Chapter 13

The Trigonometric Functions

We look at trigonometric functions and their derivatives.

13.1 Definitions of the Trigonometric Functions

Consider an angle with origin (vertex) at the origin of the coordinate system and two *rays* where the *initial side* ray is along the x-axis and *terminal side* ray is at the end of a rotation of angle θ. *Acute*, *right*, *obtuse* and *straight* angles occur when $0^o < \theta < 90^o$, $\theta = 90^o$, $90^o < \theta < 180^o$ and $\theta = 180^o$ respectively. For *radius*, r and *arc (length)*, s, of a circle, *radian measure* of θ is defined as $\frac{s}{r}$; where, notice, if radius of circle is one (1), a *unit* circle, radian measure equals arc length s. An angle can be measured in either *degrees* or *radians*, where

$$1 \text{ radian} \;=\frac{180^o}{\pi}, \qquad 1^o \;= \frac{\pi}{180} \text{ radians.}$$

Let r be distance from origin to point (x, y) on terminal side ray. Then

$$\begin{aligned}
\sin\theta &= \frac{y}{r} & \csc\theta &= \frac{r}{y}, \quad (y \neq 0) \\
\cos\theta &= \frac{x}{r} & \sec\theta &= \frac{r}{x}, \quad (x \neq 0) \\
\tan\theta &= \frac{y}{x}, \quad (x \neq 0) & \cot\theta &= \frac{x}{y}, \quad (y \neq 0)
\end{aligned}$$

where notation t or x can be used instead of θ; for example, $\sin t$ or $\sin x$ could be used instead of $\sin\theta$. Also, some trigonometric identities are:

$$\begin{aligned}
\csc\theta &= \frac{1}{\sin\theta} & \sec\theta &= \frac{1}{\cos\theta} & \cot\theta &= \frac{1}{\tan\theta} \\
\tan\theta &= \frac{\sin\theta}{\cos\theta} & \cot\theta &= \frac{\cos\theta}{\sin\theta} & \sin^2\theta + \cos^2\theta &= 1
\end{aligned}$$

Values for trigonometric functions are typically found using a calculator but some values can be found for triangles with special angles given in the Figure. For example, for $30^o - 60^o - 90^o$ triangle, where $60^o = 60 \cdot \left(\frac{\pi}{180}\right) = \frac{\pi}{3}$, so $\sin 60^o = \sin\frac{\pi}{3} = \frac{y}{r} = \frac{\sqrt{3}}{2}$.

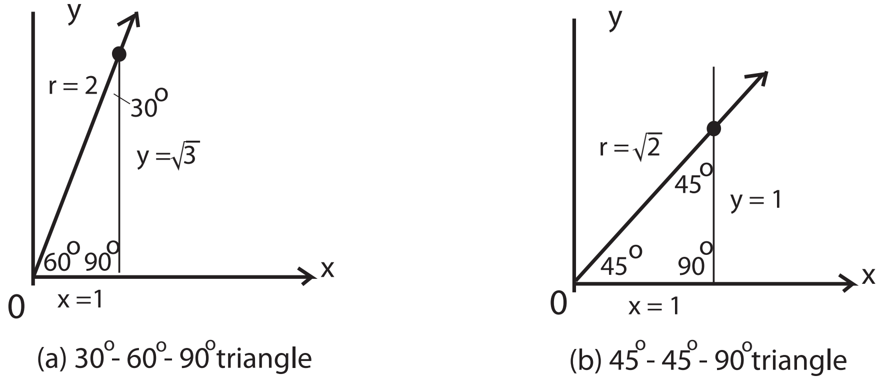

Figure 13.1 (Trigonometric functions and special angles)

A trigonometric function is *periodic* because it is a function $y = f(x)$ with real number a such that $f(x) = f(x + a)$ for all x; *smallest* a, when the function repeats itself, is the *period* of the function. Periods of both $\sin x$ and $\cos x$ are 2π; their *amplitudes* (half their range ("height") from -1 to 1) are both 1. Furthermore, constants a, b, c, d transform graphs of both $a\sin(bx + c) + d$ and $a\cos(bx + c) + d$ in the following ways:

- amplitude a increases (decreases) "height" of graph for $|a|$ large (small) when $a < 0$, graphs reflected in x-axis ("flipped"),
- constant b (assume $b > 0$) affects period; for example, graph of $y = \sin(bx)$ looks like $y = \sin x$ but with period $T = \frac{2\pi}{b}$ if $0 < b < 1$, period completed more slowly (longer period) than when $b = 1$ if $b > 1$, period completed more rapidly (shorter period) than when $b = 1$
- horizontal shift c moves graphs *left* ($c > 0$) or *right* ($c < 0$)
- vertical shift d moves graphs up ($d > 0$) or down ($d < 0$)

Phase shift, $\frac{c}{b}$, gives number of units $\sin bx$ or $\cos bx$ are shifted horizontally; for example, if $c = 2\pi$, $b = 1$, then $\sin(x + 2\pi)$ is $\frac{c}{b} = \frac{2\pi}{1} = 2\pi$ units *left* of $\sin x$, whereas if $c = 2\pi$, $b = 2$, then $\sin(2x + 2\pi)$ is $\frac{2\pi}{2} = \pi$ units *left* of $\sin 2x$.

Exercise 13.1 (Definitions of the Trigonometric Functions)

1. *Introduction to angles, radians and trigonometric functions.*

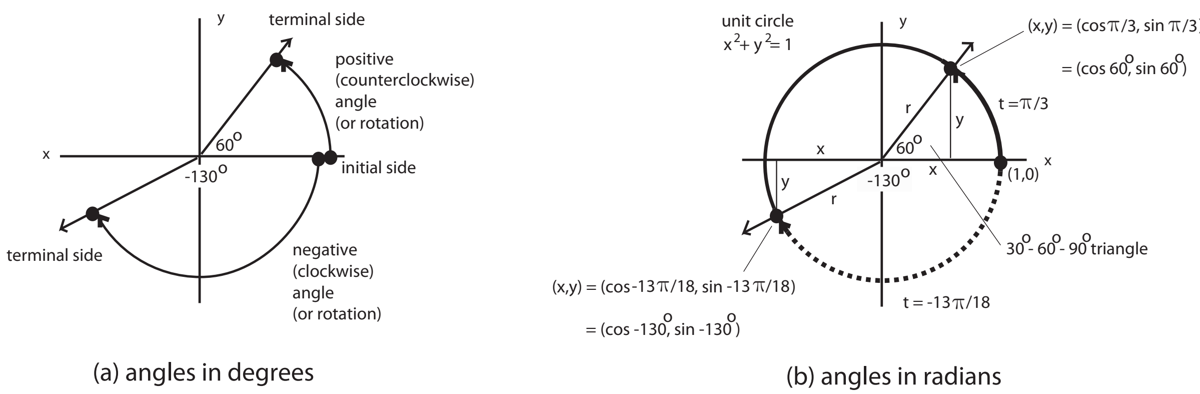

Figure 13.2 (Angles, radians and trigonometric functions)

(a) (i) **True** (ii) **False**.
Origin vertex, and two rays, initial side and terminal side, form an *angle*.

(b) Consider Figure (a). Rotating ray is called (i) **terminal side** (ii) **initial side**, whereas positive half of the x–axis is called *initial side*.

(c) Counterclockwise rotation of 60^o is (i) **positive** (ii) **negative** rotation, whereas clockwise rotation of -130^o is a negative rotation.

(d) Angle of 90^o equals (i) $\mathbf{-270^o}$ (ii) $\mathbf{270^o}$ (iii) $\mathbf{-180^o}$.
Hint: $270 - 180 = 90$.

(e) 60^o *angle in figure (b).*

i. Angle 60^o traces *arc length* (i) $-\frac{\mathbf{13\pi}}{\mathbf{18}}$ (ii) $\frac{\boldsymbol{\pi}}{\mathbf{3}}$ radians along *unit* circle.

ii. Horizontal distance $x = \cos 60^o =$ (i) **0.5** (ii) **0.87**
Calculator: MODE DEGREE ENTER COS 60 ENTER

iii. Vertical distance $y = \sin 60^o \approx$ (i) **0.5** (ii) **0.87**
Calculator: SIN 60 ENTER

iv. Horizontal distance $x = \cos\frac{\pi}{3} =$ (i) **0.5** (ii) **0.87**
Calculator: MODE RADIAN ENTER COS 2nd $\pi/3$ ENTER

v. Vertical distance $x = \sin\frac{\pi}{3} =$ (i) **0.5** (ii) **0.87**
Calculator: SIN 2nd $\pi/3$ ENTER

vi. (i) **True** (ii) **False** Since $\cos 60^o = \cos\frac{\pi}{3}$ and $\sin 60^o = \sin\frac{\pi}{3}$ point (x, y) can be determined by evaluating trigonometric functions using *either* degrees or radians.

vii. Since this is $30^o - 60^o - 90^o$ triangle where $x = 1$, $y = \sqrt{3}$ and $r = 2$,
$\cos 60^o = \cos\frac{\pi}{3} = \frac{x}{r} = \frac{1}{2} =$ (i) **0.87** (ii) **0.5** see special triangle figure
$\sin 60^o = \sin\frac{\pi}{3} = \frac{y}{r} = \frac{\sqrt{3}}{2} \approx$ (i) **0.87** (ii) **0.5** see special triangle figure
Notice $30^o - 60^o - 90^o$ triangle assumes radius $r = 2$, not 1, as given in figure (b).
If radius was specified as 1, $r = \frac{2}{2} = 1$, then $x = \frac{1}{2}$ and $y = \frac{\sqrt{3}}{2}$.

(f) -130^o *angle in figure (b).*

i. Angle -130^o traces arc length (i) $\boldsymbol{\frac{\pi}{3}}$ (ii) $\boldsymbol{-\frac{13\pi}{18}}$ radians.

ii. Horizontal distance $x = \cos(-130^o) =$ (i) $\mathbf{-0.64}$ (ii) $\mathbf{-0.77}$

Calculator: MODE DEGREE ENTER COS -130 ENTER

iii. Vertical distance $y = \sin(-130^o) \approx$ (i) $\mathbf{-0.64}$ (ii) $\mathbf{-0.77}$

Calculator: SIN -130 ENTER

iv. Horizontal distance $x = \cos\left(-\frac{13\pi}{18}\right) =$ (i) $\mathbf{-0.64}$ (ii) $\mathbf{-0.77}$

Calculator: MODE RADIAN ENTER COS 2nd $-\frac{13\pi}{18}$ ENTER

v. Vertical distance $x = \sin\left(-\frac{13\pi}{18}\right) =$ (i) $\mathbf{-0.64}$ (ii) $\mathbf{-0.77}$

Calculator: SIN 2nd $-\frac{13\pi}{18}$ ENTER

vi. (i) **True** (ii) **False**

Since $\cos(-130^o) = \cos\left(-\frac{13\pi}{18}\right)$ and $\sin(-130^o) = \sin\left(-\frac{13\pi}{18}\right)$ point (x, y) can be determined by evaluating trigonometric functions using *either* degrees or radians.

vii. (i) **True** (ii) **False**

This case does *not* involve either the $30^o - 60^o - 90^o$ triangle or the $45^o - 45^o - 90^o$ triangle and so neither can be used to determine the $\cos\theta$ or $\sin\theta$ functions unlike for the 60^o case.

2. *Converting Degrees To Radians.*

$$1^o = \frac{\pi}{180} \text{ radians}$$

(a) Angle of 180^o equivalent to $180 \times \frac{\pi}{180} =$ (i) $\boldsymbol{\frac{\pi}{2}}$ (ii) $\boldsymbol{\pi}$ (iii) $\boldsymbol{\frac{3\pi}{2}}$ radians.

(b) Angle of 90^o equivalent to $90 \times \frac{\pi}{180} =$ (i) $\boldsymbol{\frac{\pi}{2}}$ (ii) $\boldsymbol{\pi}$ (ii) $\boldsymbol{\frac{3\pi}{2}}$ radians.

(c) Angle of -90^o equivalent to $-90 \times \frac{\pi}{180} =$ (i) $\boldsymbol{-\frac{\pi}{2}}$ (ii) $\boldsymbol{-\pi}$ (iii) $\boldsymbol{-\frac{3\pi}{2}}$

(d) Angle of 60^o equivalent to $60 \times \frac{\pi}{180} =$ (one or more!)
(i) $\boldsymbol{\left(\frac{60^o}{180^o}\right)\pi}$ (ii) $\boldsymbol{\frac{\pi}{3}}$ (iii) $\mathbf{0.33}\boldsymbol{\pi}$ (iv) $\mathbf{1.047}$ radians.

(e) Angle of 75^o equivalent to $75 \times \frac{\pi}{180} =$ (one or more!)
(i) $\boldsymbol{\left(\frac{75^o}{180^o}\right)\pi}$ (ii) $\mathbf{0.42}\boldsymbol{\pi}$ (iii) $\mathbf{1.309}$ radians.

(f) An angle of 106^o equivalent to $106 \times \frac{\pi}{180} =$ (one or more!)
(i) $\boldsymbol{\left(\frac{106^o}{180^o}\right)\pi}$ (ii) $\mathbf{0.59}\boldsymbol{\pi}$ (iii) $\mathbf{1.85}$ radians.

(g) An angle of 166^o equivalent to $166 \times \frac{\pi}{180} =$ (one or more!)
(i) $\boldsymbol{\left(\frac{166^o}{180^o}\right)\pi}$ (ii) $\mathbf{0.92}\boldsymbol{\pi}$ (iii) $\mathbf{2.90}$ radians.

(h) An angle of -466^o equivalent to $-466 \times \frac{\pi}{180} =$ (one or more!)
(i) $\boldsymbol{\left(\frac{-466^o}{180^o}\right)\pi}$ (ii) $\mathbf{-2.59}\boldsymbol{\pi}$ (iii) $\mathbf{-8.13}$ radians.

3. *Converting Radians To Degrees.*

$$1 \text{ radian} = \frac{180^o}{\pi}$$

(a) Radian π equivalent to $\pi \times \frac{180}{\pi}$ = (i) $\mathbf{90^o}$ (ii) $\mathbf{180^o}$ (iii) $\mathbf{360^o}$ degrees.

(b) Radian $\frac{3\pi}{2}$ equivalent to $\frac{3\pi}{2} \times \frac{180}{\pi}$ = (i) $\mathbf{90^o}$ (ii) $\mathbf{270^o}$ (iii) $\mathbf{360^o}$

(c) Radian $-\frac{3\pi}{2}$ equivalent to $-\frac{3\pi}{2} \times \frac{180}{\pi}$ =
(i) $\mathbf{-90^o}$ (ii) $\mathbf{-270^o}$ (iii) $\mathbf{-360^o}$ degrees.

(d) Radian 2π equivalent to $2\pi \times \frac{180}{\pi}$ = (one or more)
(i) $\left(\frac{\mathbf{2\pi}}{\boldsymbol{\pi}}\right) \mathbf{180^o}$ (ii) $\mathbf{2(180^o)}$ (iii) $\mathbf{360^o}$ degrees.

(e) Radian $\frac{5\pi}{9}$ equivalent to $\frac{5\pi}{9} \times \frac{180}{\pi}$ = (one or more)
(i) $\left(\frac{\mathbf{5\pi/9}}{\boldsymbol{\pi}}\right) \mathbf{180^o}$ (ii) $\mathbf{\frac{5}{9}(180^o)}$ (iii) $\mathbf{100^o}$ degrees.

(f) Radian $\frac{7\pi}{2}$ equivalent to $\frac{7\pi}{2} \times \frac{180}{\pi}$ = (one or more)
(i) $\left(\frac{\mathbf{7\pi/2}}{\boldsymbol{\pi}}\right) \mathbf{180^o}$ (ii) $\mathbf{\frac{7}{2}(180^o)}$ (iii) $\mathbf{630^o}$ degrees.

(g) Radian of 1.3 equivalent to $1.3 \times \frac{180}{\pi}$ = (one or more)
(i) $\left(\frac{\mathbf{1.3}}{\boldsymbol{\pi}}\right) \mathbf{180^o}$ (ii) $\mathbf{(0.41)(180^o)}$ (iii) $\mathbf{74.5^o}$ degrees.

4. *Evaluating trigonometric functions using the definitions.*

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y}, \quad (y \neq 0)$$
$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x}, \quad (x \neq 0)$$
$$\tan\theta = \frac{y}{x}, \quad (x \neq 0) \qquad \cot\theta = \frac{x}{y}, \quad (y \neq 0)$$

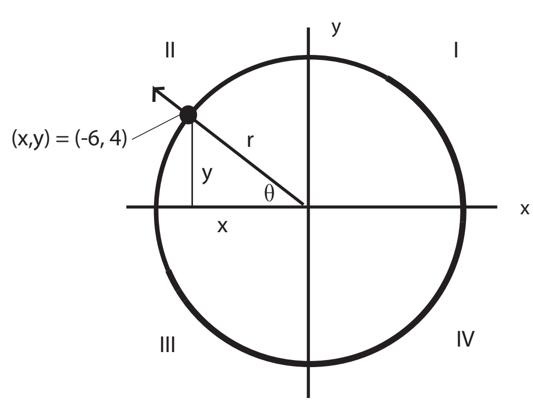

Figure 13.3 (Evaluating trigonometric functions at $(x, y) = (-6, 4)$)

(a) Since $x = -6$, $y = 4$,
$r = \sqrt{x^2 + y^2} = \sqrt{(-6)^2 + 4^2} \approx$ (i) **7.211** (ii) **8.211** (iii) **9.211**

(b) $\sin\theta = \frac{y}{r} \approx \frac{4}{7.211} \approx$ (i) $\mathbf{-0.83}$ (ii) $\mathbf{0.55}$ (iii) $\mathbf{-1.50}$
$\sin\theta$ is (i) **positive** (ii) **negative**

(c) $\cos\theta = \frac{x}{r} \approx \frac{-6}{7.211} \approx$ (i) $\mathbf{0.55}$ (ii) $\mathbf{-0.83}$ (iii) $\mathbf{-1.50}$
$\cos\theta$ is (i) **positive** (ii) **negative**

(d) $\tan\theta = \frac{y}{x} \approx \frac{4}{-6} \approx$ (i) $\mathbf{0.55}$ (ii) $\mathbf{-0.83}$ (iii) $\mathbf{-0.67}$
$\tan\theta$ is (i) **positive** (ii) **negative**

(e) $\csc\theta = \frac{r}{y} \approx \frac{7.211}{4} \approx$ (i) $\mathbf{-0.83}$ (ii) $\mathbf{-1.50}$ (iii) $\mathbf{1.80}$
$\csc\theta$ is (i) **positive** (ii) **negative**

(f) $\sec\theta = \frac{r}{x} \approx \frac{7.211}{-6} \approx$ (i) $\mathbf{-1.20}$ (ii) $\mathbf{0.55}$ (iii) $\mathbf{-1.50}$
$\sec\theta$ is (i) **positive** (ii) **negative**

(g) $\cot\theta = \frac{x}{y} \approx \frac{-6}{4} =$ (i) $\mathbf{0.55}$ (ii) $\mathbf{-1.50}$ (iii) $\mathbf{-0.83}$
$\cot\theta$ is (i) **positive** (ii) **negative**

(h) *Where is* $(x, y) = (-6, 4)$*?*
Point $(x, y) = (-6, 4)$ is in quadrant (i) **I** (ii) **II** (iii) **III** (iv) **IV**

(i) *What is* θ*?*
$\theta = \sin^{-1}\left(\frac{y}{r}\right) \approx \sin^{-1}\left(\frac{4}{7.211}\right) \approx$
(i) $\mathbf{-43.31}$ (ii) $\mathbf{-53.31}$ (iii) $\mathbf{-33.31}$ degrees
MODE DEGREE ENTER 2nd SIN^{-1} 4 / 7.211 ENTER

or $\theta = \tan^{-1}\left(\frac{y}{x}\right) \approx \tan^{-1}\left(\frac{-4}{6}\right) \approx$
(i) $\mathbf{-33.31}$ (ii) $\mathbf{-43.31}$ (iii) $\mathbf{-53.31}$ degrees
2nd TAN^{-1} -4 / 6 ENTER
Any of the other trigonometric functions such as $\cos^{-1}$, $\csc^{-1}$ could also be used as well.

5. *Evaluating an inverse trigonometric function.*

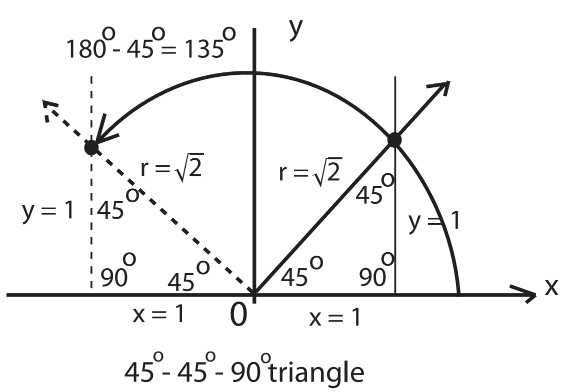

Figure 13.4 (Evaluating inverse trigonometric functions

Find all θ between 0 and 2π where $\sin\theta = \frac{1}{\sqrt{2}}$

(a) *Solve using special angles.*

Since $\sin\theta = \frac{y}{r} = \frac{1}{\sqrt{2}}$, so $y = 1$ and $r = \sqrt{2}$, this is a
(i) $\mathbf{30^o - 60^o - 90^o}$ **triangle**
(ii) $\mathbf{45^o - 45^o - 90^o}$ **triangle**

also, since both $y = 1$ and, of course, radius $r = \sqrt{2}$ are positive, this means θ could only be in *two* quadrants:
(i) **I** (ii) **II** (iii) **III** (iv) **IV**
Look at the figure.

Specifically $\theta =$ (choose two!)
(i) $\mathbf{45^o}$ (ii) $\mathbf{135^o}$ $180^o - 45^o = 135^o$ (iii) $\mathbf{90^o}$

(b) *Solve using calculator.*

since $\sin\theta = \frac{y}{r} = \frac{1}{\sqrt{2}}$; that is, since both $y = 1$ and $r = \sqrt{2}$ are positive, this means θ could only be in *two* quadrants:
(i) **I** (ii) **II** (iii) **III** (iv) **IV**
Look at the figure.

since $\sin\theta = \frac{1}{\sqrt{2}}$, $\theta = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) =$ (i) $\mathbf{45^o}$ (ii) $\mathbf{135^o}$ (iii) $\mathbf{90^o}$
MODE DEGREES ENTER 2nd SIN^{-1} $\frac{1}{\sqrt{2}}$

but θ could also be in quadrant II, so
$\theta =$ (i) $\mathbf{45^o}$ (ii) $\mathbf{135^o}$ $180^o - 45^o = 135^o$ (iii) $\mathbf{90^o}$

6. *Transforming graphs of* $\cos t$ *and* $\sin t$. Constants a, b, c, d transform graphs of both $a\sin(bx + c) + d$ and $a\cos(bx + c) + d$ in the following ways:

- amplitude a increases (decreases) "height" of graph for $|a|$ large (small) when $a < 0$, graphs reflected in x-axis ("flipped"),
- constant b (assume $b > 0$) affects period;
  graph of $y = \sin(bx)$ looks like $y = \sin x$ but with period $T = \frac{2\pi}{b}$
  if $0 < b < 1$, period completed more rapidly (shorter period) than $b = 1$
  if $b > 1$, period completed more slowly (longer period) than when $b = 1$
- horizontal shift c moves graphs *left* ($c > 0$) or *right* ($c < 0$)
- vertical shift d moves graphs up ($d > 0$) or down ($d < 0$)

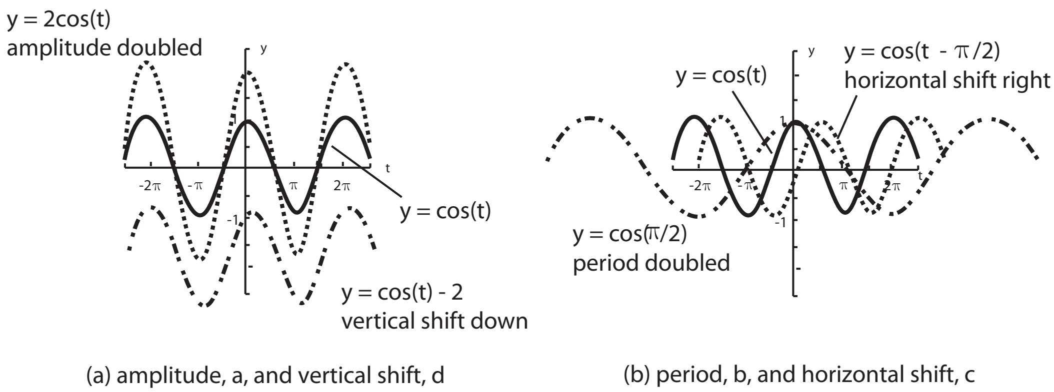

Figure 13.5 (Transforming graphs of $\cos t$ and $\sin t$)

(GRAPH using $Y_1 = \cos(X)$, $Y_2 = 2\cos(X)$, $Y_3 = \cos X - 2$, $Y_4 = \cos(X)$ and $Y_5 = \cos(X - \pi/2)$, $Y_6 = \cos(X/2)$, with WINDOW -4π 4π π −2 2 1)

(a) *Amplitude stretching or shrinking:* *changing a in either $a\sin(bx + c) + d$ or $a\cos(bx + c) + d$.*

i. In graph (a) of figure, *amplitude* of function $2\cos t$ is (i) **one–half** (ii) **two times** that of function $\cos t$.

ii. Amplitude of function $3\sin t$ is (i) **one–third** (ii) **three times** that of function $\sin t$.

iii. Amplitude of function $\frac{1}{5}\sin t$ is (i) **one–fifth** (ii) **five times** that of function $\sin t$.

(b) *Vertical shift up or down:* *changing d in either $a\sin(bx + c) + d$ or $a\cos(bx + c) + d$.*

i. In graph (a) of figure, function $\cos(t) - 2$ is a *vertical shift* of 2 units (i) **above** (ii) **below** function $\cos t$.

ii. Function $\sin(t) + \pi$ is a vertical shift of π units (i) **above** (ii) **below** function $\sin t$.

(c) *Horizontal shift left or right:* *changing c in either $a\sin(bx + c) + d$ or $a\cos(bx + c) + d$.*

i. In graph (b) of figure, function $\cos\left(t - \frac{\pi}{2}\right)$ is a *horizontal shift* of $\pi/2$ units to the (i) **right** (ii) **left** of function $\cos t$.
(Seems to shift "wrong way" to right, instead of to left. Check it out with your calculator.)

ii. Function $\sin\left(t + 3\frac{\pi}{2}\right)$ is a horizontal shift of $3\frac{\pi}{2}$ units to the (i) **right** (ii) **left** of function $\cos t$.

(d) *Period increased or decreased:* *changing b in either $a\sin(bx + c) + d$ or $a\cos(bx + c) + d$.*

i. In graph (b) of figure, *period* of $\cos \frac{t}{2}$ is
(i) **one–half** (ii) **two times** that of function $\cos t$.
(Seems to go wrong way, increases, instead of decreases. Think about it: t in $\cos \frac{t}{2}$ has to be twice as large as t in $\cos t$ "to go the same distance". Check it out with your calculator.)

ii. Period of function $\cos 2t$ is
(i) **one–half** (ii) **two times** that of function $\cos t$.

iii. Period of function $\sin 4t$ is
(i) **one–fourth** (ii) **four times** that of the function $\sin t$.

(e) *More on the period.*

i. (i) **True** (ii) **False**. Both $\sin t$ and $\cos t$ have a period of $T = 2\pi$; that is, they "repeat" themselves every 2π,

$$\begin{aligned} \sin(t + 2\pi) &= \sin t \\ \cos(t + 2\pi) &= \cos t \end{aligned}$$

ii. $\cos \pi =$ (one or more)
(i) $\boldsymbol{\cos 3\pi}$ (ii) $\boldsymbol{\cos(\pi + 2\pi)}$ (ii) $\boldsymbol{\cos(\pi + 4\pi)}$

iii. $\cos \frac{3\pi}{2} =$ (one or more)
(i) $\boldsymbol{\cos\left(\frac{3\pi}{2} + 2\pi\right)}$ (ii) $\boldsymbol{\cos\left(\frac{3\pi}{2} + 4\pi\right)}$ (iii) $\boldsymbol{\cos\left(\frac{3\pi}{2} + 6\pi\right)}$

iv. $\sin \frac{3\pi}{2} =$ (one or more)
(i) $\boldsymbol{\sin\left(\frac{3\pi}{2} + 2\pi\right)}$ (ii) $\boldsymbol{\sin\left(\frac{3\pi}{2} + 4\pi\right)}$ (iii) $\boldsymbol{\sin\left(\frac{3\pi}{2} + 5\pi\right)}$

v. $\sin 0 =$ (one or more) (i) $\boldsymbol{\cos(0 + 2\pi)}$ (ii) $\boldsymbol{\sin 2\pi}$ (iii) $\boldsymbol{\sin \frac{\pi}{2}}$

vi. Period of $\cos \frac{t}{2}$, where $b = \frac{1}{2}$ is $T = \frac{2\pi}{\frac{1}{2}} =$ (i) $\boldsymbol{\frac{1}{2}\pi}$ (ii) $\boldsymbol{2\pi}$ (iii) $\boldsymbol{4\pi}$

vii. Period $\cos \frac{t}{3}$, where $b = \frac{1}{3}$ is $T = \frac{2\pi}{\frac{1}{3}} =$ (i) $\boldsymbol{2\pi}$ (ii) $\boldsymbol{4\pi}$ (iii) $\boldsymbol{6\pi}$

viii. Period of $\cos \frac{t}{4}$, where $b = \frac{1}{4}$ is $T = \frac{2\pi}{\frac{1}{4}} =$ (i) $\boldsymbol{2\pi}$ (ii) $\boldsymbol{4\pi}$ (iii) $\boldsymbol{8\pi}$

ix. Period of $\cos 2t$, where $b = 2$ is $T = \frac{2\pi}{2} =$ (i) $\boldsymbol{\pi}$ (ii) $\boldsymbol{\frac{1}{2}\pi}$ (iii) $\boldsymbol{\frac{1}{4}\pi}$

x. Period of $\sin \frac{3t}{2}$, where $b = \frac{3}{2}$ is $T = \frac{2\pi}{\frac{3}{2}} =$ (i) $\boldsymbol{\frac{2}{3}\pi}$ (ii) $\boldsymbol{\frac{4}{3}\pi}$ (iii) $\boldsymbol{\frac{2}{5}\pi}$

(f) *Mixing it up: $a \sin(bt + c) + d$ and $a \cos(bt + c) + d$.*

i. Function $3 \sin\left(\frac{t}{2} + \pi\right) + 2$ is/has
amplitude 3 which is (i) **three times** (ii) **one–third** that of $\sin t$
(i) **above** (ii) **below** $\sin t$ by 2 units
shifted π units to the (i) **right** (ii) **left** of $\sin t$
a period 4π which is (i) **two times** (ii) **one–half** that of $\sin t$

ii. Function $\frac{1}{2} \cos\left(\frac{3t}{2} - 2\right) - \pi$ is/has
amplitude $\frac{1}{2}$ which is (i) **twice** (ii) **one–half** amplitude of $\sin t$
(i) **above** (ii) **below** $\sin t$ by π units
shifted 2 units to the (i) **right** (ii) **left** of $\cos t$
period $\frac{2\pi}{\frac{3}{2}} = \frac{4\pi}{3}$ (i) **two–thirds** (ii) **three-halves** that of $\cos t$

13.2 Derivatives of Trigonometric Functions

Trigonometric identities:

$$\sin^2 x + \cos^2 x = 1, \quad \tan x = \frac{\sin x}{\cos x}$$

Sum–difference identities

$$\begin{aligned}
\cos(x+y) &= \cos x \cos y - \sin x \sin y \\
\cos(x-y) &= \cos x \cos y + \sin x \sin y \\
\sin(x+y) &= \sin x \cos y + \cos x \sin y \\
\sin(x-y) &= \sin x \cos y - \cos x \sin y
\end{aligned}$$

Derivatives of trigonometric functions include:

$$\begin{array}{rl}
D_x[\sin x] = \cos x & \quad D_x[\csc x] = -\cot x \csc x \\
D_x[\cos x] = -\sin x & \quad D_x[\sec x] = \tan x \sec x \\
D_x[\tan x] = \sec^2 x & \quad D_x[\cot x] = -\csc^2 x
\end{array}$$

Exercise 13.2 (Derivatives of Trigonometric Functions)

1. *Trigonometric function and chain rule.* $y = \cos^2 x$.
   Let $f[g(x)] = \cos^2 x = (\cos x)^2$
   with inner function $g(x) = \cos x$ and outer function $f(x) = x^2$
   and $g'(x) =$ (i) $\boldsymbol{x^2}$ (ii) $\boldsymbol{-\sin^2 x}$ (iii) $\boldsymbol{-\sin x}$
   and $f'(x) =$ (i) $\boldsymbol{2x}$ (ii) $\boldsymbol{-\sin x}$ (iii) $\boldsymbol{x^3}$
   and so by chain rule

   $$f'[g(x)] \cdot g'(x) = f'[\cos x] \cdot (-\sin x) = 2(\cos x)(-\sin x) =$$

   (i) $\boldsymbol{-2\cos x \sin x}$ (ii) $\boldsymbol{2\cos^2 x}$ (iii) $\boldsymbol{-2\sin^2 x}$

2. *Trigonometric function and chain rule.* $y = \cot(3x^2 - x)$.
   Let $f[g(x)] = \cot(3x^2 - x)$, $g(x) = 3x^2 - x$ and $f(x) = \cot x$
   and $g'(x) =$ (i) $\boldsymbol{6x - 1}$ (ii) $\boldsymbol{3x^2}$ (iii) $\boldsymbol{-\csc^2 x}$
   and $f'(x) =$ (i) $\boldsymbol{-\sin x}$ (ii) $\boldsymbol{-\csc^2 x}$ (iii) $\boldsymbol{\cos x}$
   and so by chain rule

   $$f'[g(x)] \cdot g'(x) = f'[3x^2 - x] \cdot (6x - 1) =$$

   (i) $\boldsymbol{12\cot^3 x \csc^2 x}$ (ii) $\boldsymbol{-\csc^2(3x^2 - x)(6x - 1)}$ (iii) $\boldsymbol{3\csc^3 x}$

3. $y = \sec^4 x + 3\cot^3 x$

   First let $f[g(x)] = \sec^4 x$, $g(x) = \sec x$ and $f(x) = x^4$
   and $g'(x) =$ (i) $\boldsymbol{4x^3}$ (ii) $\boldsymbol{\tan x \sec x}$ (iii) $\boldsymbol{-\csc^2 x}$
   and $f'(x) =$ (i) $\boldsymbol{-\sin x}$ (ii) $\boldsymbol{\tan x \sec x}$ (iii) $\boldsymbol{4x^3}$
   and so by chain rule

   $$f'[g(x)] \cdot g'(x) = f'[\sec x] \cdot \tan x \sec x = 4(\sec x)^3 \cdot \tan x \sec x =$$

   (i) $\boldsymbol{12\cot^3 x \csc^2 x}$ (ii) $\boldsymbol{\tan^3 x \sec x}$ (iii) $\boldsymbol{4\tan x \sec^4 x}$

   then let $f[g(x)] = 3\cot^3 x$, $g(x) = \cot x$ and $f(x) = 3x^2$
   and $g'(x) =$ (i) $\boldsymbol{6x}$ (ii) $\boldsymbol{-\csc^2 x}$ (iii) $\boldsymbol{\csc^2 x}$
   and $f'(x) =$ (i) $\boldsymbol{6x}$ (ii) $\boldsymbol{-\csc^2 x}$ (iii) $\boldsymbol{\cos x}$
   and so by chain rule

   $$f'[g(x)] \cdot g'(x) = f'[\cot x] \cdot (-\csc^2 x) = 6(\cot x) \cdot (-\csc^2 x) =$$

   (i) $\boldsymbol{12\cot^3 x \csc^2 x}$ (ii) $\boldsymbol{-6(\cot x)(\csc^2 x)}$ (iii) $\boldsymbol{-\csc^2 (3x^2 - x)(6x - 1)}$

   and so $\frac{dy}{dx} =$
   (i) $\boldsymbol{4\tan x \sec^4 x - 6(\cot x)(\csc^2 x)}$
   (ii) $\boldsymbol{4\sec^4 x \tan x - 9\cot^2 x \csc^2 x}$
   (iii) $\boldsymbol{12\sec^2 x - 18\cot \csc^2 x}$

4. *Trigonometric functions and product rule.* $y = \csc x \cos x$.

   Let $u(x) = \csc x$ and $v(x) = \cos x$

   then $u'(x) =$
   (i) $\boldsymbol{-\cot x \csc x}$
   (ii) $\boldsymbol{-\sin x}$
   (iii) $\boldsymbol{\cot x \csc x}$

   and $v'(x) =$
   (i) $\boldsymbol{-\sin x}$
   (ii) $\boldsymbol{-\cot x \csc x}$
   (iii) $\boldsymbol{\tan x}$

   and so $v(x)u'(x) =$
   (i) $\boldsymbol{(\cos x)\,(-\csc x)}$
   (ii) $\boldsymbol{(\cos x)\,(-\cot x \csc x)}$
   (iii) $\boldsymbol{-\cot x \csc x}$

and $u(x)v'(x) =$
(i) $\boldsymbol{(\csc x)(-\tan x)}$
(ii) $\boldsymbol{(\csc x)(-\sin x)}$
(iii) $\boldsymbol{(\sec x)(-\sin x)}$

and so $f'(x) = v(x) \cdot u'(x) + u(x) \cdot v'(x) =$
(i) $\boldsymbol{(-\cot x \csc x) + (\csc x)(-\sin x)}$
(ii) $\boldsymbol{-\cos x \cot x \csc x - \csc x \sin x}$
(iii) $\boldsymbol{(\cos x)\,(-\cot x \csc x) - (\csc x)(-\sin x)}$

5. *Trigonometric functions and product rule.* $y = \sin(2x^3 - 4x)\cos^2 x$.

   Let $u(x) = \sin(2x^3 - 4x)$ and $v(x) = \cos^2 x$

   then $u'(x) = f'[g(x)] \cdot g'(x) =$
   (i) $\boldsymbol{-\sin(2x^3 - 4x)}$
   (ii) $\boldsymbol{\cot x \csc x}$
   (iii) $\boldsymbol{\cos(2x^3 - 4x) \cdot (6x - 4)}$

   and $v'(x) = f'[g(x)] \cdot g'(x) =$
   (i) $\boldsymbol{2(\cos x) \cdot (-\sin x)}$
   (ii) $\boldsymbol{-\cot x \csc x}$
   (iii) $\boldsymbol{\tan x}$

   and so $v(x)u'(x) =$
   (i) $\boldsymbol{(\cos x)\,(-\csc x)}$
   (ii) $\boldsymbol{-\cot x \csc x}$
   (iii) $\boldsymbol{(\cos^2 x)\,(\cos(2x^3 - 4x) \cdot (6x - 4))}$

   and $u(x)v'(x) =$
   (i) $\boldsymbol{(\sin(2x^3 - 4x))(2(\cos x) \cdot (-\sin x))}$
   (ii) $\boldsymbol{(\csc x)(-\tan x)}$
   (iii) $\boldsymbol{(\sec x)(-\sin x)}$

   and so $f'(x) = v(x) \cdot u'(x) + u(x) \cdot v'(x) =$
   (i) $\boldsymbol{(-\cot x \csc x) + (\csc x)(-\sin x)}$
   (ii) $\boldsymbol{(6x - 4)\cos^2 x \cos(2x^3 - 4x) - 2\sin(2x^3 - 4x)\cos x \sin x}$
   (iii) $\boldsymbol{\cos^2 x(6x^2 - 4) - (2x^2 - 4x)\sin x}$

6. *Trigonometric functions and quotient rule.* $y = \frac{\sin(2x^3-4x)}{\cos^2 x}$.

Let $u(x) = \sin(2x^3 - 4x)$ and $v(x) = \cos^2 x$

then $u'(x) = f'[g(x)] \cdot g'(x) =$
(i) $\boldsymbol{-\sin(2x^3 - 4x)}$
(ii) $\boldsymbol{\cos(2x^3 - 4x) \cdot (6x - 4)}$
(iii) $\boldsymbol{\cot x \csc x}$

and $v'(x) = f'[g(x)] \cdot g'(x) =$
(i) $\boldsymbol{-\cot x \csc x}$
(ii) $\boldsymbol{\tan x}$
(iii) $\boldsymbol{2(\cos x) \cdot (-\sin x)}$

and so $v(x)u'(x) =$
(i) $\boldsymbol{(\cos x)\,(-\csc x)}$
(ii) $\boldsymbol{(\cos^2 x)\,(\cos(2x^3 - 4x) \cdot (6x - 4))}$
(iii) $\boldsymbol{-\cot x \csc x}$

and $u(x)v'(x) =$
(i) $\boldsymbol{(\csc x)(-\tan x)}$
(ii) $\boldsymbol{(\sin(2x^3 - 4x))(2(\cos x) \cdot (-\sin x))}$
(iii) $\boldsymbol{(\sec x)(-\sin x)}$

and so $f'(x) = \frac{v(x) \cdot u'(x) - u(x) \cdot v'(x)}{[v(x)]^2} =$
(i) $\boldsymbol{(-\cot x \csc x) + (\csc x)(-\sin x)}$
(ii) $\boldsymbol{\frac{(6x-4)\cos^2 x \cos(2x^3-4x) - 2\sin(2x^3-4x)\cos x \sin x}{\cos^2 x}}$
(iii) $\boldsymbol{\frac{(6x-4)\cos^2 x \cos(2x^3-4x) - 2\sin(2x^3-4x)\cos x \sin x}{\cos^4 x}}$

7. *Application: temperature of patient.* Temperature of a patient is

$$T(t) = 102.3\ln(3t) + 3\sin\left(\frac{\pi}{4}t\right)$$

Determine rate of change in temperature $T'(t)$ and also $T'(4)$.

(a) $T'(t)$

First let $f[g(t)] = 102.3\ln 3t$, $g(t) = 3t$ and $f(t) = 102.3\ln t$
and $g'(t) =$ (i) $\boldsymbol{\frac{102.3}{t}}$ (ii) $\boldsymbol{e^t}$ (iii) $\boldsymbol{3}$
and $f'(t) =$ (i) $\boldsymbol{3}$ (ii) $\boldsymbol{\frac{102.3}{t}}$ (iii) $\boldsymbol{\tan t \sec t}$
and so by chain rule

$$f'[g(t)] \cdot g'(t) = f'[3t] \cdot 3 = \frac{102.3}{3t} \cdot 3 =$$

(i) $\boldsymbol{\frac{1}{3t}}$ (ii) $\boldsymbol{\frac{102.3}{t}}$ (iii) **102.3**

then let $f[g(x)] = 3\sin\left(\frac{\pi}{4}t\right)$, $g(t) = \frac{\pi}{4}t$ and $f(t) = 3\sin t$
and $g'(t) =$ (i) $\boldsymbol{3\cos t}$ (ii) $\boldsymbol{\frac{\pi}{4}}$ (iii) $\boldsymbol{\cos t}$
and $f'(t) =$ (i) $\boldsymbol{-\csc^2 t}$ (ii) $\boldsymbol{3\cos t}$ (iii) $\boldsymbol{\pi\cos t}$
and so by chain rule

$$f'[g(t)] \cdot g'(t) = f'\left[\frac{\pi}{4}t\right] \cdot \frac{\pi}{4} = 3\cos\left(\frac{\pi}{4}t\right) \cdot \frac{\pi}{4} =$$

(i) $\boldsymbol{\cos\left(\frac{\pi}{4}t\right)}$ (ii) $\boldsymbol{\frac{3\pi}{4}\cos\left(\frac{\pi}{4}t\right)}$ (iii) $\boldsymbol{\frac{3\pi}{4}}$

and so $\frac{dy}{dt}$ = (i) $\boldsymbol{\frac{3\pi}{4}\cos\left(\frac{\pi}{4}t\right)}$ (ii) $\boldsymbol{\frac{102.3}{t}}$ (iii) $\boldsymbol{\frac{102.3}{t} + \frac{3\pi}{4}\cos\left(\frac{\pi}{4}t\right)}$

(b) $T'(4) = \frac{102.3}{4} + \frac{3\pi}{4}\cos\left(\frac{4\pi}{4}\right) \approx$ (i) **37.93** (ii) **27.93** (iii) **47.93**