True** (ii) **False** If $f(x) = e^x$, then

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{e^{x+h}-e^x}{h} \\ &= \lim_{h\to 0} \frac{e^x e^h - e^x}{h} \\ &= e^x \lim_{h\to 0} \frac{e^h-1}{h} = e^x(1) = e^x \quad \text{(using limit result above)} \end{aligned}$$

2. *Exponential functions.*

(a) If $f(x) = 5^x$ then
$f'(x) =$ (i) $\left(\mathbf{ln^3 5}\right) \mathbf{5^x}$ (ii) $(\mathbf{ln\,5})\,\mathbf{5^x}$ (iii) $\left(\mathbf{ln^2 5}\right) \mathbf{5^x}$

(b) *Exponential and chain rule.* $y = 4 \cdot 3^{-7x}$.

Let $f[g(x)] = 4 \cdot 3^{-7x}$ and $g(x) = -7x$ and $f(x) = 4 \cdot 3^x$

and $g'(x) =$ (i) $\mathbf{-7}$ (ii) $\mathbf{3x}$ (iii) $\mathbf{4 \cdot 3^x}$
and $f'(x) =$ (i) $\mathbf{4 \cdot (ln\,3)\,3^x}$ (ii) $\mathbf{(ln\,3)\,3^x}$ (iii) $\mathbf{4 \cdot (ln\,3)}$
and so by chain rule

$$f'[g(x)] \cdot g'(x) = f'[-7x] \cdot (-7) = 4 \cdot (\ln 3)\, 3^{-7x}\,(-7) =$$

(i) $\mathbf{-7 \cdot 3^{-7x}\, ln\,3}$ (ii) $\mathbf{-28 \cdot 3^{-7x}\, ln\,3}$ (iii) $\mathbf{-14 \cdot 3^{-7x}\, ln\,3}$

(c) $y = 5(9)^{2x} - x^3$.

Let $f[g(x)] = 5 \cdot 9^{2x}$ and $g(x) = 2x$ and $f(x) = 5 \cdot 9^x$
and $g'(x) =$ (i) $\mathbf{2}$ (ii) $\mathbf{2x}$ (iii) $\mathbf{5 \cdot 9^x}$
and $f'(x) =$ (i) $\mathbf{(ln\,9)\,9^x}$ (ii) $\mathbf{5 \cdot (ln\,9)}$ (iii) $\mathbf{5 \cdot (ln\,9)\,9^x}$
and so by chain rule

$$f'[g(x)] \cdot g'(x) = f'[2x] \cdot (2) = 5 \cdot (\ln 9)\, 9^{2x}\,(2) =$$

(i) $\mathbf{(\ln 9)\, 9^{2x}}$ (ii) $\mathbf{10 \cdot (\ln 9)\, 9^{2x}}$ (iii) $\mathbf{10 \cdot (\ln 9)}$

also since $\frac{d}{dx}\left(x^3\right) =$ (i) $\boldsymbol{3x^2}$ (ii) $\boldsymbol{x^3}$ (iii) $\boldsymbol{3x^3}$

then $\frac{dy}{dx} =$
(i) $\boldsymbol{(\ln 9)\, 9^{2x} - 3x^2}$
(ii) $\boldsymbol{10 \cdot (\ln 9)\, 9^{2x}}$
(iii) $\boldsymbol{10 \cdot (\ln 9)\, 9^{2x} - 3x^2}$

(d) *Exponential and product rule.* $y = x^3(5.5)^x$.

Let $u(x) = x^3$ and $v(x) = 5.5^x$
then $u'(x) =$ (i) $\boldsymbol{3x^2}$ (ii) $\boldsymbol{3x}$ (iii) $\boldsymbol{e^{2x}}$
and $v'(x) =$ (i) $\boldsymbol{5.5^x}$ (ii) $\boldsymbol{(\ln 5.5)}$ (iii) $\boldsymbol{(\ln 5.5)\, 5.5^x}$

and so $v(x)u'(x) =$
(i) $\boldsymbol{5.5^x}$
(ii) $\boldsymbol{3x^3}$
(iii) $\boldsymbol{(5.5^x)\,(3x^2)}$

and $u(x)v'(x) =$
(i) $\boldsymbol{(x^3)\,(\ln 5.5)\, 5.5^x}$
(ii) $\boldsymbol{(\ln 5.5)\, 5.5^x}$
(iii) $\boldsymbol{(3x^3)\,(\ln 5.5)}$

and so $f'(x) = v(x) \cdot u'(x) + u(x) \cdot v'(x) =$
(i) $\boldsymbol{(5.5^x)\,(3x^2) + (3x^3)\,(\ln 5.5)}$
(ii) $\boldsymbol{(5.5^x) + (3x^3)\,(\ln 5.5)\, 5.5^x}$
(iii) $\boldsymbol{(5.5^x)\,(3x^2) + (x^3)\,(\ln 5.5)\, 5.5^x}$
which simplifies to

$$\left(x^2\right)(5.5^x)\,(3 + x \ln 5.5)$$

(e) *Exponential and quotient rule.* $y = \frac{x^3}{(5.5)^x}$.

Let $u(x) = x^3$ and $v(x) = 5.5^x$
then $u'(x) =$ (i) $\boldsymbol{3x}$ (ii) $\boldsymbol{3x^2}$ (iii) $\boldsymbol{e^{2x}}$
and $v'(x) =$ (i) $\boldsymbol{(\ln 5.5)\, 5.5^x}$ (ii) $\boldsymbol{5.5^x}$ (iii) $\boldsymbol{(\ln 5.5)}$

and so $v(x)u'(x) =$
(i) $\boldsymbol{5.5^x}$
(ii) $\boldsymbol{3x^2}$
(iii) $\boldsymbol{(5.5^x)\,(3x^2)}$

and $u(x)v'(x) =$
(i) $\mathbf{(x^3)\,(\ln 5.5)\,5.5^x}$
(ii) $\mathbf{(\ln 5.5)\,5.5^x}$
(iii) $\mathbf{(3x^3)\,(\ln 5.5)}$

and so $f'(x) = \frac{v(x)\cdot u'(x) - u(x)\cdot v'(x)}{[v(x)]^2} =$
(i) $\mathbf{\frac{(5.5^x)\left(3x^2\right) - \left(x^3\right)(\ln 5.5) 5.5^x}{[5.5^x]^2}}$
(ii) $\mathbf{(5.5^x)\,(3x^3) - (3x^3)\,(\ln 5.5)}$
(iii) $\mathbf{(5.5^x) + (3x^3)\,(\ln 5.5)\,5.5^x}$
which simplifies to

$$\frac{(x^2)\,(3 - x\ln 5.5)}{5.5^x}$$

3. *Application: radioactive decay.* Quantity (in ounces) present at time t (in years)

$$Q(t) = 500(5^{-0.2t})$$

Determine the rate of change of quantity, $Q'(t)$, after $t = 4$ years.

(a) Let $f[g(t)] = 500 \cdot 5^{-0.2t}$ and $g(t) = -0.2t$ and $f(t) = 500 \cdot 5^t$

and $g'(t) =$ (i) $\mathbf{-0.2t}$ (ii) $\mathbf{500 \cdot 5^t}$ (iii) $\mathbf{-0.2}$
and $f'(t) =$ (i) $\mathbf{500 \cdot (\ln 5)\, 5^t}$ (ii) $\mathbf{(\ln 5)\, 5^t}$ (iii) $\mathbf{500 \cdot (\ln 5)}$
and so by chain rule

$$f'[g(t)] \cdot g'(t) = f'[-0.2t] \cdot (-0.2) = 500 \cdot (\ln 5)\, 5^{-0.2t}\,(-0.2) =$$

(i) $\mathbf{-100 \cdot (\ln 5)\, 5^{-0.2t}}$ (ii) $\mathbf{(\ln 5)\, 5^{-0.2t}}$ (iii) $\mathbf{-100 \cdot (\ln 5)}$

(b) So $Q'(4) = -100 \cdot (\ln 5)\, 5^{-0.2(4)} \approx$
(i) $\mathbf{-34.4}$ (ii) $\mathbf{-44.4}$ (iii) $\mathbf{-24.4}$ years.

4.5 Derivatives of Logarithmic Functions

Derivative of *(natural) logarithmic function* is:

$$\frac{d}{dx} \ln|x| = \frac{1}{x} = x^{-1}$$

whereas derivative of the *logarithmic function* is:

$$\frac{d}{dx}[\log_a |x|] = \frac{1}{(\ln a)x} = ((\ln a)x)^{-1}.$$

Exercise 4.5 (Derivatives of Logarithmic Functions)

1. *Natural logarithmic functions.*

   (a) If $f(x) = \ln x$ then
   $f'(x) =$ (i) $\mathbf{\frac{1}{x^2}}$ (ii) $\mathbf{\frac{1}{2x}}$ (iii) $\mathbf{\frac{1}{x}}$

   (b) *Natural logarithmic function and chain rule.* $y = \ln 2x$

   Let $f[g(x)] = \ln 2x$ and $g(x) = 2x$ and $f(x) = \ln x$

   and $g'(x) =$ (i) $\mathbf{2x}$ (ii) $\mathbf{2}$ (iii) $\mathbf{\ln 2x}$
   and $f'(x) =$ (i) $\mathbf{\frac{1}{x^2}}$ (ii) $\mathbf{\frac{1}{x}}$ (iii) $\mathbf{\frac{1}{2x}}$
   and so by chain rule

   $$f'[g(x)] \cdot g'(x) = f'[2x] \cdot (2) = \frac{1}{2x}(2) =$$

   (i) $\mathbf{\frac{1}{x}}$ (ii) $\mathbf{\frac{1}{2x}}$ (iii) $\mathbf{\frac{1}{3x}}$

   (i) **True** (ii) **False**
   Derivative of $\ln x$ equals derivative of $\ln 2x$, $\frac{d}{dx}(\ln x) = \frac{d}{dx}(\ln 2x)$
   (This is true in general: $\frac{d}{dx}(\ln x) = \frac{d}{dx}(\ln ax)$, a a real number.)

   (c) *Natural logarithmic function and chain rule.* $y = 4\ln(7\sqrt{x})$

   Let $f[g(x)] = 4\ln(7\sqrt{x})$ and $g(x) = 7\sqrt{x} = 7x^{\frac{1}{2}}$ and $f(x) = 4\ln x$

   and $g'(x) =$ (i) $\mathbf{\frac{5}{2}x^{-\frac{1}{2}}}$ (ii) $\mathbf{\frac{7}{2}x^{-\frac{1}{2}}}$ (iii) $\mathbf{\frac{3}{2}x^{-\frac{1}{2}}}$
   and $f'(x) =$ (i) $\mathbf{\frac{4}{x^2}}$ (ii) $\mathbf{\frac{1}{4x}}$ (iii) $\mathbf{\frac{4}{x}}$
   and so by chain rule

   $$f'[g(x)] \cdot g'(x) = f'\left[7x^{\frac{1}{2}}\right] \cdot \left(\frac{7}{2}x^{-\frac{1}{2}}\right) = \frac{4}{7x^{\frac{1}{2}}}\left(\frac{7}{2}x^{-\frac{1}{2}}\right) =$$

   (i) $\mathbf{\frac{2}{x}}$ (ii) $\mathbf{-\frac{7}{2}x^{-3/2}}$ (iii) $\mathbf{\frac{7}{2}x^{-\frac{1}{2}}}$

   (d) $y = 5e^{2x} - 4\ln(7\sqrt{x})$.

   Let $f[g(x)] = 4\ln(7\sqrt{x})$ and $g(x) = 7\sqrt{x} = 7x^{\frac{1}{2}}$ and $f(x) = 4\ln x$

   and $g'(x) =$ (i) $\mathbf{\frac{5}{2}x^{-\frac{1}{2}}}$ (ii) $\mathbf{\frac{3}{2}x^{-\frac{1}{2}}}$ (iii) $\mathbf{\frac{7}{2}x^{-\frac{1}{2}}}$
   and $f'(x) =$ (i) $\mathbf{\frac{4}{x}}$ (ii) $\mathbf{\frac{4}{x^2}}$ (iii) $\mathbf{\frac{1}{4x}}$
   and so by chain rule

   $$f'[g(x)] \cdot g'(x) = f'\left[7x^{\frac{1}{2}}\right] \cdot \left(\frac{7}{2}x^{-\frac{1}{2}}\right) = \frac{4}{7x^{\frac{1}{2}}}\left(\frac{7}{2}x^{-\frac{1}{2}}\right) =$$

(i) $\boldsymbol{-\frac{7}{2}x^{-3/2}}$ (ii) $\boldsymbol{\frac{7}{2}x^{-\frac{1}{2}}}$ (iii) $\boldsymbol{\frac{2}{x}}$

and also let $f[g(x)] = 5e^{2x}$ and $g(x) = 2x$ and $f(x) = 5e^x$
and $g'(x) =$ (i) $\boldsymbol{2x^2}$ (ii) $\boldsymbol{4x}$ (iii) $\boldsymbol{2}$
and $f'(x) =$ (i) $\boldsymbol{x}$ (ii) $\boldsymbol{5e^x}$ (iii) $\boldsymbol{e^{2x}}$
and so by chain rule

$$f'[g(x)] \cdot g'(x) = f'[2x] \cdot (2) = 5e^{2x}(2) =$$

(i) $\boldsymbol{10e^{2x}}$ (ii) $\boldsymbol{5xe^{2x}}$ (iii) $\boldsymbol{2xe^{2x}}$

So $\frac{dy}{dx} = \frac{2}{x} - 10e^{2x} =$ (i) $\boldsymbol{\frac{2+10xe^{2x}}{x}}$ (ii) $\boldsymbol{\frac{2-10xe^{2x}}{x}}$ (iii) $\boldsymbol{\frac{10xe^{2x}}{x}}$

(e) *Natural logarithmic function and product rule.* $y = x^3 \ln x$.

Let $u(x) = x^3$ and $v(x) = \ln x$

then $u'(x) =$ (i) $\boldsymbol{3x}$ (ii) $\boldsymbol{3x^2}$ (iii) $\boldsymbol{\ln x}$
and $v'(x) =$ (i) $\boldsymbol{\frac{1}{x}}$ (ii) $\boldsymbol{\frac{1}{2x}}$ (iii) $\boldsymbol{3x^2}$

and so $v(x)u'(x) =$
(i) $\boldsymbol{(\ln x)\,(3x^2)}$
(ii) $\boldsymbol{3x^3}$
(iii) $\boldsymbol{(\ln 2x)\,(3x^3)}$

and $u(x)v'(x) =$
(i) $\boldsymbol{(3x^2)\left(\frac{1}{x}\right)}$
(ii) $\boldsymbol{(x^3)\left(\frac{1}{2x}\right)}$
(iii) $\boldsymbol{(x^3)\left(\frac{1}{x}\right)}$

and so $f'(x) = v(x) \cdot u'(x) + u(x) \cdot v'(x) =$
(i) $\boldsymbol{(\ln x)\,(3x^2) + (x^3)\left(\frac{1}{x}\right)}$
(ii) $\boldsymbol{(\ln x)\,(x^3) + (x^3)\left(\frac{1}{x}\right)}$
(iii) $\boldsymbol{(\ln 2x)\,(3x^2) + (x^3)\left(\frac{1}{x}\right)}$
which simplifies to

$$3x^2 \ln x + x^2$$

(f) *Natural logarithmic function and product rule.* $y = (2x+1)^2 \ln(3x^5 - 5x^2)$.

Let $u(x) = (2x+1)^2$ and $v(x) = \ln(3x^5 - 5x^2)$

then $u'(x) = f'[g(x)] \cdot g'(x) = f'[2x+1] \cdot (2) =$
(i) $\boldsymbol{4x + 4}$

(ii) $\mathbf{2(2x+1)^{2-1}(2) = 4(2x+1) = 8x+4}$ (chain rule)
(iii) $\mathbf{x+1}$

and $v'(x) = f'[g(x)] \cdot g'(x) = f'[3x^5 - 5x^2] \cdot (15x^4 - 10x) =$
(i) $\mathbf{e^{3x^5-5x^2}(15x^4 - 10x) = (15x^4 - 10x)e^{3x^5-5x^2}}$ (chain rule)
(ii) $\mathbf{e^{3x^5-5x^2}}$
(iii) $\mathbf{\left(\frac{1}{3x^5-5x^2}\right)(15x^4 - 10x) = \frac{15x^3-10}{3x^4-5x}}$ (chain rule)

and so $v(x)u'(x) =$
(i) $\mathbf{\ln(3x^5 - 5x^2)(8x+4)}$
(ii) $\mathbf{\ln(3x^5 - 5x^2)(8x)}$
(iii) $\mathbf{\ln(5x^2)(8x+4)}$

and $u(x)v'(x) =$
(i) $\mathbf{(2x+1)^2\left(\frac{15x^3-10}{3x^4-5x}\right)}$
(ii) $\mathbf{(2x+1)\left(\frac{15x^3-10}{3x^4-5x}\right)}$
(iii) $\mathbf{(2x+1)^2\left(\frac{10}{3x^4-5x}\right)}$

and so $f'(x) = v(x) \cdot u'(x) + u(x) \cdot v'(x) =$
(i) $\mathbf{\ln(3x^5 - 5x^2)(8x+4) + (2x+1)\left(\frac{15x^3-10}{3x^4-5x}\right)}$
(ii) $\mathbf{\ln(3x^5)(8x+4) + (2x+1)^2\left(\frac{15x^3-10}{3x^4-5x}\right)}$

(iii) $\mathbf{\ln(3x^5 - 5x^2)(8x+4) + (2x+1)^2\left(\frac{15x^3-10}{3x^4-5x}\right)}$

(g) *Application: growth rate.* Number of insects at time t (in hours) is

$$N(t) = \frac{0.1t + 4}{\ln t}.$$

Determine growth rate, $N'(t)$, after $t = 3$ hours.

Let $u(t) = 0.1t + 4$ and $v(t) = \ln t$
then $u'(t) =$ (i) $\mathbf{0.1}$ (ii) $\mathbf{0.1t}$ (iii) $\mathbf{0.t^2}$
and $v'(t) =$ (i) $\mathbf{\frac{1}{2t}}$ (ii) $\mathbf{\frac{1}{t}}$ (iii) $\mathbf{\frac{1}{3t}}$

and so $v(t)u'(t) =$ (i) $\mathbf{(\ln t)(0.1)}$ (ii) $\mathbf{(\ln 2t)(0.1)}$ (iii) $\mathbf{(\ln t)(0.1t)}$

and $u(t)v'(t) =$ (i) $\mathbf{(0.1t+4)\left(\frac{1}{3t}\right)}$ (ii) $\mathbf{(0.1t)\left(\frac{1}{t}\right)}$ (iii) $\mathbf{(0.1t+4)\left(\frac{1}{t}\right)}$

and so $N'(t) = \frac{v(t) \cdot u'(t) - u(t) \cdot v'(t)}{[v(t)]^2} =$
(i) $\mathbf{\frac{(\ln 2t)(0.1) - (0.1t+4)\left(\frac{1}{t}\right)}{[\ln t]^2}}$

(ii) $\frac{\mathbf{(\ln t)(0.1)-(0.1t)\left(\frac{1}{t}\right)}}{\mathbf{[\ln t]^2}}$
(iii) $\frac{\mathbf{(\ln t)(0.1)-(0.1t+4)\left(\frac{1}{t}\right)}}{\mathbf{[\ln t]^2}}$
which simplifies to

$$\frac{0.1t\ln t - 0.1t - 4}{2t\ln t}$$

and so $N'(3) = \frac{0.1(3)\ln(3)-0.1(3)-4}{2(3)\ln 3} \approx$ (i) $\mathbf{-18.1}$ (ii) $\mathbf{-19.1}$ (iii) $\mathbf{-20.1}$

(h) (i) **True** (ii) **False** Recall

$$|x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases}$$

and so

$$\frac{d}{dx}\ln|x| = \frac{1}{x}$$

means, if $x > 0$,

$$\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln x = \frac{1}{x}$$

and also, if $x < 0$, and using the chain rule,

$$\frac{d}{dx}\ln|x| = f'[g(x)]\cdot g'(x) = f'[-x]\cdot(-1) = \frac{1}{-x}(-1) = \frac{1}{x}$$

(Since $\ln x$, $x < 0$, does *not* exist, then also derivative of $\ln x$, $x < 0$, does *not* exist, however, derivative of $\ln|x| = \ln -x$, $x < 0$, does exist and is $\frac{1}{x}$.)

2. *Logarithmic functions.*

(a) If $f(x) = \log_5 x$ then
$f'(x) =$ (i) $\mathbf{\frac{1}{\ln 5}}$ (ii) $\mathbf{\frac{1}{(\ln 5)x}}$ (iii) $\mathbf{\frac{1}{x}}$

(b) *Logarithmic function and chain rule.* $y = \log_{12} 2x$.

Let $f[g(x)] = \log_{12} 2x$ and $g(x) = 2x$ and $f(x) = \log_{12} x$

and $g'(x) =$ (i) $\mathbf{2}$ (ii) $\mathbf{2x}$ (iii) $\mathbf{\log_{12} 2x}$
and $f'(x) =$ (i) $\mathbf{\frac{1}{x}}$ (ii) $\mathbf{\frac{1}{(\ln 12)x}}$ (iii) $\mathbf{\frac{1}{\ln 12}}$
and so by chain rule

$$f'[g(x)]\cdot g'(x) = f'[2x]\cdot(2) = \frac{1}{(\ln 12)\,2x}(2) =$$

(i) $\mathbf{\frac{1}{x}}$ (ii) $\mathbf{\frac{1}{x\ln 12}}$ (iii) $\mathbf{\frac{1}{\ln 12}}$

(c) *Logarithmic function and chain rule.* $y = \log_7(\ln x)$.

Let $f[g(x)] = \log_7(\ln x)$ and $g(x) = \ln x$ and $f(x) = \log_7 x$

and $g'(x) =$ (i) $\boldsymbol{\frac{1}{x}}$ (ii) $\boldsymbol{\frac{1}{12x}}$ (iii) $\boldsymbol{\frac{7}{x}}$
and $f'(x) =$ (i) $\boldsymbol{\frac{1}{(\ln 7)x}}$ (ii) $\boldsymbol{\frac{1}{7x}}$ (iii) $\boldsymbol{\frac{1}{\ln 7}}$
and so by chain rule

$$f'[g(x)] \cdot g'(x) = f'\left[\ln x\right] \cdot \left(\frac{1}{x}\right) = \frac{1}{(\ln 7)(\ln x)}\left(\frac{1}{x}\right) =$$

(i) $\boldsymbol{\frac{1}{(\ln x)(\ln 7)}}$ (ii) $\boldsymbol{\frac{1}{x(\ln 7)}}$ (iii) $\boldsymbol{\frac{1}{x(\ln x)(\ln 7)}}$

(d) *Logarithmic function and quotient rule.* $y = \frac{x^3}{\log_3 x}$.

Let $u(x) = x^3$ and $v(x) = \log_3 x$
then $u'(x) =$ (i) $\boldsymbol{3x^3}$ (ii) $\boldsymbol{e^{2x}}$ (iii) $\boldsymbol{3x^2}$
and $v'(x) =$ (i) $\boldsymbol{\frac{1}{\ln 3}}$ (ii) $\boldsymbol{\frac{1}{(\ln 3)x}}$ (iii) $\boldsymbol{\frac{1}{x}}$

and so $v(x)u'(x) =$
(i) $\boldsymbol{\log_3 x^3}$
(ii) $\boldsymbol{(\log_3 x)\,(3x^2)}$
(iii) $\boldsymbol{(\log_3 x^2)\,(3x^3)}$

and $u(x)v'(x) =$
(i) $\boldsymbol{\left(\frac{1}{(\ln 3)x}\right)}$
(ii) $\boldsymbol{(x^3)\left(\frac{1}{(\ln 3)}\right)}$
(iii) $\boldsymbol{(x^3)\left(\frac{1}{(\ln 3)x}\right)}$

and so $f'(x) = \frac{v(x)\cdot u'(x) - u(x)\cdot v'(x)}{[v(x)]^2} =$
(i) $\boldsymbol{\frac{(\log_3 x) - (x^3)\left(\frac{1}{(\ln 3)x}\right)}{[\log_3 x]^2}}$
(ii) $\boldsymbol{\frac{(\log_3 x)(3x^2) - (x^3)\left(\frac{1}{(\ln 3)x}\right)}{[\log_3 x]^2}}$
(iii) $\boldsymbol{\frac{(\log_3 x)(3x^2) - \left(\frac{1}{(\ln 3)x}\right)}{[\log_3 x]^2}}$
which simplifies to

$$\frac{3x^3\,(\ln 3)\,(\log_3 x) - x}{(\ln 3)\,(\log_3 x)^2}$$

(e) *Application.* Adult body surface area, S, (in cm^2) related to weight, w, is

$$S(w) = w^w.$$

Determine change in surface area, $S'(w)$, at $w = 2$ kilograms.

i. *Use your calculator.*

Set WINDOW 0 3 1 0 30 10 1

Type $Y_1 = X^X$

GRAPH TRACE 2 ENTER 2nd CALC dy/dx ENTER

$S'(2) = dy/dx \approx$ (i) **5.75** (ii) **6.77** (iii) **7.85**

ii. *Attempt analytical approach.*

Let $f(w) = w^w$,
so $f'(w) =$ (i) **huh?** (ii) $\boldsymbol{w^w (\ln w + 1)}$ (iii) $\boldsymbol{5w^w}$

because, so far, although we know how to differentiate functions like $y = 5^w$ or $y = w^5$, we do not yet know how to differentiate $y = w^w$; later we will use *implicit* differentiation to show the derivative of $y = w^w$ is $\frac{dy}{dx} = w^w (\ln w + 1)$.