9.5 Total Differentials and Approximations

For function $z = f(x, y)$ whose partial derivatives exists, *total differential* of z is

$$dz = f_x(x, y) \cdot dx + f_y(x, y) \cdot dy,$$

where dz is sometimes written df. On the one hand, the *exact* value of function is

$$f(x + \Delta x, y + \Delta y) = f(x, y) + \Delta z.$$

On the other hand, *if* differentials dx and dy are *small*, *then* $dz \approx \Delta z$, and so the value of the function could be *(linearly) approximated* by,

$$\begin{aligned} f(x + \Delta x, y + \Delta y) &\approx f(x, y) + dz \\ &= f(x, y) + f_x(x, y) \cdot dx + f_y(x, y) \cdot dy. \end{aligned}$$

Total differentials can be generalized. For a function $f = f(x, y, z)$ whose partial derivatives exists, the total differential of f is given by

$$df = f_x(x, y, z) \cdot dx + f_y(x, y, z) \cdot dy + f_z(x, y, z) \cdot dz.$$

Exercise 9.5 (Total Differentials and Approximations)

1. *Differentials* $\Delta z \approx dz$ *and* $dz = f_x(x, y) \cdot dx + f_y(x, y) \cdot dy$.

   (a) $f(x, y) = x^2 + 3xy + y^2$, $x = 2$, $y = 0$, $dx = 0.01$, $dy = -0.01$.

   since $f_x(x, y) = 2x^{2-1} + 3x^{1-1}y + 0 =$
   (i) $\boldsymbol{2x^2}$ (ii) $\boldsymbol{3x + 2y}$ (iii) $\boldsymbol{2x + 3y}$,

   and $f_y(x, y) = 0 + 3xy^{1-1} + 2y^{2-1} =$
   (i) $\boldsymbol{2x^2}$ (ii) $\boldsymbol{3x + 2y}$ (iii) $\boldsymbol{2x + 3y}$,

   $$\begin{aligned} dz &= f_x(x, y) \cdot dx + f_y(x, y) \cdot dy \\ &= (2x + 3y)dx + (3x + 2y)dy \\ &= (2(2) + 3(0))(0.01) + (3(2) + 2(0))(-0.01) = \end{aligned}$$

   (i) $\mathbf{-0.02}$ (ii) $\mathbf{0}$ (iii) $\mathbf{0.02}$
   exact $\Delta z = f(2.01, -0.01) - f(2, 0) = \left[2.01^2 + 3(2.01)(-0.01) + (-0.01)^2\right] - \left[2^2 + 3(2)(0) + (0)^2\right] =$ -0.0201, so approximation minus exact: $-0.02 - (-0.0201) = 0.0001$

(b) $f(x,y) = 3x^3 + 3y^2 - 4\ln y$, $x = 0$, $y = 1$, $dx = 0.01$, $dy = -0.01$.

since $f_x(x,y) = 3 \cdot 3x^{3-1} + 0 - 0 =$
(i) $\boldsymbol{9x^2}$ (ii) $\boldsymbol{6y - \frac{4}{y}}$ (iii) $\boldsymbol{2x + 3y}$,

and $f_y(x,y) = 0 + 3 \cdot 2y^{2-1} - 4 \cdot \frac{1}{y} =$
(i) $\boldsymbol{9x^2}$ (ii) $\boldsymbol{6y - \frac{4}{y}}$ (iii) $\boldsymbol{2x + 3y}$,

$$\begin{aligned} dz &= f_x(x,y) \cdot dx + f_y(x,y) \cdot dy \\ &= (9x^2)dx + \left(6y - \frac{4}{y}\right) dy \\ &= (9(0)2)(0.01) + \left(6(1) - \frac{4}{1}\right)(-0.01) = \end{aligned}$$

(i) $\mathbf{-0.02}$ (ii) $\mathbf{0}$ (iii) $\mathbf{0.02}$

exact $\Delta z = f(2.01, -0.01) - f(2,0) = \left[3(0.01)^3 + 3(0.99)^2 - 4\ln(0.99)\right] - \left[3(0)^3 + 3(1)^2 - 4\ln(1)\right] \approx$ -0.0195, so approximation minus exact: $-0.02 - (-0.0195) = -0.0005$

(c) $f(x,y) = \sqrt{3x + y^2}$, $x = 2$, $y = 0$, $dx = 0.01$, $dy = 0.01$.

for $f_x(x,y)$, let $f[g(x)] = (3x + y^2)^{\frac{1}{2}}$
with "inner" function $g(x,y) =$ (i) $\boldsymbol{3x + y^2}$ (ii) $\boldsymbol{3x}$ (iii) $\boldsymbol{3x + x^2}$
and "outer" function $f(x,y) =$ (i) $\boldsymbol{x^{\frac{1}{2}}}$ (ii) $\boldsymbol{x^{\frac{3}{2}}}$ (iii) $\boldsymbol{x^{\frac{5}{2}}}$
with derivative $g_x(x,y) =$ (i) $\mathbf{3}$ (ii) $\boldsymbol{2y}$ (iii) $\boldsymbol{3x}$
and derivative $f_x(x,y) =$ (i) $\boldsymbol{\frac{3}{2}x^{\frac{1}{2}}}$ (ii) $\boldsymbol{\frac{3}{2}x^{\frac{3}{2}}}$ (iii) $\boldsymbol{\frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}}$
and so by chain rule

$$\begin{aligned} f_x(x,y) &= f_x[g(x,y)] \cdot g_x(x,y) = f_x[3x + y^2] \cdot (3) \\ &= \frac{1}{2\sqrt{3x + y^2}}(3) = \end{aligned}$$

(i) $\boldsymbol{\frac{y}{\sqrt{3x+y^2}}}$ (ii) $\boldsymbol{\frac{3}{2\sqrt{3x+y^2}}}$ (iii) $\boldsymbol{\frac{3x}{3\sqrt{3x+y^2}}}$

for $f_y(x,y)$, also let $f[g(x)] = (3x + y^2)^{\frac{1}{2}}$
with "inner" function $g(x,y) =$ (i) $\boldsymbol{3x + y^2}$ (ii) $\boldsymbol{3x}$ (iii) $\boldsymbol{3x + x^2}$
and "outer" function $f(x,y) =$ (i) $\boldsymbol{y^{\frac{1}{2}}}$ (ii) $\boldsymbol{y^{\frac{3}{2}}}$ (iii) $\boldsymbol{y^{\frac{5}{2}}}$
with derivative $g_y(x,y) =$ (i) $\mathbf{3}$ (ii) $\boldsymbol{2y}$ (iii) $\boldsymbol{3x}$
and derivative $f_y(x,y) =$ (i) $\boldsymbol{\frac{3}{2}y^{\frac{1}{2}}}$ (ii) $\boldsymbol{\frac{3}{2}y^{\frac{3}{2}}}$ (iii) $\boldsymbol{\frac{1}{2}y^{-\frac{1}{2}} = \frac{1}{2\sqrt{y}}}$
and so by chain rule

$$\begin{aligned} f_y(x,y) &= f_y[g(x,y)] \cdot g_y(x,y) = f_y[3x + y^2] \cdot (2y) \\ &= \frac{1}{2\sqrt{3x + y^2}}(2y) = \end{aligned}$$

(i) $\boldsymbol{\frac{y}{\sqrt{3x+y^2}}}$ (ii) $\boldsymbol{\frac{3}{2\sqrt{3x+y^2}}}$ (iii) $\boldsymbol{\frac{3x}{3\sqrt{3x+y^2}}}$

so

$$\begin{aligned} dz &= f_x(x,y)\cdot dx + f_y(x,y)\cdot dy \\ &= \left(\frac{3}{2\sqrt{3x+y^2}}\right) dx + \left(\frac{y}{\sqrt{3x+y^2}}\right) dy \\ &= \left(\frac{3}{2\sqrt{3(2)+(0)^2}}\right)(0.01) + \left(\frac{0}{\sqrt{3(2)+(0)^2}}\right)(0.01) \approx \end{aligned}$$

(i) **0.005** (ii) **0.006** (iii) **0.007**

(d) $f(x,y) = \frac{4x-y}{5x+4y}$, $x = 0$, $y = 1$, $dx = 0.01$ and $dy = 0.03$

for $f_x(x,y)$, let $u(x,y) = 4x - y$ and $v(x,y) = 5x + 4y$.
then, $u_x(x,y) =$ (i) $\boldsymbol{4x^2}$ (ii) **4** (iii) $\boldsymbol{-1}$
and $v_x(x,y) =$ (i) $\boldsymbol{5x^2}$ (ii) **5** (iii) $\boldsymbol{3+4x}$
and so

$$f_x(x,y) = \frac{v(x,y)\cdot u_x(x,y) - u(x,y)\cdot v_x(x,y)}{[v(x,y)]^2} = \frac{(5x+4y)(4)-(4x-y)(5)}{(5x+4y)^2} =$$

(i) $\boldsymbol{\frac{21}{(5x+4y)^2}}$ (ii) $\boldsymbol{\frac{21y}{(5x+4y)^2}}$ (iii) $\boldsymbol{\frac{20}{(5x+4y)^2}}$

for $f_y(x,y)$, let $u(x,y) = 4x - y$ and $v(x,y) = 5x + 4y$.
then, $u_y(x,y) =$ (i) $\boldsymbol{4x^2}$ (ii) **4** (iii) $\boldsymbol{-1}$
and $v_y(x,y) =$ (i) $\boldsymbol{5x^2}$ (ii) **5** (iii) **4**
and so

$$f_y(x,y) = \frac{v(x,y)\cdot u_y(x,y) - u(x,y)\cdot v_y(x,y)}{[v(x,y)]^2} = \frac{(5x+4y)(-1)-(4x-y)(4)}{(5x+4y)^2} =$$

(i) $\boldsymbol{\frac{21}{(5x+4y)^2}}$ (ii) $\boldsymbol{\frac{21y}{(5x+4y)^2}}$ (iii) $\boldsymbol{\frac{-21x}{(5x+4y)^2}}$

$$\begin{aligned} dz &= f_x(x,y)\cdot dx + f_y(x,y)\cdot dy \\ &= \left(\frac{21y}{(5x+4y)^2}\right) dx + \left(\frac{-21x}{(5x+4y)^2}\right) dy \\ &= \left(\frac{21(1)}{(5(0)+4(1))^2}\right)(0.01) + \left(\frac{-21(0)}{(5(0)+4(1))^2}\right)(0.03) \approx \end{aligned}$$

(i) **0.011** (ii) **0.012** (iii) **0.013**

2. *Approximation* $f(x+\Delta x, y+\Delta y) \approx f(x,y) + f_x(x,y)\cdot dx + f_y(x,y)\cdot dy$.

   (a) *Approximate* $\sqrt{8.01^2 + 14.97^2}$.

let

$$\begin{aligned}\Delta z &= f(x+\Delta x, y+\Delta y) - f(x,y) \\ &= \sqrt{(x+\Delta x)^2 + (y+\Delta y)^2} - \sqrt{x^2+y^2} \\ &= \sqrt{8.01^2 + 14.97^2} - \sqrt{8^2+15^2} \\ &= \sqrt{8.01^2 + 14.97^2} - \sqrt{17^2} \\ &= \sqrt{8.01^2 + 14.97^2} - 17\end{aligned}$$

where we take advantage of the *pythagorean triple* $8^2 + 15^2 = 17^2$ $(64 + 225 = 289)$ relationship

implying $f(x,y) =$ (i) $\boldsymbol{\sqrt{x^2+y^2}}$ (ii) $\boldsymbol{x^2+y^2}$ (iii) $\boldsymbol{x+y}$

and $x =$ (i) **8** (ii) **15** (iii) **17**
and $\Delta x = dx = 8.01 - 8 =$ (i) **0.01** (ii) **0.1** (iii) $\mathbf{-0.01}$

and $y =$ (i) **8** (ii) **15** (iii) **17**
and $\Delta y = dy = 14.97 - 15 =$ (i) **0.03** (ii) $\mathbf{-0.03}$ (iii) **0.01**

so

$$\begin{aligned}\sqrt{8.01^2 + 14.97^2} &= \Delta z + 17 \\ &\approx dz + 17 \quad \text{since } dz \approx \Delta z \\ &= f_x(x,y)\cdot dx + f_y(x,y)\cdot dy + 17\end{aligned}$$

for $f_x(x,y)$, let $f[g(x)] = (x^2+y^2)^{\frac{1}{2}}$
with "inner" function $g(x,y) =$ (i) $\boldsymbol{x^2+y^2}$ (ii) $\boldsymbol{3x}$ (iii) $\boldsymbol{3x+x^2}$
and "outer" function $f(x,y) =$ (i) $\boldsymbol{x^{\frac{1}{2}}}$ (ii) $\boldsymbol{x^{\frac{3}{2}}}$ (iii) $\boldsymbol{x^{\frac{5}{2}}}$
with derivative $g_x(x,y) =$ (i) **3** (ii) $\boldsymbol{2y}$ (iii) $\boldsymbol{2x}$
and derivative $f_x(x,y) =$ (i) $\boldsymbol{\frac{3}{2}x^{\frac{1}{2}}}$ (ii) $\boldsymbol{\frac{3}{2}x^{\frac{3}{2}}}$ (iii) $\boldsymbol{\frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}}$
and so by chain rule

$$\begin{aligned}f_x(x,y) &= f_x[g(x,y)]\cdot g_x(x,y) = f_x[x^2+y^2]\cdot(2x) \\ &= \frac{1}{2\sqrt{x^2+y^2}}(2x) =\end{aligned}$$

(i) $\boldsymbol{\frac{y}{\sqrt{x^2+y^2}}}$ (ii) $\boldsymbol{\frac{x}{\sqrt{x^2+y^2}}}$ (iii) $\boldsymbol{\frac{3x}{3\sqrt{3x+y^2}}}$

for $f_y(x,y)$, also let $f[g(x)] = (x^2+y^2)^{\frac{1}{2}}$

with "inner" function $g(x, y) =$ (i) $\boldsymbol{x^2 + y^2}$ (ii) $\boldsymbol{3x}$ (iii) $\boldsymbol{3x + x^2}$
and "outer" function $f(x, y) =$ (i) $\boldsymbol{y^{\frac{1}{2}}}$ (ii) $\boldsymbol{y^{\frac{3}{2}}}$ (iii) $\boldsymbol{y^{\frac{5}{2}}}$
with derivative $g_y(x, y) =$ (i) $\boldsymbol{3}$ (ii) $\boldsymbol{2y}$ (iii) $\boldsymbol{2x}$
and derivative $f_y(x, y) =$ (i) $\boldsymbol{\frac{3}{2}y^{\frac{1}{2}}}$ (ii) $\boldsymbol{\frac{3}{2}y^{\frac{3}{2}}}$ (iii) $\boldsymbol{\frac{1}{2}y^{-\frac{1}{2}} = \frac{1}{2\sqrt{y}}}$
and so by chain rule

$$\begin{aligned} f_y(x, y) &= f_y[g(x, y)] \cdot g_y(x, y) = f_y[x^2 + y^2] \cdot (2y) \\ &= \frac{1}{2\sqrt{3x + y^2}}(2y) = \end{aligned}$$

(i) $\boldsymbol{\frac{y}{\sqrt{x^2+y^2}}}$ (ii) $\boldsymbol{\frac{x}{\sqrt{x^2+y^2}}}$ (iii) $\boldsymbol{\frac{3x}{3\sqrt{3x+y^2}}}$

and so

$$\begin{aligned} \sqrt{8.01^2 + 14.97^2} &\approx f_x(x, y) \cdot dx + f_y(x, y) \cdot dy + 17 \\ &= \left(\frac{x}{\sqrt{x^2 + y^2}}\right) dx + \left(\frac{y}{\sqrt{x^2 + y^2}}\right) dy + 17 \\ &= \left(\frac{8}{\sqrt{(8)^2 + (15)^2}}\right)(0.01) + \left(\frac{15}{\sqrt{8^2 + 15^2}}\right)(-0.03) + 17 \\ &= \left(\frac{8}{17}\right)(0.01) + \left(\frac{15}{17}\right)(-0.03) + 17 \approx \end{aligned}$$

(i) **16.9782** (ii) **16.9822** (iii) **16.9932**

exact $\sqrt{8.01^2 + 14.97^2} \approx 16.9783$, so approximation minus exact: $16.9783 - 16.9782 = 0.0001$

(b) *Approximate* $\sqrt{7.01^2 + 24.01^2}$.

let

$$\begin{aligned} \Delta z &= f(x + \Delta x, y + \Delta y) - f(x, y) \\ &= \sqrt{(x + \Delta x)^2 + (y + \Delta y)^2} - \sqrt{x^2 + y^2} \\ &= \sqrt{7.01^2 + 24.01^2} - \sqrt{7^2 + 24^2} \\ &= \sqrt{7.01^2 + 24.01^2} - \sqrt{25^2} \\ &= \sqrt{7.01^2 + 24.01^2} - 25 \end{aligned}$$

where we take advantage of the pythagorean triple $7^2 + 24^2 = 25^2$ $(49 + 576 = 625)$ relationship

implying $f(x, y) =$ (i) $\boldsymbol{\sqrt{x^2 + y^2}}$ (ii) $\boldsymbol{x^2 + y^2}$ (iii) $\boldsymbol{x + y}$

and $x =$ (i) **7** (ii) **24** (iii) **25** and $\Delta x = dx = 7.01 - 7 =$ (i) $\mathbf{0.01}$ (ii) $\mathbf{0.1}$ (iii) $\mathbf{-0.01}$

and $y =$ (i) $\mathbf{7}$ (ii) $\mathbf{24}$ (iii) $\mathbf{25}$
and $\Delta y = dy = 24.01 - 24 =$ (i) $\mathbf{0.03}$ (ii) $\mathbf{-0.03}$ (iii) $\mathbf{0.01}$

so

$$\begin{aligned} \sqrt{7.01^2 + 24.01^2} &= \Delta z + 25 \\ &\approx dz + 25 \quad \text{since } dz \approx \Delta z \\ &= f_x(x,y) \cdot dx + f_y(x,y) \cdot dy + 25 \end{aligned}$$

for $f_x(x,y)$, let $f[g(x)] = (x^2 + y^2)^{\frac{1}{2}}$
with "inner" function $g(x,y) =$ (i) $\boldsymbol{x^2 + y^2}$ (ii) $\boldsymbol{3x}$ (iii) $\boldsymbol{3x + x^2}$
and "outer" function $f(x,y) =$ (i) $\boldsymbol{x^{\frac{1}{2}}}$ (ii) $\boldsymbol{x^{\frac{3}{2}}}$ (iii) $\boldsymbol{x^{\frac{5}{2}}}$
with derivative $g_x(x,y) =$ (i) $\mathbf{3}$ (ii) $\boldsymbol{2y}$ (iii) $\boldsymbol{2x}$
and derivative $f_x(x,y) =$ (i) $\boldsymbol{\frac{3}{2}x^{\frac{1}{2}}}$ (ii) $\boldsymbol{\frac{3}{2}x^{\frac{3}{2}}}$ (iii) $\boldsymbol{\frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}}$
and so by chain rule

$$\begin{aligned} f_x(x,y) &= f_x[g(x,y)] \cdot g_x(x,y) = f_x[x^2 + y^2] \cdot (2x) \\ &= \frac{1}{2\sqrt{x^2 + y^2}}(2x) = \end{aligned}$$

(i) $\boldsymbol{\frac{y}{\sqrt{x^2+y^2}}}$ (ii) $\boldsymbol{\frac{x}{\sqrt{x^2+y^2}}}$ (iii) $\boldsymbol{\frac{3x}{3\sqrt{3x+y^2}}}$

for $f_y(x,y)$, also let $f[g(x)] = (x^2 + y^2)^{\frac{1}{2}}$
with "inner" function $g(x,y) =$ (i) $\boldsymbol{x^2 + y^2}$ (ii) $\boldsymbol{3x}$ (iii) $\boldsymbol{3x + x^2}$
and "outer" function $f(x,y) =$ (i) $\boldsymbol{y^{\frac{1}{2}}}$ (ii) $\boldsymbol{y^{\frac{3}{2}}}$ (iii) $\boldsymbol{y^{\frac{5}{2}}}$
with derivative $g_y(x,y) =$ (i) $\mathbf{3}$ (ii) $\boldsymbol{2y}$ (iii) $\boldsymbol{2x}$
and derivative $f_y(x,y) =$ (i) $\boldsymbol{\frac{3}{2}y^{\frac{1}{2}}}$ (ii) $\boldsymbol{\frac{3}{2}y^{\frac{3}{2}}}$ (iii) $\boldsymbol{\frac{1}{2}y^{-\frac{1}{2}} = \frac{1}{2\sqrt{y}}}$
and so by chain rule

$$\begin{aligned} f_y(x,y) &= f_y[g(x,y)] \cdot g_y(x,y) = f_y[x^2 + y^2] \cdot (2y) \\ &= \frac{1}{2\sqrt{3x + y^2}}(2y) = \end{aligned}$$

(i) $\boldsymbol{\frac{y}{\sqrt{x^2+y^2}}}$ (ii) $\boldsymbol{\frac{x}{\sqrt{x^2+y^2}}}$ (iii) $\boldsymbol{\frac{3x}{3\sqrt{3x+y^2}}}$

and so

$$\begin{aligned} \sqrt{7.01^2 + 24.01^2} &\approx f_x(x,y) \cdot dx + f_y(x,y) \cdot dy + 17 \\ &= \left(\frac{x}{\sqrt{x^2 + y^2}}\right) dx + \left(\frac{y}{\sqrt{x^2 + y^2}}\right) dy + 17 \end{aligned}$$

$$= \left(\frac{7}{\sqrt{(7)^2+(24)^2}}\right)(0.01) + \left(\frac{24}{\sqrt{7^2+24^2}}\right)(0.01) + 25$$
$$= \left(\frac{7}{25}\right)(0.01) + \left(\frac{24}{25}\right)(0.01) + 25 =$$

(i) **25.0124** (ii) **25.0224** (iii) **25.0234**

exact $\sqrt{7.01^2+24.01^2} \approx 25.0124$, so approximation minus exact: $25.0124 - 25.01240092 = -0.0000009$

(c) *Approximate* $1.03 \ln 1.02$.

let

$$\begin{aligned} \Delta z &= f(x+\Delta x, y+\Delta y) - f(x,y) \\ &= (x+\Delta x)\ln(y+\Delta y) - x\ln y \\ &= 1.03\ln 1.02 - 1\ln 1 \\ &= 1.03\ln 1.02 \end{aligned}$$

where we take advantage of the fact $1 \ln 1 = 1 \cdot 0 = 0$

implying $f(x,y) =$ (i) $\boldsymbol{x \ln y}$ (ii) $\boldsymbol{xy}$ (iii) $\boldsymbol{\ln x + y}$

and $x =$ (i) **1** (ii) **1.02** (iii) **1.03**
and $\Delta x = dx = 1.02 - 1 =$ (i) **0.01** (ii) **0.02** (iii) **0.03**

and $y =$ (i) **1** (ii) **1.02** (iii) **1.03**
and $\Delta y = dy = 1.03 - 1 =$ (i) **0.01** (ii) **0.02** (iii) **0.03**

so

$$\begin{aligned} 1.03\ln 1.02 &= \Delta z \\ &\approx dz \quad \text{since } dz \approx \Delta z \\ &= f_x(x,y)\cdot dx + f_y(x,y)\cdot dy \end{aligned}$$

for $f_x(x,y)$, let $u(x,y) = x$ and $v(x,y) = \ln y$.
then, $u_x(x,y) =$ (i) **1** (ii) **0** (iii) $\mathbf{-1}$
and $v_x(x,y) =$ (i) **0** (ii) $\boldsymbol{\frac{1}{y}}$ (iii) $\boldsymbol{3+4x}$
and so

$$f_x(x,y) = v(x,y)\cdot u_x(x,y) + u(x,y)\cdot v_x(x,y) = (\ln x)(1) + (\ln y)(0) =$$

(i) $\boldsymbol{\ln x}$ (ii) $\boldsymbol{\frac{21y}{(5x+4y)^2}}$ (iii) $\boldsymbol{\frac{20}{(5x+4y)^2}}$

for $f_y(x,y)$, let $u(x,y) = x$ and $v(x,y) = \ln y$.
then, $u_y(x,y) =$ (i) $\mathbf{1}$ (ii) $\mathbf{0}$ (iii) $\mathbf{-1}$
and $v_y(x,y) =$ (i) $\mathbf{0}$ (ii) $\boldsymbol{\frac{1}{y}}$ (iii) $\mathbf{3 + 4x}$
and so

$$f_y(x,y) = v(x,y) \cdot u_y(x,y) + u(x,y) \cdot v_y(x,y) = (\ln y)(0) + (x)\left(\frac{1}{y}\right) =$$

(i) $\boldsymbol{\ln x}$ (ii) $\boldsymbol{\frac{x}{y}}$ (iii) $\boldsymbol{xy}$

and so

$$\begin{aligned} 1.03 \ln 1.02 &= f_x(x,y) \cdot dx + f_y(x,y) \cdot dy \\ &= (\ln x)\, dx + \left(\frac{x}{y}\right) dy \\ &= (\ln 1)\,(0.03) + \left(\frac{1}{1}\right)(0.02) \\ &= (0)(0.03) + 1(0.02) = \end{aligned}$$

(i) $\mathbf{0.02}$ (ii) $\mathbf{0.03}$ (iii) $\mathbf{0.04}$
exact $1.03 \ln 1.02 \approx 0.0204$, so approximation minus exact: $0.02 - 0.0204 = -0.0004$

3. *Application: temperature of flying bird*
   The temperature function for a bird in flight is given by

$$T(x,y,z) = 0.09x^2 + 1.4xy + 95z^2$$

Use differential $dT = T_x(x,y,z) \cdot dx + T_y(x,y,z) \cdot dy + T_z(x,y,z) \cdot dz$ to approximate change in temperature when head wind x increases from 1 meters per second to 2 meters per second, bird heart rate y increases from 50 beats per minute to 55 beats per minute and flapping rate z increases from 3 flaps per second to 4 flaps per second.

Since $T_x(x,y,z) = 0.09 \cdot 2x^{2-1} + 1.4x^{1-1}y + 0 =$
(i) $\mathbf{190z}$ (ii) $\mathbf{1.4x}$ (iii) $\mathbf{0.18x + 1.4y}$,

and $T_y(x,y,z) = 0 + 1.4xy^{1-1} + 0 =$
(i) $\mathbf{190z}$ (ii) $\mathbf{1.4x}$ (iii) $\mathbf{0.18x + 1.4y}$,

and $T_z(x,y,z) = 0 + 0 + 95 \cdot 2z^{2-1} =$
(i) $\mathbf{190z}$ (ii) $\mathbf{1.4x}$ (iii) $\mathbf{0.18x + 1.4y}$,

since x increases from 1 to 2,
$dx = 2 - 1 =$ (i) **5** (ii) **1** (iii) **−1**

and y increases from 50 to 55,
$dx = 55 - 50 =$ (i) **5** (ii) **1** (iii) **−1**

and z increases from 3 to 4,
$dx = 4 - 3 =$ (i) **5** (ii) **1** (iii) **−1**

and so

$$\begin{aligned} dT &= T_x(x,y,z)\cdot dx + T_y(x,y,z)\cdot dy + T_z(x,y,z)\cdot dz \\ &= (0.18x + 1.4y)dx + (1.4x)dy + (190z)dz \\ &= (0.18(1) + 1.4(50))(1) + (1.4(1))(5) + (190(3))(1) = \end{aligned}$$

(i) **647.08** (ii) **647.18** (iii) **647.28**

exact $\Delta T = T(2,55,4) - f(1,50,3) = \left[0.09(2)^2 + 1.4(2)(55) + 95(4)^2\right] - \left[0.09(1)^2 + 1.4(1)(50) + 95(3)^2\right] =$ 749.27, so approximation minus exact: $749.27 - 647.28 = 102.09$, large because $dx = 5$ large

9.6 Double Integrals

Double integral of $f(x,y)$ over rectangular region R in $a \le x \le b$, $c \le y \le d$,

$$\iint_R f(x,y)\,dy\,dx = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

where, notice, integrating first over y from c to d, then over x from a to b equals integrating first over x from a to b, then over y from c to d.

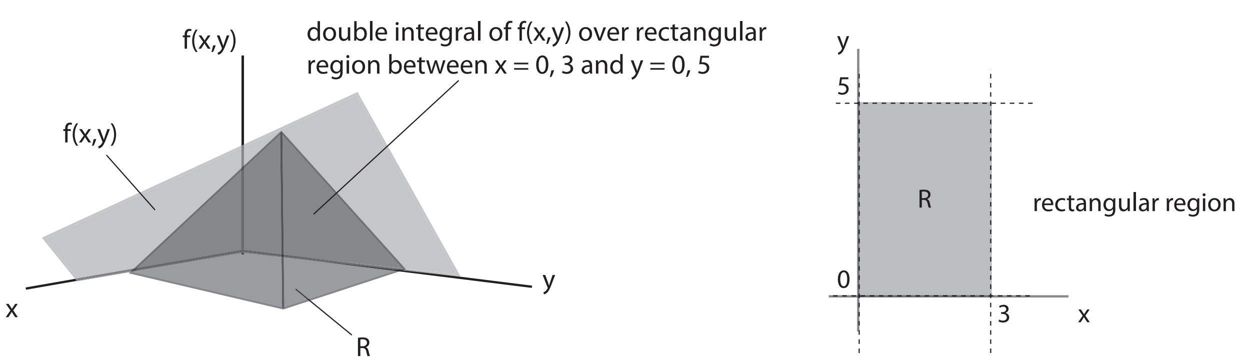


Figure 9.21 (double integration over rectangular region)

If $z = f(x, y)$ is never negative in rectangular region, double integral gives *volume* of under $f(x, y)$ over region R. Double integral of $f(x, y)$ over one type of *variable region* R where $a \leq x \leq b$, $g(x) \leq y \leq h(x)$ is

$$\int_a^b \int_{g(x)}^{h(x)} f(x, y)\, dy\, dx,$$

or double integral of $f(x, y)$ over variable region R where $g(y) \leq x \leq h(y)$, $c \leq y \leq d$

$$\int_c^d \int_{g(y)}^{h(y)} f(x, y)\, dx\, dy$$

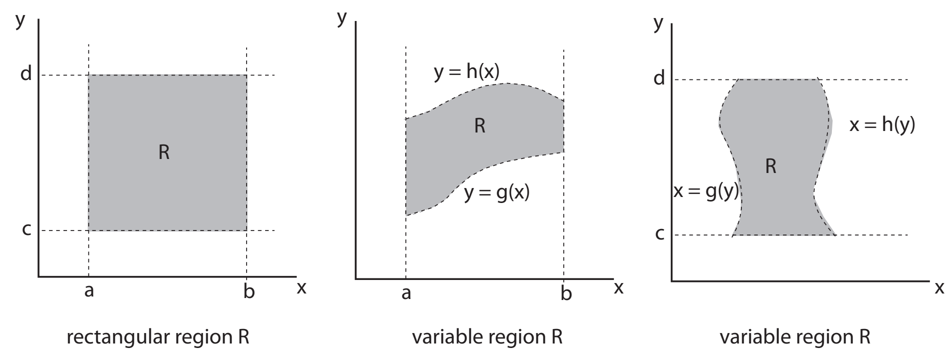


Figure 9.22 (Double integrals over rectangular and variable regions)

If a double integration with variable regions is difficult to solve one way, it is often possible to interchange limits of integration to make the integration easier, but care must be taken to change the limits of integration accordingly.

Exercise 9.6 (Double Integrals)

1. *Area* $\int\int_R 1\, dy\, dx = \int\int_R 1\, dx\, dy$ *of rectangular region* $0 \leq x \leq 3$, $0 \leq y \leq 5$

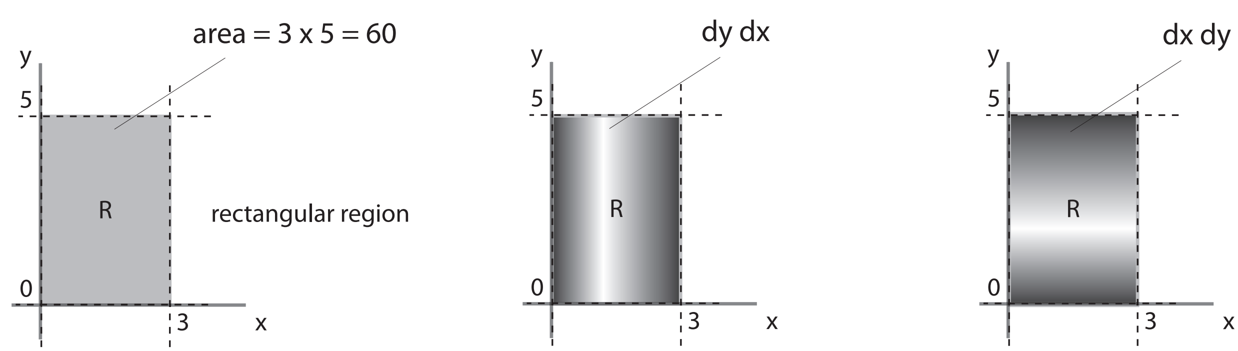


Figure 9.23 (Area of rectangle bounded by $0 \leq x \leq 3$, $0 \leq y \leq 5$)

Calculate area using order of integration $dy\,dx$,

$$\int_0^3 \int_0^5 1\,dy\,dx = \underbrace{\int_0^3 \underbrace{\left[\int_0^5 1\,dy\right]}_{\text{inner integral}} dx}_{\text{outer integral}}$$

first calculate inner integral over $0 \le y \le 5$

$$\begin{aligned} \int_0^5 dy &= \int_0^5 1\,dy \\ &= 1\int_0^5 y^0\,dy \\ &= 1\left(\frac{1}{0+1}y^{0+1}\right)_{y=0}^{y=5} \\ &= (y)_{y=0}^{y=5} \\ &= (5-0) = \end{aligned}$$

(i) **5** (ii) **0** (iii) **20**

then, using inner integral result, calculate outer integral over $0 \le x \le 3$

$$\begin{aligned} \int_0^3 \left[\int_0^5 dy\right] dx &= \int_0^3 [5]\,dx \\ &= 5\int_0^3 x^0\,dx \\ &= 5\left(\frac{1}{0+1}x^{0+1}\right)_{x=0}^{x=3} \\ &= 5(x)_{x=0}^{x=3} \\ &= 5\,(3-0) = \end{aligned}$$

(i) **15** (ii) **20** (iii) **25**

notice this result matches result in figure above, where area $= 3 \times 5 = 15$, or volume $= 1 \times 2 \times 5 = 15$

Reversing the order of integration from $dy\,dx$ to $dx\,dy$,

$$\int_0^5 \int_0^3 dx\,dy = \underbrace{\int_0^5 \underbrace{\left[\int_0^3 1\,dx\right]}_{\text{inner integral}} dy}_{\text{outer integral}}$$

first calculate inner integral over $0 \le x \le 3$

$$\int_0^3 dx = \int_0^3 1\,dx$$

$$\begin{aligned}
&= 1\int_0^3 x^0\,dx \\
&= 1\left(\frac{1}{0+1}x^{0+1}\right)_{x=0}^{x=3} \\
&= (x)_{x=0}^{x=3} \\
&= (3-0) =
\end{aligned}$$

(i) **3** (ii) **5** (iii) **7**

then, using inner integral result, calculate outer integral over $0 \le y \le 5$

$$\begin{aligned}
\int_0^5 \left[\int_0^3 dx\right] dy &= \int_0^5 [3]\,dy \\
&= 3\int_0^5 y^0\,dy \\
&= 3\left(\frac{1}{0+1}y^{0+1}\right)_{y=0}^{y=5} \\
&= 3(x)_{y=0}^{y=5} \\
&= 3\,(5-0) =
\end{aligned}$$

(i) **15** (ii) **20** (iii) **25**

notice this result matches previous result, which is not always true, but will be the case in this course

2. *Volume* $\int\int_R 4\,dy\,dx$ *on rectangular region* $0 \le x \le 3$, $0 \le y \le 5$

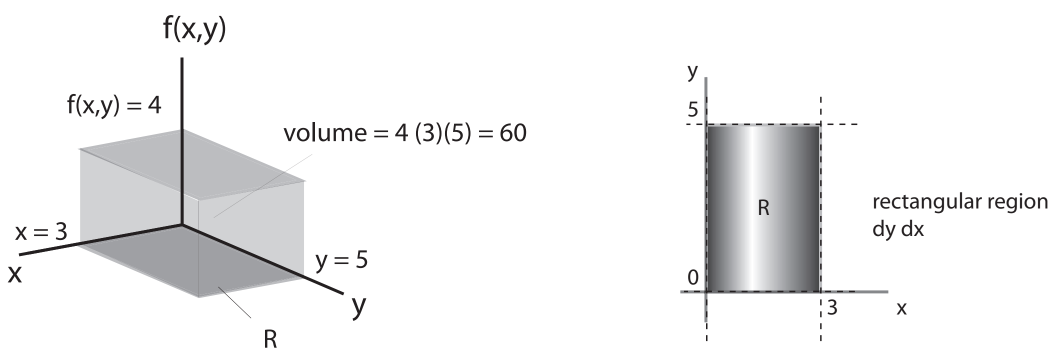


Figure 9.24 (Double integral $f(x,y) = 4$ over $0 \le x \le 3$, $0 \le y \le 5$)

calculate

$$\int_0^3 \int_0^5 4\,dy\,dx = \underbrace{\int_0^3 \underbrace{\left[\int_0^5 4\,dy\right]}_{\text{inner integral}} dx}_{\text{outer integral}}$$

first calculate inner integral over $0 \leq y \leq 5$

$$\begin{aligned} \int_0^5 4\,dy &= 4\int_0^5 y^0\,dy \\ &= 4\left(\frac{1}{0+1}y^{0+1}\right)_{y=0}^{y=5} \\ &= 4(y)_{y=0}^{y=5} \\ &= 4\,(5-0) = \end{aligned}$$

(i) **5** (ii) **0** (iii) **20**

then, using inner integral result, calculate outer integral over $0 \leq x \leq 3$

$$\begin{aligned} \int_0^3 \left[\int_0^5 4\,dy\right] dx &= \int_0^3 [20]\;dx \\ &= 20\int_0^3 x^0\,dx \\ &= 20\left(\frac{1}{0+1}x^{0+1}\right)_{x=0}^{x=3} \\ &= 20(x)_{x=0}^{x=3} \\ &= 20\,(3-0) = \end{aligned}$$

(i) **70** (ii) **60** (iii) **80**

notice this result matches result in figure above, where volume = $3 \times 5 \times 4 = 60$

3. $\int\int_R 4\,dy\,dx$ *on rectangular region* $0 \leq x \leq 3$, $0 \leq y \leq 5$

calculate

$$\int_0^5 \int_0^3 4\,dx\,dy = \underbrace{\int_0^5 \underbrace{\left[\int_0^3 4\,dx\right]}_{\text{inner integral}} dy}_{\text{outer integral}}$$

where, notice, although similar to previous question, inner and outer integrals have been switched

first calculate inner integral over $0 \leq x \leq 3$

$$\begin{aligned} \int_0^3 4\,dx &= 4\int_0^3 x^0\,dx \\ &= 4\left(\frac{1}{0+1}x^{0+1}\right)_{x=0}^{x=3} \\ &= 4(x)_{x=0}^{y=3} \\ &= 4\,(3-0) = \end{aligned}$$

(i) **12** (ii) **13** (iii) **14**

then, using inner integral result, calculate outer integral over $0 \leq y \leq 5$

$$\begin{aligned}
\int_0^5 \left[\int_0^3 4\,dx\right] dy &= \int_0^4 [12]\;dy \\
&= 12\int_0^4 y^0\,dy \\
&= 12\left(\frac{1}{0+1}y^{0+1}\right)_{y=0}^{y=5} \\
&= 12(y)_{y=0}^{y=5} \\
&= 12\,(5-0) =
\end{aligned}$$

(i) **70** (ii) **60** (iii) **80**

notice this result, $\int_0^3 \int_0^5 4\,dy\,dx = 60$, matches previous result, $\int_0^5 \int_0^3 4\,dx\,dy = 60$; although both required about the same amount of work to solve, sometimes one is easier to solve than the other

4. $\int_0^3 \int_0^5 (3x+4y)\,dy\,dx$

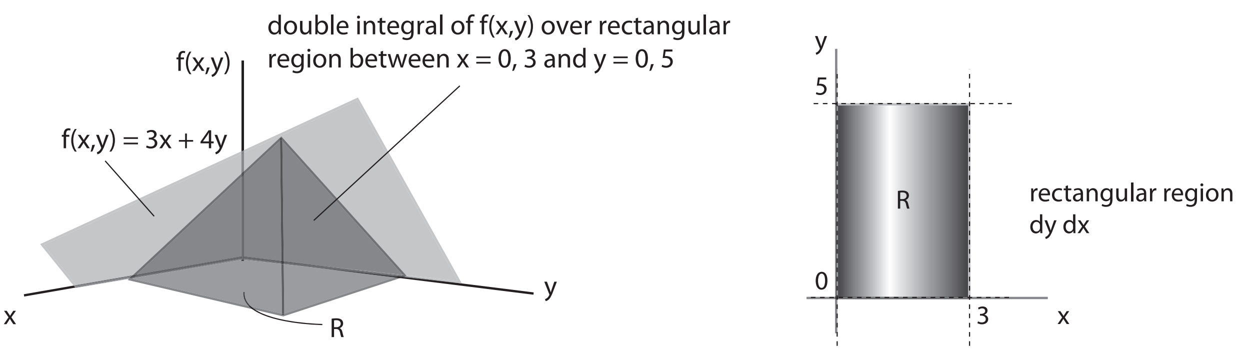


Figure 9.25 ($\int_0^3 \int_0^5 (3x+4y)\,dy\,dx$)

calculate inner integral over $0 \leq y \leq 5$

$$\begin{aligned}
\int_0^3 \int_0^5 (3x+4y)\,dy\,dx &= \int_0^3 \left[\int_0^5 (3x+4y)\,dy\right] dx \\
&= \int_0^3 \left[\int_0^5 (3xy^0+4y^1)\,dy\right] dx \\
&= \int_0^3 \left[\left(3x\cdot\frac{1}{0+1}y^{0+1} + 4\cdot\frac{1}{1+1}y^{1+1}\right)_{y=0}^{y=5}\right] dx \\
&= \int_0^3 \left[\left(3xy+2y^2\right)_{y=0}^{y=5}\right] dx \\
&= \int_0^3 \left[\left(3x(5)+2(5)^2\right) - \left(3x(0)+4(0)^2\right)\right] dx =
\end{aligned}$$

(i) $\boldsymbol{\int_0^3 (15y + 50)\, dx}$ (ii) $\boldsymbol{\int_0^3 15x\, dx}$ (iii) $\boldsymbol{\int_0^3 (15x + 50)\, dx}$

notice, unlike previous questions, inner integration was solved "inside" outer integral, not separately as before

continuing, calculate outer integral over $0 \leq x \leq 3$

$$\begin{aligned}
\int_0^3 \int_0^5 (3x+4y)\, dy\, dx &= \int_0^3 (15x+50)\, dx \\
&= \left(15 \cdot \frac{1}{1+1} x^{1+1} + 50 \cdot \frac{1}{0+1} x^{0+1}\right)_{x=0}^{x=3} \\
&= \left(\frac{15}{2} x^2 + 50x\right)_{x=0}^{x=3} \\
&= \left(\frac{15}{2}(3)^2 + 50(3)\right) - \left(\frac{9}{2}(0)^2 + 36(0)\right) =
\end{aligned}$$

(i) **217.5** (ii) **218.5** (iii) **219.5**

5. $\int_0^5 \int_0^3 (3x + 4y)\, dy\, dx$

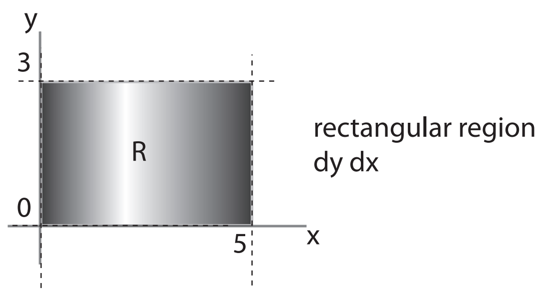


Figure 9.26 (region of integration: $0 \leq x \leq 5$, $0 \leq y \leq 3$)

calculate inner integral over $0 \leq y \leq 3$

$$\begin{aligned}
\int_0^5 \int_0^3 (3x+4y)\, dy\, dx &= \int_0^5 \left[\int_0^3 (3x+4y)\, dy\right] dx \\
&= \int_0^5 \left[\int_0^3 (3xy^0+4y^1)\, dy\right] dx \\
&= \int_0^5 \left[\left(3x \cdot \frac{1}{0+1} y^{0+1} + 4 \cdot \frac{1}{1+1} y^{1+1}\right)_{y=0}^{y=3}\right] dx \\
&= \int_0^5 \left[\left(3xy + 2y^2\right)_{y=0}^{y=3}\right] dx \\
&= \int_0^5 \left[\left(3x(3) + 2(3)^2\right) - \left(3x(0) + 4(0)^2\right)\right] dx =
\end{aligned}$$

(i) $\boldsymbol{\int_0^5 9x\, dx}$ (ii) $\boldsymbol{\int_0^5 (9y + 18)\, dx}$ (iii) $\boldsymbol{\int_0^5 (9x + 18)\, dx}$

continuing, calculate outer integral over $0 \leq x \leq 5$

$$\begin{aligned}
\int_0^5 \int_0^3 (3x + 4y)\, dy\, dx &= \int_0^5 (9x + 18)\, dx \\
&= \left(9 \cdot \frac{1}{1+1} x^{1+1} + 18 \cdot \frac{1}{0+1} x^{0+1} \right)_{x=0}^{x=5} \\
&= \left(\frac{9}{2} x^2 + 18x \right)_{x=0}^{x=5} \\
&= \left(\frac{9}{2}(5)^2 + 18(5) \right) - \left(\frac{9}{2}(0)^2 + 36(0) \right) =
\end{aligned}$$

(i) **202.5** (ii) **203.5** (iii) **204.5**

notice $\int_0^5 \int_0^3 (3x+4y)\, dy\, dx = 202.5$, here, does *not* equal previous result, $\int_0^3 \int_0^5 (3x+4y)\, dy\, dx = 217.5$. Why?

6. $\int_1^5 \int_2^3 (3x + 4y)\, dy\, dx$

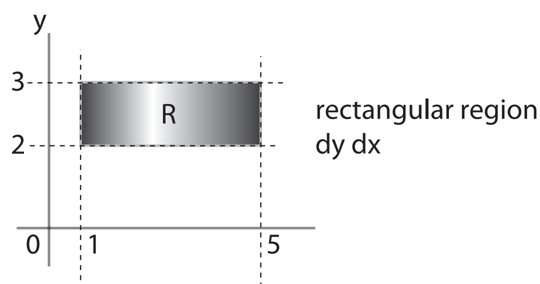


Figure 9.27 (region of integration: $1 \leq x \leq 5$, $2 \leq y \leq 3$,)

calculate inner integral over $2 \leq y \leq 3$

$$\begin{aligned}
\int_1^5 \int_2^3 (3x + 4y)\, dy\, dx &= \int_1^5 \left[\int_2^3 (3x + 4y)\, dy \right] dx \\
&= \int_1^5 \left[\int_2^3 (3xy^0 + 4y^1)\, dy \right] dx \\
&= \int_1^5 \left[\left(3x \cdot \frac{1}{0+1} y^{0+1} + 4 \cdot \frac{1}{1+1} y^{1+1} \right)_{y=2}^{y=3} \right] dx \\
&= \int_1^5 \left[\left(3xy + 2y^2 \right)_{y=2}^{y=3} \right] dx \\
&= \int_1^5 \left[\left(3x(3) + 2(3)^2 \right) - \left(3x(2) + 4(2)^2 \right) \right] dx =
\end{aligned}$$

(i) $\boldsymbol{\int_1^5 (3y + 2)\, dx}$ (ii) $\boldsymbol{\int_1^5 (3x + 2)\, dx}$ (iii) $\boldsymbol{\int_1^5 (9x + 18)\, dx}$

continuing, calculate outer integral over $1 \leq x \leq 5$

$$\begin{aligned}
\int_1^5 \int_2^3 (3x + 4y)\, dy\, dx &= \int_1^5 (3x + 2)\, dx \\
&= \left(3 \cdot \frac{1}{1+1} x^{1+1} + 2 \cdot \frac{1}{0+1} x^{0+1}\right)_{x=1}^{x=5} \\
&= \left(\frac{3}{2} x^2 + 2x\right)_{x=1}^{x=5} \\
&= \left(\frac{3}{2}(5)^2 + 2(5)\right) - \left(\frac{3}{2}(1)^2 + 2(1)\right) =
\end{aligned}$$

(i) **45** (ii) **46** (iii) **44**

7. $\int_0^5 \int_0^3 3x\, dy\, dx$

calculate inner integral over $0 \leq y \leq 3$

$$\begin{aligned}
\int_0^5 \int_0^3 3x\, dy\, dx &= \int_0^5 \left[\int_0^3 3x\, dy\right] dx \\
&= \int_0^5 \left[\int_0^3 3xy^0\, dy\right] dx \\
&= \int_0^5 \left[3x \left(\frac{1}{0+1} y^{0+1}\right)_{y=0}^{y=3}\right] dx \quad \text{3x is constant with respect to } y \\
&= \int_0^5 \left[3x\left((3) - (0)\right)\right] dx =
\end{aligned}$$

(i) $\int_0^5 \boldsymbol{9x\, dx}$ (ii) $\int_0^5 \boldsymbol{6x\, dx}$ (iii) $\int_0^5 \boldsymbol{9y\, dx}$

continuing, calculate outer integral over $0 \leq x \leq 5$

$$\begin{aligned}
\int_0^5 \int_0^3 3x\, dy\, dx &= \int_0^5 9x\, dx \\
&= 9\left(\frac{1}{1+1} x^{1+1}\right)_{x=0}^{x=5} \\
&= 9\left(\frac{1}{2}(5)^2 - \frac{1}{2}(0)^2\right) =
\end{aligned}$$

(i) **111.5** (ii) **112.5** (iii) **113.5**

Since $f(x, y) = 3x$ is never negative over region $0 \leq x \leq 5$ and $0 \leq y \leq 3$, value of double integral is a *volume* in this case.

8. $\int_0^3 \int_0^4 (x^2 + y^2)\, dy\, dx$

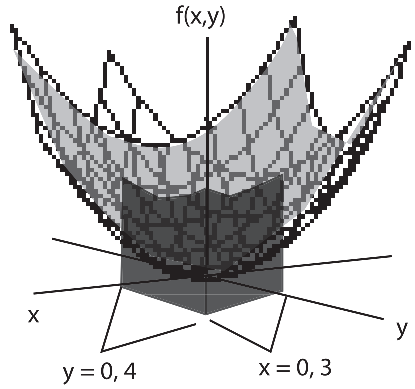


Figure 9.28 ($\int_0^3 \int_0^4 (x^2+y^2)\,dy\,dx$)

calculate inner integral over $0 \leq y \leq 4$

$$\begin{aligned}
\int_0^3 \int_0^4 (x^2+y^2)\,dy\,dx &= \int_0^3 \left[\int_0^4 (x^2+y^2)\,dy\right] dx \\
&= \int_0^3 \left[\int_0^4 (x^2y^0+y^2)\,dy\right] dx \\
&= \int_0^3 \left[\left(x^2 \cdot \frac{1}{0+1}y^{0+1} + \frac{1}{2+1}y^{2+1}\right)_{y=0}^{y=4}\right] dx \\
&= \int_0^3 \left[\left(x^2y + \frac{1}{3}y^3\right)_{y=0}^{y=4}\right] dx \\
&= \int_0^3 \left[\left(x^2(4) + \frac{1}{3}(4)^3\right) - \left(x^2(0) + \frac{1}{3}(0)^3\right)\right] dx =
\end{aligned}$$

(i) $\boldsymbol{\int_0^3 \left(4x^2 + \frac{64}{3}\right) dx}$ (ii) $\boldsymbol{\int_0^3 \left(2x^2 + \frac{64}{3}\right) dx}$ (iii) $\boldsymbol{\int_0^3 (9x^2 + 64)\, dx}$

continuing, calculate outer integral over $0 \leq x \leq 3$

$$\begin{aligned}
\int_0^3 \int_0^4 (x^2+y^2)\,dy\,dx &= \int_0^3 \left(4x^2 + \frac{64}{3}\right) dx \\
&= \left(4 \cdot \frac{1}{2+1}x^{2+1} + \frac{64}{3} \cdot \frac{1}{0+1}x^{0+1}\right)_{x=0}^{x=3} \\
&= \left(\frac{4}{3}x^3 + \frac{64}{3}x\right)_{x=0}^{x=3} \\
&= \left(\frac{4}{3}(3)^3 + \frac{64}{3}(3)\right) - \left(\frac{4}{3}(0)^3 + \frac{64}{3}(0)\right) =
\end{aligned}$$

(i) **99** (ii) **100** (iii) **101**

9. $\int_0^1 \int_0^1 (2x^2 - 2y^2)\,dy\,dx$

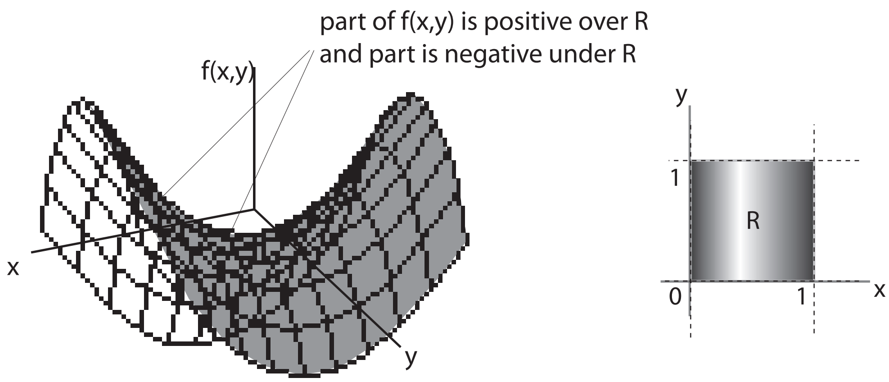


Figure 9.29 ($\int_0^1 \int_0^1 (2x^2 - 2y^2)\, dy\, dx$)

calculate inner integral over $0 \leq y \leq 1$

$$\begin{aligned}
\int_0^1 \int_0^1 (2x^2 - 2y^2)\, dy\, dx &= \int_0^1 \left[\int_0^1 (2x^2 - 2y^2)\, dy\right] dx \\
&= \int_0^1 \left[\int_0^1 (2x^2y^0 - 2y^2)\, dy\right] dx \\
&= \int_0^1 \left[\left(2x^2 \cdot \frac{1}{0+1} y^{0+1} - 2 \cdot \frac{1}{2+1} y^{2+1}\right)_{y=0}^{y=1}\right] dx \\
&= \int_0^1 \left[\left(2x^2y - \frac{2}{3}y^3\right)_{y=0}^{y=1}\right] dx \\
&= \int_0^1 \left[\left(2x^2(1) - \frac{2}{3}(1)^3\right) - \left(2x^2(0) + \frac{2}{3}(0)^3\right)\right] dx =
\end{aligned}$$

(i) $\boldsymbol{\int_0^1 \left(4x^2 + \frac{2}{3}\right) dx}$ (ii) $\boldsymbol{\int_0^1 \left(2x^2 - \frac{2}{3}\right) dx}$ (iii) $\boldsymbol{\int_0^1 \left(2x^2 + \frac{2}{3}\right) dx}$

continuing, calculate outer integral over $0 \leq x \leq 1$

$$\begin{aligned}
\int_0^1 \int_0^1 (2x^2 - 2y^2)\, dy\, dx &= \int_0^1 \left(2x^2 - \frac{2}{3}\right) dx \\
&= \left(2 \cdot \frac{1}{2+1} x^{2+1} - \frac{2}{3} \cdot \frac{1}{0+1} x^{0+1}\right)_{x=0}^{x=1} \\
&= \left(\frac{2}{3}x^3 - \frac{2}{3}x\right)_{x=0}^{x=1} \\
&= \left(\frac{2}{3}(1)^3 - \frac{2}{3}(1)\right) - \left(\frac{2}{3}(0)^3 - \frac{2}{3}(0)\right) =
\end{aligned}$$

(i) $\mathbf{-1}$ (ii) $\mathbf{0}$ (iii) $\mathbf{1}$

Negative part of integral below $f(x, y)$ exactly cancels positive integral below surface, so total integral is zero.

10. $\int_0^1 \int_0^1 e^{3x+4y}\, dx\, dy$ (notice dx and dy have been reversed)

calculate inner integral over $0 \leq x \leq 1$

$$\begin{aligned}
\int_0^1 \int_0^1 e^{3x+4y}\, dx\, dy &= \int_0^1 \left[\int_0^1 e^{3x+4y}\, dx\right] dy \\
&= \int_0^1 \left[\int_0^1 e^{3x+4y} \cdot \frac{1}{3} \cdot 3\, dx\right] dy \\
&= \int_0^1 \left[\int e^u \frac{1}{3}\, du\right] dy \quad \text{let } u = 3x + 4y, \text{ so } du_x = 3dx \\
&= \int_0^1 \left[\left(\frac{1}{3} e^u\right)\right] dy \quad \text{recall } \int e^u\, du = e^u \\
&= \int_0^1 \left[\left(\frac{1}{3} e^{3x+4y}\right)_{x=0}^{x=1}\right] dy \quad \text{because } u = 3x + 4y \\
&= \int_0^1 \left[\left(\frac{1}{3} e^{3(1)+4y}\right) - \left(\frac{1}{3} e^{3(0)+4y}\right)\right] dy =
\end{aligned}$$

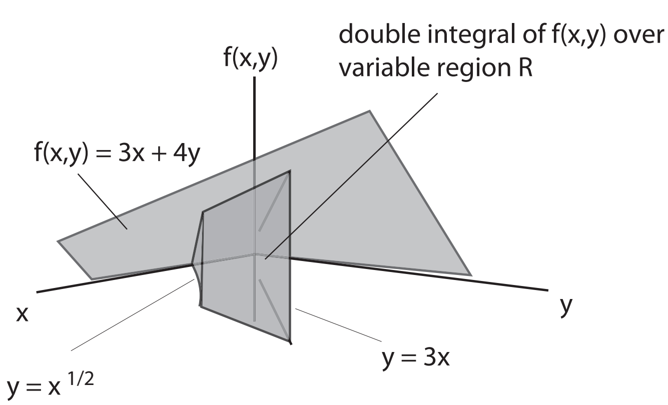
(i) $\frac{1}{3}\int_0^1 \left(e^{4y+3} + e^{4y}\right) dy$ (ii) $\frac{1}{3}\int_0^1 \left(e^{4y+3}\right) dy$ (iii) $\frac{1}{3}\int_0^1 \left(e^{4y+3} - e^{4y}\right) dy$

continuing, calculate outer integral over $0 \leq y \leq 1$

$$\begin{aligned}
\int_0^1 \int_0^1 e^{3x+4y}\, dx\, dy &= \frac{1}{3}\int_0^1 \left(e^{4y+3} - e^{4y}\right) dy \\
&= \frac{1}{3}\left(\frac{1}{4} \cdot e^{4y+3} - \frac{1}{4} \cdot e^{4y}\right)_{y=0}^{y=1} \quad \text{following steps similar to above} \\
&= \frac{1}{3} \cdot \frac{1}{4}\left(e^{4y+3} - e^{4y}\right)_{y=0}^{y=1} \\
&= \frac{1}{12}\left[\left(e^{4(1)+3} - e^{4(1)}\right) - \left(e^{4(0)+3} - e^{4(0)}\right)\right] =
\end{aligned}$$

(i) $\frac{1}{12}\left[e^7 - e^4 - e^3 + 1\right]$ (ii) $\frac{1}{12}\left[e^7 - e^4\right]$ (iii) $\frac{1}{12}\left[-e^3 + 1\right]$

11. $\int_1^3 \int_{\sqrt{x}}^{3x} (3x + 4y)\, dy\, dx$ (notice *variable* limits of integration for y)



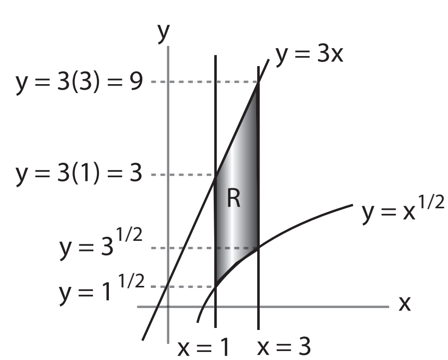


Figure 9.30 ($\int_1^3 \int_{\sqrt{x}}^{3x} (3x+4y)\, dy\, dx$)

calculate inner integral over $\sqrt{x} \leq y \leq 3x$

$$\begin{aligned}
\int_1^3 \int_{\sqrt{x}}^{3x} (3x+4y)\, dy\, dx &= \int_1^3 \left[\int_{\sqrt{x}}^{3x} (3x+4y)\, dy \right] dx \\
&= \int_1^3 \left[\int_{\sqrt{x}}^{3x} (3xy^0+4y^1)\, dy \right] dx \\
&= \int_1^3 \left[\left(3x \cdot \frac{1}{0+1} y^{0+1} + 4 \cdot \frac{1}{1+1} y^{1+1} \right)_{y=\sqrt{x}}^{y=3x} \right] dx \\
&= \int_1^3 \left[\left(3xy + 2y^2 \right)_{y=\sqrt{x}}^{y=3x} \right] dx \\
&= \int_1^3 \left[\left(3x(3x) + 2(3x)^2 \right) - \left(3x(\sqrt{x}) + 4(\sqrt{x})^2 \right) \right] dx =
\end{aligned}$$

(i) $\boldsymbol{\int_1^3 (9x^2 - 4x)\, dx}$ (ii) $\boldsymbol{\int_1^3 (-3x^{\frac{3}{2}} - 4x)\, dx}$ (iii) $\boldsymbol{\int_1^3 (27x^2 - 3x^{\frac{3}{2}} - 4x)\, dx}$

continuing, calculate outer integral over $1 \leq x \leq 3$

$$\begin{aligned}
\int_1^3 \int_{\sqrt{x}}^{3x} (3x+4y)\, dy\, dx &= \int_1^3 (27x^2 - 3x^{\frac{3}{2}} - 4x)\, dx \\
&= \left(27 \cdot \frac{1}{2+1} x^{2+1} - 3 \cdot \frac{1}{\frac{3}{2}+1} x^{\frac{3}{2}+1} - 4 \cdot \frac{1}{0+1} x^{0+1} \right)_{x=1}^{x=3} \\
&= \left(\frac{27}{3} x^3 - \frac{6}{5} x^{\frac{5}{2}} - 4x \right)_{x=1}^{x=3} \\
&= \left(\frac{27}{3}(3)^3 - \frac{6}{5}(3)^{\frac{5}{2}} - 4(3) \right) - \left(\frac{27}{3}(1)^3 - \frac{6}{5}(1)^{\frac{5}{2}} - 4(1) \right) \approx
\end{aligned}$$

(i) **218.5** (ii) **228.5** (iii) **208.5**

12. $\int \int_R \frac{1}{y}\, dy\, dx$ *bounded by variable region* $y = x$, $y = \frac{1}{x}$, $x = 2$

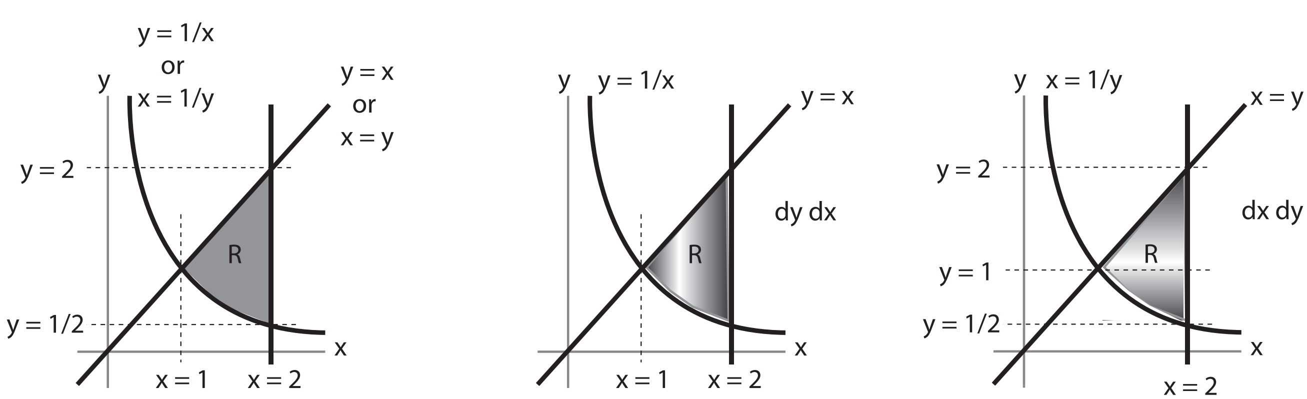


Figure 9.31 ($\int\int_R \frac{1}{y}\, dy\, dx$, $y = x$, $y = \frac{1}{x}, x = 2$)

A possible way of writing this question is (choose *two*!)

(i) $\boldsymbol{\int_1^2 \int_{\frac{1}{x}}^{x} \frac{1}{y}\, dy\, dx}$ (ii) $\boldsymbol{\int_{\frac{1}{y}}^{y} \int_{\frac{1}{2}}^{2} \frac{1}{y}\, dy\, dx}$ (iii) $\boldsymbol{\int_{\frac{1}{2}}^{1} \int_{\frac{1}{y}}^{2} \frac{1}{y}\, dx\, dy + \int_1^2 \int_y^2 \frac{1}{y}\, dx\, dy}$

so, on the one hand,

$$\begin{aligned}\int_1^2 \int_{\frac{1}{x}}^{x} \frac{1}{y}\, dy\, dx &= \int_1^2 \left[\int_{\frac{1}{x}}^{x} \frac{1}{y}\, dy\right] dx \\ &= \int_1^2 \left[(\ln|y|)_{y=\frac{1}{x}}^{y=x}\right] dx \\ &= \int_1^2 \left[\ln x - \ln \frac{1}{x}\right] dx =\end{aligned}$$

(i) $\boldsymbol{\int_1^2 \ln x^2\, dx}$ (ii) $\boldsymbol{\int_1^2 \ln x^{-2}\, dx}$ (iii) $\boldsymbol{\int_1^2 \ln x\, dx}$

recall, $\ln x - \ln \frac{1}{x} = \ln \frac{x}{\frac{1}{x}} = \ln x^2$

continuing, calculate outer integral over $1 \leq x \leq 2$

$$\begin{aligned}\int_1^2 \int_{\frac{1}{x}}^{x} \frac{1}{y}\, dy\, dx &= \int_1^2 \ln x^2\, dx \quad \text{solve using integration by parts, shown below} \\ &= \left(x \ln x^2 - 2x\right)_{x=1}^{x=2} \\ &= \left((2)\ln(2)^2 - 2(2)\right) - \left((1)\ln(1)^2 - 2(1)\right) =\end{aligned}$$

(i) $\mathbf{2 \ln 4 - 2}$ (ii) $\mathbf{4 \ln 4 - 2}$ (iii) $\mathbf{4 \ln 2 - 2}$

integration by parts: let

$$\int \ln x^2\, dx = \int u\, dv$$

so

$$\begin{array}{ll} u = \ln x^2 & v = x \\ du = \frac{1}{x^2} \cdot 2x = \frac{2}{x}\, dx \text{ (chain rule)} & dv = dx \end{array}$$

and so

$$uv - \int v\, du = (\ln x^2)(x) - \int x \cdot \frac{2}{x}\, dx = x \ln x^2 - \int 2\, dx = x \ln x^2 - 2x$$

and, on the other hand,

$$\int_{\frac{1}{2}}^{1} \int_{\frac{1}{y}}^{2} \frac{1}{y}\, dx\, dy + \int_1^2 \int_y^2 \frac{1}{y}\, dx\, dy = \int_{\frac{1}{2}}^{1} \left[\left(\frac{x}{y}\right)_{x=\frac{1}{y}}^{x=2}\right] dy + \int_1^2 \left[\left(\frac{x}{y}\right)_{x=y}^{x=2}\right] dy$$

$$\begin{aligned} &= \int_{\frac{1}{2}}^{1} \left[\frac{2}{y} - \frac{1}{y^2}\right] dy + \int_{1}^{2} \left[\frac{2}{y} - \frac{y}{y}\right] dy \\ &= \left(2\ln y - \frac{1}{-2+1} y^{-1}\right)_{y=\frac{1}{2}}^{y=1} + (2\ln y - y)_{y=1}^{y=2} \\ &= \left(2\ln 1 + \frac{1}{1}\right) - \left(\frac{1}{2}\ln\frac{1}{2} + \frac{1}{1/2}\right) \\ &\quad + (2\ln 2 - 2) - (2\ln 1 - 1) = \end{aligned}$$

(i) $\mathbf{2\ln 4 - 2}$ (ii) $\mathbf{4\ln 4 - 2}$ (iii) $\mathbf{4\ln 2 - 2}$

which is the same answer as before, where, recall $2\ln 2 - 2\ln\frac{1}{2} = 2\ln\frac{2}{\frac{1}{2}} = 2\ln 4$