Chapter 10

Differential Equations

10.1 Solutions to Elementary and Separable Differential Equations

Remember the *general solution* to (elementary) differential equation

$$\frac{dy}{dx} = g(x)$$

is the antiderivative $G(x)$

$$y = \int g(x)\, dx = G(x) + C.$$

With the addition of a *initial (boundary) condition*, $y(x_0)$ at $x = x_0$, the elementary differential equation becomes an *initial value problem* which has a *particular solution* where a "particular" constant C can be identified. A second but slightly more sophisticated class of differential equations, *separable differential equations*,

$$\frac{dy}{dx} = \frac{p(x)}{q(y)},$$

have (general) solution

$$\int q(y)\, dy = \int p(x)\, dx, \quad \text{or} \quad Q(y) = P(x) + C,$$

where, again, with an initial condition, $y(x_0)$ at $x = x_0$, a particular solution can be identified. Examples of separable differential equations and their solutions include the previously discussed *exponential growth (decay)*, *limited growth* and *logistic growth* models, given in the figure and table below,

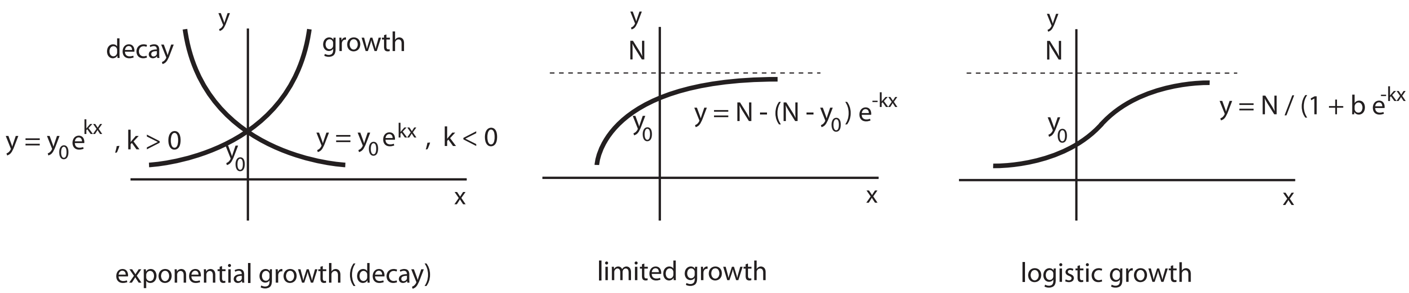

Figure 10.1 (Examples of separable differential equations)

	differential equation, initial condition	solution
exponential growth (decay)	$\frac{dy}{dx} = ky$, $y(0) = y_0$	$y = y_0 e^{ky}$
limited growth	$\frac{dy}{dx} = k(N - y)$, $y(0) = y_0$	$y = N - (N - y_0)e^{-kt}$
logistic growth	$\frac{dy}{dx} = k\left(1 - \frac{y}{N}\right) y$, $y(0) = y_0$	$y = \frac{N}{1+be^{-kt}}$, $b = \frac{N-y_0}{y_0}$

where, if $k > 0$, then k is a *growth constant* and y is an exponential growth function and if $k < 0$, then k is a *decay constant* and y is an exponential decay function and N is the *carrying (limiting) capacity.*

Exercise 10.1 (Solutions to Elementary and Separable Differential Equations)

1. *Elementary differential equation,* $\frac{dy}{dx} = 3x$

   (a) *General Solution.*
   Solve $\frac{dy}{dx} = 3x$ for y.
   Since $\frac{dy}{dx} = 3x$ could be rewritten as

$$\begin{aligned} dy &= 3x\,dx \\ \int dy &= \int 3x\,dy \quad \text{integrate both sides} \\ y &= 3 \cdot \frac{1}{1+1} x^{1+1} + C \end{aligned}$$

   or (i) $\boldsymbol{y = 3x^2 + C}$ (ii) $\boldsymbol{y = \frac{3}{2}x^2 + C}$ (iii) $\boldsymbol{y = \frac{5}{2}x^2 + C}$

   (b) *Verify the Solution.*
   Verify general solution of $\frac{dy}{dx} = 3x$ is $y = \frac{3}{2}x^2 + C$.
   Plug $y = \frac{3}{2}x^2 + C$ into the *left* side of $\frac{dy}{dx} = 3x$,

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(\frac{3}{2}x^2 + C\right) =$$

(i) $\frac{\mathbf{3}}{\mathbf{2}}\boldsymbol{x}$ (ii) $\frac{\mathbf{5}}{\mathbf{2}}\boldsymbol{x}$ (iii) $\mathbf{3}\boldsymbol{x}$
which equals *right* side of $\frac{dy}{dx} = 3x$, so solution is verified.

(c) *Particular Solution.*
Solve $\frac{dy}{dx} = 3x$ for y at $(x, y) = (1, 2)$.
Since $y = \frac{3}{2}x^2 + C$,

$$2 = \frac{3}{2}(1)^2 + C \quad \text{since } x = 1, y = 2$$

or $C =$ (i) $\frac{\mathbf{1}}{\mathbf{2}}$ (ii) $\frac{\mathbf{3}}{\mathbf{2}}$ (iii) $\frac{\mathbf{5}}{\mathbf{2}}$

and so the *particular solution* in this case is $y = \frac{3}{2}x^2 + C =$
(i) $\boldsymbol{y = 3x^2 + \frac{1}{2}}$ (ii) $\boldsymbol{y = \frac{3}{2}x^2 + \frac{1}{2}}$ (iii) $\boldsymbol{y = \frac{5}{2}x^2 + \frac{1}{2}}$

(d) *Particular Solution, Different Notation.*
Solve $f'(x) = 3x$, given $f(1) = 2$.
Since $f'(x) = 3x$ is just $\frac{dy}{dx} = 3x$, particular solution is still
(i) $\boldsymbol{f(x) = 3x^2 + \frac{1}{2}}$ (ii) $\boldsymbol{f(x) = \frac{3}{2}x^2 + \frac{1}{2}}$ (iii) $\boldsymbol{f(x) = \frac{5}{2}x^2 + \frac{1}{2}}$

(e) *Another Particular Solution.*
Solve $\frac{dy}{dx} = 3x$ for y at $(x, y) = (1, 3)$.
Since $y = \frac{3}{2}x^2 + C$,

$$3 = \frac{3}{2}(1)^2 + C$$

or $C =$ (i) $\frac{\mathbf{1}}{\mathbf{2}}$ (ii) $\frac{\mathbf{3}}{\mathbf{2}}$ (iii) $\frac{\mathbf{5}}{\mathbf{2}}$

so particular solution in this case is $y = \frac{3}{2}x^2 + C =$
(i) $\boldsymbol{y = \frac{3}{2}x^2 - \frac{1}{2}}$ (ii) $\boldsymbol{y = \frac{3}{2}x^2 + \frac{1}{2}}$ (iii) $\boldsymbol{y = \frac{3}{2}x^2 + \frac{3}{2}}$

(f) *Graphs of $\frac{dy}{dx} = 3x$.*
There are/is (i) **one** (ii) **many** curves/graphs associated with differential equation $\frac{dy}{dx} = 3x$, as shown in the picture below.

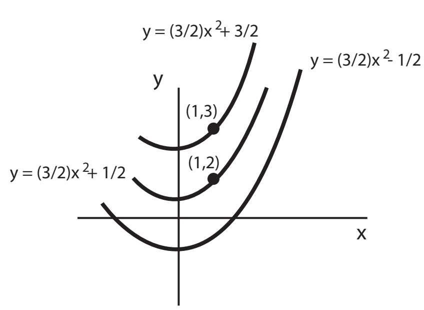

Figure 10.2 ($\frac{dy}{dx} = 3x$ or, equivalently, $f(x) = \frac{3}{2}x^2 + C$)

2. *Elementary differential equation,* $\frac{dy}{dx} - x = 3x^2$

   (a) *General Solution.*
   Solve $\frac{dy}{dx} - x = 3x^2$ for y.
   Since $\frac{dy}{dx} - x = 3x^2$, or $\frac{dy}{dx} = x + 3x^2$ could be rewritten as

$$\begin{aligned} dy &= (x + 3x^2)\,dx \\ \int dy &= \int (x + 3x^2)\,dy \quad \text{integrate both sides} \\ y &= 1 \cdot \frac{1}{1+1}x^{1+1} + 3 \cdot \frac{1}{2+1}x^{2+1} + C \end{aligned}$$

   or (i) $\boldsymbol{y = \frac{1}{2}x^2 + C}$ (ii) $\boldsymbol{y = \frac{1}{2}x^2 + x^3 + C}$ (iii) $\boldsymbol{y = x^3 + C}$

   (b) *Particular Solution.*
   Solve $\frac{dy}{dx} - x = 3x^2$, given $f(1) = 2$.
   Since $y = \frac{1}{2}x^2 + x^3 + C$,

$$2 = \frac{1}{2}(1)^2 + (1)^3 + C \quad \text{since } x = 1, y = 2$$

   or $C =$ (i) $\boldsymbol{\frac{1}{2}}$ (ii) $\boldsymbol{\frac{3}{2}}$ (iii) $\boldsymbol{\frac{5}{2}}$

   and so the *particular solution* in this case is $y = \frac{1}{2}x^2 + x^3 + C =$
   (i) $\boldsymbol{y = \frac{1}{2}x^2 + x^3 + \frac{1}{2}}$ (ii) $\boldsymbol{y = \frac{3}{2}x^2 + \frac{1}{2}}$ (iii) $\boldsymbol{y = x^3 + \frac{1}{2}}$

3. *Separable differential equation,* $\frac{dy}{dx} = xy$

   (a) *General Solution.*
   Solve $\frac{dy}{dx} = xy$ for y.
   Since

$$\begin{aligned} \frac{dy}{dx} &= xy \\ dy &= xy\,dx \\ \frac{1}{y}\,dy &= x\,dx \quad \text{separation of variables} \\ \int \frac{1}{y}\,dy &= \int x\,dx \\ \ln y &= \frac{1}{1+1}x^{1+1} + C \\ e^{\ln y} &= e^{\frac{1}{2}x^2 + C} \end{aligned}$$

so (i) $\boldsymbol{y = e^{\frac{1}{2}x^2}e^C = Me^{\frac{1}{2}x^2}}$ (ii) $\boldsymbol{y = M^2e^{\frac{1}{2}x^3}}$

(b) *Verify the Solution.*
Verify the general solution of $\frac{d}{dx}(y) = xy$ is $y = Me^{\frac{1}{2}x^2}$.
Plug $y = Me^{\frac{1}{2}x^2}$ into *left* side of $\frac{d}{dx}(y) = xy$,

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(Me^{\frac{1}{2}x^2}\right) = M \cdot \frac{1}{2} \cdot 2x^{2-1} \cdot e^{\frac{1}{2}x^2} =$$

(i) $\boldsymbol{Me^{\frac{1}{2}x^2}}$ (ii) $\boldsymbol{xe^{\frac{1}{2}x^2}}$ (iii) $\boldsymbol{Mxe^{\frac{1}{2}x^2}}$

Plug $y = Me^{\frac{1}{2}x^2}$ into the *right* side of $\frac{d}{dx}(y) = xy$,

$$x(y) = x\left(Me^{\frac{1}{2}x^2}\right) =$$

(i) $\boldsymbol{Me^{\frac{1}{2}x^2}}$ (ii) $\boldsymbol{xe^{\frac{1}{2}x^2}}$ (iii) $\boldsymbol{Mxe^{\frac{1}{2}x^2}}$
and so since the left side equals the right side, we have verified the solution.

(c) *Particular Solution.*
Solve $\frac{dy}{dx} = xy$ for y at $(x, y) = (1, 3)$.
Since $y = Me^{\frac{1}{2}x^2}$,

$$3 = Me^{\frac{1}{2}(1)^2} \quad \text{since } x = 1, y = 3$$

and so
$M =$ (i) $\boldsymbol{e^{-\frac{1}{2}}}$ (ii) $\boldsymbol{3e^{-\frac{1}{2}}}$ (iii) $\boldsymbol{3e^{\frac{1}{2}}}$

and so $y = Me^{\frac{1}{2}x^2} =$ (i) $\boldsymbol{e^{-\frac{1}{2}}e^{\frac{1}{2}x^2}}$ (ii) $\boldsymbol{3e^{-\frac{1}{2}}e^{\frac{1}{2}x^2}}$ (iii) $\boldsymbol{3e^{\frac{1}{2}}e^{\frac{1}{2}x^2}}$

4. *Separable differential equation,* $\frac{dy}{dx} = \frac{x^3}{y}$
Since $\frac{dy}{dx} = \frac{x^3}{y} = x^3 \cdot \frac{1}{y}$,

$$\begin{aligned} dy &= x^3 \cdot \frac{1}{y}\, dx \\ y\, dy &= x^3\, dx \quad \text{separation of variables} \\ \int y\, dy &= \int x^3\, dx \\ \frac{1}{1+1}y^{1+1} &= \frac{1}{3+1}x^{3+1} + C \end{aligned}$$

so $y^2 =$ (i) $\boldsymbol{2x^4 + C}$ (ii) $\boldsymbol{4x^4 - C}$ (iii) $\boldsymbol{\frac{1}{2}x^4 + C}$
notice not sure if $y = \sqrt{2x^4 + C}$ or $y = -\sqrt{2x^4 + C}$, so left as $y^2 = \frac{1}{2}x^4 + C$

5. *Separable differential equation,* $4y^3 \frac{dy}{dx} = 5x$
   Since

$$\begin{aligned} 4y^3\,dy &= 5x\,dx \quad \text{separation of variables} \\ \int 4y^3\,dy &= \int 5x\,dx \\ 4 \cdot \frac{1}{3+1} y^{3+1} &= 5 \cdot \frac{1}{1+1} x^{1+1} + C \end{aligned}$$

   and so $y^4 =$ (i) $\boldsymbol{\frac{3}{2}x^2 + C}$ (ii) $\boldsymbol{\frac{5}{2}x^2 + C}$ (iii) $\boldsymbol{\frac{7}{2}x^2 + C}$

6. *Separable differential equation,* $4y^3 \frac{dy}{dx} - 5x^2 + 3x = 0$
   Since

$$\begin{aligned} 4y^3 \frac{dy}{dx} &= 5x^2 - 3x \\ 4y^3\,dy &= \left(5x^2 - 3x\right)\,dx \quad \text{separation of variables} \\ 4 \cdot \frac{1}{3+1} y^{3+1} &= 5 \cdot \frac{1}{2+1} x^{2+1} - 3 \cdot \frac{1}{1+1} x^{1+1} + C \end{aligned}$$

   then $y^4 =$ (i) $\boldsymbol{\frac{5}{3}x^3 - \frac{3}{2}x^2 + C}$ (ii) $\boldsymbol{\frac{5}{2}x^2 + C}$ (iii) $\boldsymbol{\frac{7}{2}x^2 + C}$

7. *Separable differential equation,* $\frac{dy}{dx} = x + 3xy$

   (a) *General Solution.*
   Since $\frac{dy}{dx} = x + 3xy = x(1+3y)$,

$$\begin{aligned} dy &= x(1+3y)\,dx \\ \frac{1}{1+3y}\,dy &= x\,dx \quad \text{separation of variables} \\ \int \frac{1}{1+3y}\,dy &= \int x\,dx \\ \frac{1}{3}\int \frac{1}{1+3y}(3)\,dy &= \int x\,dx \\ \frac{1}{3}\int \frac{1}{u}\,du &= \int x\,dx \quad \text{substitute } u = 1+3y, \text{ so } du = 3\,dy \\ \frac{1}{3}\ln u &= \frac{1}{1+1}x^{1+1} + C \\ \frac{1}{3}\ln(1+3y) &= \frac{1}{2}x^2 + C \quad \text{since } u = 1+3y \\ \ln(1+3y) &= 3 \cdot \frac{1}{2}x^2 + 3C \end{aligned}$$

$$\begin{aligned} e^{\ln(1+3y)} &= e^{\frac{3}{2}x^2+3C} \\ 1+3y &= e^{\frac{3}{2}x^2}e^{3C} \end{aligned}$$

so (i) $\boldsymbol{y = \frac{1}{3}\left(Me^{\frac{3}{2}x^2} - 1\right)}$ (ii) $\boldsymbol{y = e^{3x^2}M}$ (iii) $\boldsymbol{y = e^{3x^2}M}$

(b) *Particular Solution.*
Solve $\frac{dy}{dx} = x + 3xy$ for y at $(x, y) = (2, 3)$.
Since $y = \frac{1}{3}\left(Me^{\frac{3}{2}x^2} - 1\right)$

$$3 = \frac{1}{3}\left(Me^{\frac{3}{2}(2)^2} - 1\right),$$

and so
$M =$ (i) $\boldsymbol{10e^{-6}}$ (ii) $\boldsymbol{10e^{-22}}$ (iii) $\boldsymbol{10e^{-24}}$

and so $y = \frac{1}{3}\left(Me^{\frac{3}{2}x^3} - 1\right) =$
(i) $\boldsymbol{e^{-\frac{5}{3}}e^{\frac{5}{3}x^2}}$ (ii) $\boldsymbol{4e^{-\frac{5}{3}}e^{\frac{5}{3}x^2}}$ (iii) $\boldsymbol{\frac{1}{3}\left(10e^{-6}e^{\frac{3}{2}x^2} - 1\right)}$

8. *Separable differential equation,* $3\frac{dy}{dx} = 2y + 1$
Since

$$\begin{aligned} 3\,dy &= (2y+1)\,dx \\ \frac{3}{2y+1}\,dy &= dx \quad \text{separation of variables} \\ \int \frac{3}{2y+1}\,dy &= \int x\,dx \\ 3\cdot\frac{1}{2}\int \frac{1}{2y+1}(2)\,dy &= \int x\,dx \\ \frac{3}{2}\int \frac{1}{u}du &= \int x\,dx \quad \text{substitute } u = 2y+1, \text{ so } du = 2\,dy \\ \frac{3}{2}\ln u &= \frac{1}{1+1}x^{1+1} + C \\ \frac{3}{2}\ln(2y+1) &= \frac{1}{2}x^2 + C \\ \ln(2y+1) &= \frac{2}{3}\left(\frac{1}{2}x^2 + C\right) \quad \text{since } u = 2y+1 \\ e^{\ln(2y+1)} &= e^{\frac{1}{3}x^2+\frac{2}{3}C} \\ 2y+1 &= e^{\frac{1}{3}x^2+\frac{2}{3}C} \end{aligned}$$

so $y =$ (i) $\boldsymbol{\frac{5}{3}x^3 - \frac{3}{2}x^2 + M}$ (ii) $\boldsymbol{\frac{5}{2}e^2 + M}$ (iii) $\boldsymbol{Me^{\frac{1}{3}x^2} - \frac{1}{2}}$

9. *Application: exponential cellular growth rate,* $\frac{dy}{dt} = ky$.
How many cells after 10 hours, if there are an initial 5000 cells and $k = 0.02$?

(a) *General Solution.*
Since $\frac{dy}{dt} = ky$,

$$\begin{aligned} \frac{1}{y}\,dy &= k\,dt \quad \text{separation of variables} \\ \int \frac{1}{y}\,dy &= \int k\,dt \quad \text{integrate both sides} \\ \ln y &= k \cdot \frac{1}{0+1} t^{0+1} + C \\ e^{\ln y} &= e^{k \cdot t + C} \end{aligned}$$

so (i) $\boldsymbol{y = \frac{1}{2}x^2 + M}$ (ii) $\boldsymbol{y = \frac{1}{2}x^2 + e^{Mt} + C}$ (iii) $\boldsymbol{y = Me^{kt}}$

(b) *Particular Solution.*
Solve $\frac{dy}{dt} = ky$, given $f(0) = 5000$.
Since $y = Me^{kt}$,

$$5000 = Me^{k(0)} \quad \text{since } t = 0, y = 5000$$

or $M =$ (i) **2500** (ii) **5000** (iii) **10000**

and so the particular solution is $y = Me^{kt} =$
(i) $\boldsymbol{y = 2500e^{kt}}$ (ii) $\boldsymbol{y = 5000e^{kt}}$ (iii) $\boldsymbol{y = 10000e^{kt}}$

(c) *What is y when $t = 10$ and $k = 0.02$?*

$$y = 5000e^{kt} = 5000e^{0.02(10)} \approx$$

(i) **6004** (ii) **6053** (iii) **6107**.

10.2 Linear First-Order Differential Equations

The linear first-order differential equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

has *integrating factor*

$$I(x) = e^{\int P(x)\,dx}$$

and is solved using the following steps:

- rewrite given equation in form $\frac{dy}{dx} + P(x)y = Q(x)$

- multiply result by integrating factor, $I(x)$
- replace terms on left of result with $D_x[I(x)y]$
- integrate result, solve for y

Unlike many general (non–first–order, non–linear) differential equations (that include terms such as y^2 or e^y or $y \cdot \frac{dy}{dx}$ for example), these special first–order linear differential equations can be easily solved for y and this why these equations are used in many different applications.

Exercise 10.2 (Linear First-Order Differential Equations)

1. *Solve* $xy' + 3y = 4$ *for* y

   (a) *Rewrite in form* $\frac{dy}{dx} + P(x)y = Q(x)$.

   $$\begin{aligned} xy' + 3y &= 4 \\ x\frac{dy}{dx} + 3y &= 4 \quad \text{since } y' = \tfrac{dy}{dx} \\ \frac{dy}{dx} + 3x^{-1}y &= 4x^{-1} \quad \text{dividing by } x \end{aligned}$$

   where $Q(x) = 4x^{-1}$ and $P(x) =$ (i) $\boldsymbol{3y}$ (ii) $\boldsymbol{3x^{-1}}$ (iii) $\boldsymbol{3x^{-1}y}$.

   (b) *Integrating factor* $I(x) = e^{\int P(x)\,dx}$.

   $$\begin{aligned} I(x) &= e^{\int P(x)\,dx} \\ &= e^{\int 3x^{-1}\,dx} \\ &= e^{3\ln|x|} \\ &= e^{\ln|x|^3} = \end{aligned}$$

   (i) $\boldsymbol{x^3}$ (ii) $\boldsymbol{e^{3x}}$ (iii) $\boldsymbol{\ln|x|}$.

   (c) *Further steps.*

   $$\begin{aligned} \frac{dy}{dx} + 3x^{-1}y &= 4x^{-1} \\ x^3\frac{dy}{dx} + 3x^2y &= 4x^2 \quad \text{multiply by } I(x) = x^3 \end{aligned}$$

   recognize, by product rule, $D_x[I(x)y] = D_x[x^3y] = x^3\frac{dy}{dx} + 3x^2y$ and so

   $$x^3\frac{dy}{dx} + 3x^2y = 4x^2$$

$$\begin{aligned}
D_x[x^3y] &= 4x^2 \quad \text{because } D_x[x^3y] = x^3\tfrac{dy}{dx} + 3x^2y \\
\int D_x[x^3y]\,dx &= \int 4x^2\,dx \quad \text{integrate on both sides} \\
x^3y &= 4 \cdot \frac{1}{2+1}x^{2+1} + C
\end{aligned}$$

so $y =$ (i) $\boldsymbol{\frac{4}{3} + Cx^{-3}}$ (ii) $\boldsymbol{\frac{4}{3}x + Cx^{-3}}$ (iii) $\boldsymbol{\frac{4}{3}y + C}$.

(d) *Particular Solution.*
Solve $y = \frac{4}{3} + Cx^{-3}$, given $f(1) = 1$.

$$1 = \frac{4}{3} + C \cdot (1)^{-3} \quad \text{since } x = 1, y = 1$$

or $C =$ (i) $\boldsymbol{\frac{1}{3}}$ (ii) $\boldsymbol{-\frac{1}{3}}$ (iii) $\boldsymbol{\frac{2}{3}}$

and so the particular solution is $y = \frac{4}{3} + Cx^{-3} = \frac{4}{3} - \frac{1}{3x^3}$, or
(i) $\boldsymbol{y = \frac{4x^3-1}{3x^3}}$ (ii) $\boldsymbol{y = 1}$ (iii) $\boldsymbol{y = -\frac{1}{3x^2}}$

(e) *Why this method works, where $I(x)$ comes from.*
on the one hand,

$$\begin{aligned}
\frac{dy}{dx} + P(x)y &= Q(x) \\
I(x)\frac{dy}{dx} + I(x)P(x)y &= I(x)Q(x) \quad \text{multiplying by } I(x)
\end{aligned}$$

and, on the other hand, derivative of product $I(x) \cdot y$ is

$$I(x)\frac{dy}{dx} + I'(x)y$$

so comparing left side of first equation with second equation,

$$\begin{aligned}
I'(x) &= I(x)P(x) \\
\frac{I'(x)}{I(x)} &= P(x) \\
\int \frac{I'(x)}{I(x)}\,dx &= \int P(x)\,dx + C \\
\ln|I(x)| &= e^{\int P(x)\,dx + C} \\
I(x) &= \pm e^C e^{\int P(x)\,dx}
\end{aligned}$$

so letting $C = 0$ and so $e^C = 1$ and using only the positive value,
$I(x) =$ (i) $\boldsymbol{e^{\int P(x)\,dx}}$ (ii) $\boldsymbol{-e^{\int P(x)\,dx}}$ (iii) $\boldsymbol{1}$

2. *Solve $x^2y' + 2xy = -3x^2$ for y*

   (a) *Rewrite in form $\frac{dy}{dx} + P(x)y = Q(x)$.*

$$\begin{aligned} x^2y' + 2xy &= -3x^2 \\ x^2\frac{dy}{dx} + 2xy &= -3x^2 \quad \text{since } y' = \tfrac{dy}{dx} \\ \frac{dy}{dx} + 2x^{-1}y &= -3 \quad \text{dividing by } x^2 \end{aligned}$$

   where $Q(x) = -3$ and $P(x) =$ (i) $\boldsymbol{2x^{-1}}$ (ii) $\boldsymbol{3x^{-1}}$ (iii) $\boldsymbol{2x^{-1}y}$.

   (b) *Integrating factor $I(x) = e^{\int P(x)\,dx}$.*

$$\begin{aligned} I(x) &= e^{\int P(x)\,dx} \\ &= e^{\int 2x^{-1}\,dx} \\ &= e^{2\ln|x|} \\ &= e^{\ln|x|^2} = \end{aligned}$$

   (i) $\boldsymbol{x^2}$ (ii) $\boldsymbol{e^{2x}}$ (iii) $\boldsymbol{\ln|2x|}$.

   (c) *Further steps.*

$$\begin{aligned} \frac{dy}{dx} + 2x^{-1}y &= -3 \\ x^2\frac{dy}{dx} + 2xy &= -3x^2 \quad \text{multiply by } I(x) = x^2 \\ D_x[x^2y] &= -3x^2 \quad \text{because } D_x[x^2y] = x^2\tfrac{dy}{dx} + 2xy \\ \int D_x[x^2y]\,dx &= \int -3x^2\,dx \quad \text{integrate on both sides} \\ x^2y &= -3\cdot\frac{1}{2+1}x^{2+1} + C \end{aligned}$$

   so $y =$ (i) $\boldsymbol{\frac{4}{3}x + Cx^{-2}}$ (ii) $\boldsymbol{-x + \frac{C}{x^2}}$ (iii) $\boldsymbol{x + Cx^2}$.

   (d) *Particular Solution.*
   Solve $y = -x + \frac{C}{x^2}$, given $f(1) = 0$.

$$0 = -(1) + \frac{C}{1^2} \quad \text{since } x = 1, y = 0$$

   or $C =$ (i) **1** (ii) **0** (iii) **2**

   and so the particular solution is $y = -x + \frac{C}{x^2}$, or
   (i) $\boldsymbol{y = -\frac{1}{x}}$ (ii) $\boldsymbol{y = -x + \frac{1}{x}}$ (iii) $\boldsymbol{y = -x}$

3. *Solve* $y' = -y + x$ *for* y

   (a) *Rewrite in form* $\frac{dy}{dx} + P(x)y = Q(x)$.

$$\begin{aligned} y' &= -y + x \\ y' + y &= x \\ \frac{dy}{dx} + y &= x \quad \text{since } y' = \tfrac{dy}{dx} \end{aligned}$$

   where $Q(x) = x$ and $P(x) =$ (i) $\boldsymbol{x}$ (ii) $\boldsymbol{y}$ (iii) $\mathbf{1}$.

   (b) *Integrating factor* $I(x) = e^{\int P(x)\,dx}$.

$$\begin{aligned} I(x) &= e^{\int P(x)\,dx} \\ &= e^{\int 1\,dx} = \end{aligned}$$

   (i) $\boldsymbol{x}$ (ii) $\boldsymbol{e^x}$ (iii) $\mathbf{ln}\,|\boldsymbol{x}|$.

   (c) *Further steps.*

$$\begin{aligned} \frac{dy}{dx} + y &= x \\ e^x\frac{dy}{dx} + e^x y &= e^x x \quad \text{multiply by } I(x) = e^x \\ D_x[e^x y] &= e^x x \quad \text{because } D_x[e^x y] = e^x \tfrac{dy}{dx} + e^x y \\ \int D_x[e^x y]\,dx &= \int e^x x\,dx \quad \text{integrate on both sides} \\ e^x y &= xe^x - e^x + C \quad \text{integrate by parts, left hand side, see below} \end{aligned}$$

   so $y =$ (i) $\boldsymbol{x + e^x + C}$ (ii) $\boldsymbol{-x + Ce^x}$ (iii) $\boldsymbol{x - 1 + Ce^{-x}}$.

   for last step, left hand side, integrate by parts, where $u = x,\ du = 1dx,\ v = e^x,\ dv = e^x dx$
   and so $\int u\,dv = uv - \int v\,du = x \cdot e^x - \int e^x\,1dx = xe^x - e^x + C$

4. *Solve* $y' = y - 2x$ *for* y

   (a) *Rewrite in form* $\frac{dy}{dx} + P(x)y = Q(x)$.

$$\begin{aligned} y' &= y - 2x \\ y' - y &= -2x \\ \frac{dy}{dx} - y &= -2x \quad \text{since } y' = \tfrac{dy}{dx} \end{aligned}$$

   where $Q(x) = -2x$ and $P(x) =$ (i) $\boldsymbol{-2x}$ (ii) $\boldsymbol{-y}$ (iii) $\mathbf{-1}$.

(b) *Integrating factor* $I(x) = e^{\int P(x)\,dx}$.

$$\begin{aligned} I(x) &= e^{\int P(x)\,dx} \\ &= e^{\int -1\,dx} = \end{aligned}$$

(i) $\boldsymbol{-x}$ (ii) $\boldsymbol{e^{-x}}$ (iii) $\mathbf{ln}\,|\boldsymbol{x}|$.

(c) *Further steps.*

$$\begin{aligned} \frac{dy}{dx} - y &= -2x \\ e^{-x}\frac{dy}{dx} - e^{-x}y &= -2xe^{-x} \quad \text{multiply by } I(x) = e^{-x} \\ D_x[e^{-x}y] &= -2xe^{-x} \quad \text{because } D_x[e^{-x}y] = e^{-x}\tfrac{dy}{dx} - e^{-x}y \\ \int D_x[e^{-x}y]\,dx &= \int -2xe^{-x}\,dx \quad \text{integrate on both sides} \\ \int D_x[e^{-x}y]\,dx &= -2\int xe^{-x}\,dx \quad \text{pull } -2 \text{ outside of integration} \\ e^{-x}y &= -2\left(-xe^{-x} - e^{-x} + C\right) \quad \text{integrate by parts, left hand side, see below} \end{aligned}$$

so $y =$ (i) $\boldsymbol{x + e^{x} + C}$ (ii) $\boldsymbol{-x + Ce^{x}}$ (iii) $\boldsymbol{2x + 2 + Ce^{x}}$.

for last step, left hand side, integrate by parts, where $u = x,\ du = 1dx,\ v = -e^{-x},\ dv = e^{-x}dx$
and so $\int u\,dv = uv - \int v\,du = x\left(-e^{-x}\right) - \int\left(-e^{-x}\right) 1dx = -xe^{-x} - e^{-x} + C$

(d) *Particular Solution.*
Solve $y = 2x + 2 + Ce^{x}$, given $f(0) = 1$.

$$1 = 2(0) + 2 + Ce^{0} \quad \text{since } x = 0, y = 1$$

or $C =$ (i) $\mathbf{-1}$ (ii) $\mathbf{0}$ (iii) $\mathbf{-2}$

and so the particular solution is $y = 2x + 2 + Ce^{x}$, or
(i) $\boldsymbol{y = -\frac{1}{x}}$ (ii) $\boldsymbol{y = -x + \frac{1}{x}}$ (iii) $\boldsymbol{y = 2x + 2 - e^{x}}$

5. *Solve* $xy' = 4$ *for* y

(a) *Rewrite in form* $\frac{dy}{dx} + P(x)y = Q(x)$.

$$\begin{aligned} xy' &= 4 \\ x\frac{dy}{dx} &= 4 \quad \text{since } y' = \tfrac{dy}{dx} \\ \frac{dy}{dx} &= \frac{4}{x} \quad \text{divide by } x \end{aligned}$$

where $Q(x) = \frac{4}{x}$ and $P(x) =$ (i) $\boldsymbol{x}$ (ii) $\mathbf{0}$ (iii) $\mathbf{1}$.

(b) *Integrating factor* $I(x) = e^{\int P(x)\,dx}$.

$$\begin{aligned} I(x) &= e^{\int P(x)\,dx} \\ &= e^{\int 0\,dx} = \end{aligned}$$

(i) $\boldsymbol{1}$ (ii) $\boldsymbol{e^x}$ (iii) $\boldsymbol{x}$.

(c) *Further steps.*

$$\begin{aligned} \frac{dy}{dx} &= \frac{4}{x} \\ 1 \cdot \frac{dy}{dx} &= 1 \cdot \frac{4}{x} \quad \text{multiply by } I(x) = 1 \\ D_x[1 \cdot y] &= \frac{4}{x} \quad \text{because } D_x[1 \cdot y] = 1 \cdot \tfrac{dy}{dx} + 0 \cdot y = \tfrac{dy}{dx} \\ \int D_x[y]\,dx &= \int \frac{4}{x}\,dx \quad \text{integrate on both sides} \end{aligned}$$

so (i) $\boldsymbol{y = e^x + C}$ (ii) $\boldsymbol{y = 4\ln|x| + C}$ (iii) $\boldsymbol{y^2 = x - 1 + Ce^{-x}}$.

(d) *Alternative Method of Solution: Separation of Variables.*
Since

$$\begin{aligned} x\frac{dy}{dx} &= 4 \\ dy &= \frac{4}{x}\,dx \quad \text{separation of variables} \\ \int 1\,dy &= \int \frac{4}{x}\,dx \end{aligned}$$

so (i) $\boldsymbol{y = e^x + C}$ (ii) $\boldsymbol{y = 4\ln|x| + C}$ (iii) $\boldsymbol{y^2 = x - 1 + Ce^{-x}}$.

(e) (i) **True** (ii) **False**. Some first–order linear differential equations can be solved using the separation technique described previously, but, then, many cannot. Also, some differential equations which are neither first–order, nor linear, can be solved by the separation technique.

6. *Application: Population Growth* Solve $\frac{dP}{dt} = kP + 2400e^{-t}$ for P where $P(0) = 900,000$ and $k = 0.03$.

   (a) *Rewrite in form* $\frac{dP}{dt} + P(t)P = Q(t)$.

$$\begin{aligned} \frac{dP}{dt} &= kP + 2400e^{-t} \\ \frac{dP}{dt} - kP &= 2400e^{-t} \end{aligned}$$

where $Q(t) = 2400e^{-t}$ and $P(t) =$ (i) $\boldsymbol{-kP}$ (ii) $\mathbf{0}$ (iii) $\boldsymbol{-k}$.

(b) *Integrating factor* $I(t) = e^{\int P(t)\,dt}$.

$$\begin{aligned} I(t) &= e^{\int P(t)\,dt} \\ &= e^{\int -k\,dt} \\ &= e^{-k\cdot\frac{1}{1}t^1} = \end{aligned}$$

(i) $\boldsymbol{x^t}$ (ii) $\boldsymbol{e^{-k}}$ (iii) $\boldsymbol{e^{-kt}}$.

(c) *Further steps.*

$$\begin{aligned} \frac{dP}{dt} - kP &= 2400e^{-t} \\ e^{-kt}\frac{dP}{dt} - kPe^{-kt} &= 2400e^{-t}e^{-kt} \quad \text{multiply by } I(t) = e^{-kt} \\ D_x[e^{-kt}P] &= 2400e^{-t-kt} \quad \text{because } D_x[e^{-kt}P] = e^{-kt}\tfrac{dP}{dt} - kPe^{-kt} \\ \int D_x[e^{-kt}P]\,dt &= \int 2400e^{-(1+k)t}\,dt \quad \text{integrate on both sides} \\ e^{-kt}P &= 2400 \cdot \frac{1}{-(1+k)}e^{-(1+k)t} + C \end{aligned}$$

so, since $k = 0.03$,
$P =$ (i) $\boldsymbol{\frac{-2400}{1+0.03}e^{-t}}$ (ii) $\boldsymbol{\frac{-2400}{1+0.03}e^{t} + C}$ (iii) $\boldsymbol{\frac{-2400}{1+0.03}e^{-t} + Ce^{kt}}$.

(d) *Particular Solution.*
Solve $y = \frac{-2400}{1+0.03}e^{-t} + Ce^{kt}$, given $P(0) = 900000$.

$$900000 = \frac{-2400}{1+0.03}e^{-(0)} + Ce^{k(0)} \quad \text{since } t = 0, P = 900000$$

or $C \approx$ (i) **900000** (ii) **902330** (iii) **902440**

and so the particular solution is
(i) $\boldsymbol{y = 902330c^{0.03t}}$
(ii) $\boldsymbol{y = 902330c^{0.03t} + 2330e^{-t}}$
(iii) $\boldsymbol{y = 902330c^{0.03t} - 2330e^{-t}}$