Lecture Notes Workbook

For Mathematics 213

Finite Mathematics I

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by

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Preface

This course provides an introduction to a few finite mathematical topics, including, linear programming, probability and statistics.

This workbook is a necessary component for a student to successfully complete this course. Without the workbook, a student will not be able to participate in the course.

- This workbook is *based* on the text.
- Although the material covered in each is very similar, the *presentation* of the material in the workbook is quite different from the presentation given in the text. The text consists essentially of definitions, formulas, worked out examples and exercises; this workbook, on the other hand, consists *solely* of exercises to be worked out by the student.
- The overheads presented during each lecture are based *exclusively* on the workbook. A student fills in this workbook during the lecture.
- This attendance workbook essentially mimics what goes on during the lectures.
- There are different kinds of exercises, including multiple choice, true/false, matching and fill–in–the–blank.
- Each week, I recommend you read the text, answer questions given here in attendance workbook, look over TI–84+ instructions and then do either quiz or homework assignment, in that order.

On the one hand, the workbook is, as you will see, quite a bit more elaborate than typical lecture notes, which are usually a summary of what the instructor finds important in a recommended course text. On the other hand, this workbook is not quite a text, because although it has many exercises, it does not have quite enough exercises to qualify it as a complete text. I should also point out that this workbook, unfortunately, possesses a number of typographical errors. In short, this workbook aspires to be text and, in the next few years, when enough exercises have been collected, and when most of the typographical errors have been weeded out, it will become a text.

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Chapter 1

Applications of Linear Functions

We will first look at some of the definitions and formulas related to the Cartesian coordinate system and how to draw points and lines in this system. We will then look at both the graphical as well as algebraic description of linear functions. Economic and statistical examples will then be given.

1.1 The Cartesian Plane and Graphing

We will look at points on the Cartesian coordinate system in this section.

Exercise 1.1 (The Cartesian Plane and Graphing)

1. *Cartesian plane.* Consider rectangular coordinate system (also called Cartesian plane) below with four points.

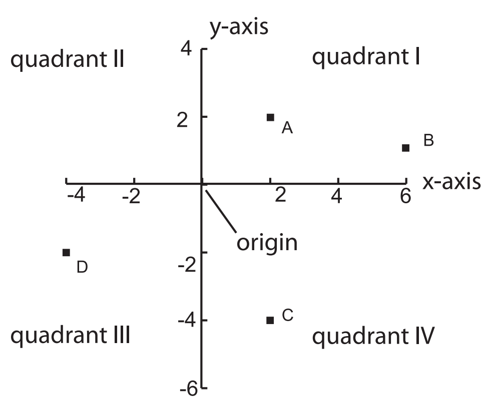


Figure 1.1 (Points on rectangular coordinate system)

   (a) Rectangular coordinate system has **2** / **3** / **4** quadrants.

   (b) The x–coordinate and y–coordinate for four points, A, B, C and D are

      i. (2,2), (6,1), (2,-4), (-4,-2)

ii. (2,2), (1,6), (2,-4), (-4,-2)
iii. (2,2), (6,1), (-4,2), (-4,-2)
iv. (2,2), (6,1), (2,-4), (-2,-4)

An alternate way of describing these four points is the following table.

x	y
-4	-2
2	-4
2	2
6	1

2. *Equations, Functions, Identities and Graphs.*
An *equation* is a statement where two mathematical expressions are equal. An equation is only true for certain values of the variables; these values are called *solutions*. An *identity* is a equation which is true for all values of the variables. A *function*[1] is a rule f which assigns to variable x one, and only one value, $f(x)$. The x is *independent* variable; all possible values of x is *domain*. The y is *dependent* variable; all possible values of y is *range*.

(a) *What is it (more than one answer could be correct)?*

i. $y = 2x + 3$
equation / function / solution

ii. $y = -2x^2 + 2x - 2$
equation / function / solution

iii. circle, centered $(h, k) = (1, 3)$ with radius $r = 5$, $(x-1)^2+(y-3)^2 = 5^2$
equation / function / solution

iv. $(x, y) = (0, 3)$ for $y = 2x + 3$
equation / function / solution

v. $(x, y) = (1, 5)$ for $y = 2x + 3$
equation / function / solution

(b) *TI–84+: Graphing Functions.*

i. Graph $y = 2x + 3$ with domain $-10 \leq x \leq 10$

Clear previous plots and functions to prevent conflicts: press $Y =$, clear any functions along side $Y =$s, turn off all three plots on top of screen–arrow up, press enter to un-black any active plot.

Enter $2x + 3$ beside $Y_1 =$; "x" is "X,Tθ,n" button.

Enter domain: Press WINDOW, set Xmin to -10, Xmax to 10, Xscl to 1, Yscl to 1, Xres to 1.

Graph: Press ZOOM, ZoomFit.

[1]Function "$f(x)$" is often written simply as "y". In the special case when a vertical line passes through a graph of an equation only once, the equation is called a function.

ii. Evaluate $y = 2x + 3$ with domain $-10 \leq x \leq 10$ at $x = 2$
$y = 2(2) + 3 =$ **5 / 6 / 7**
Press TRACE, press 2, ENTER. $X = 2$ and $Y = 7$ appear bottom of screen.

iii. Graph $y = -2x^2 + 2x - 2$, domain $-10 \leq x \leq 10$, evaluate at $x = 2$
$y = -2(2)^2 + 2(2) - 2 =$ **−5 / −6 / −7**
First clear $y = 2x + 3$: press $Y =$ and clear it.
Enter $-2x^2 + 2x - 2$ beside $Y_1 =$; use "(-)" button for first negative, "-" button for second negative, "x^2" for square.
Graph: Press ZOOM, ZoomFit.
Press TRACE, press 2, ENTER. $X = 2$ and $Y = -6$ appear bottom of screen.

1.2 Equations of Straight Lines

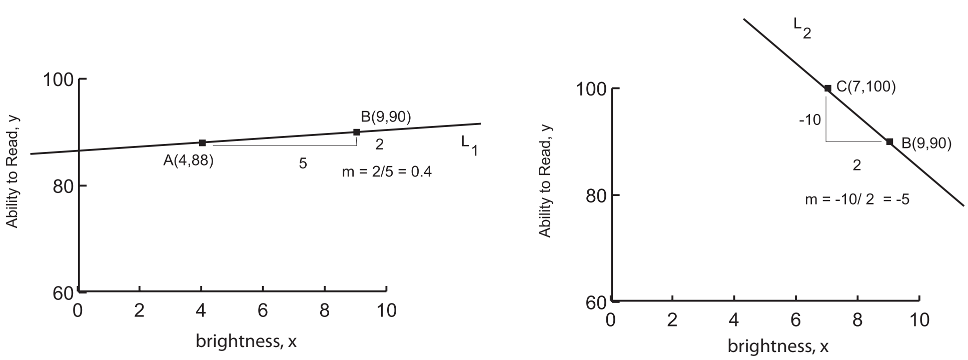
We will now look at (straight) lines. After reviewing what a slope of a line is, we look at different equations of lines:

- slope-intercept $y = mx + b$
- point-slope[2] $y - y_1 = m(x - x_1)$
- general form[3] $Ax + By = C$

All *equations* of lines, except vertical, are also linear *functions*.

Exercise 1.2 (Equations of Straight Lines)

1. *Slope: reading ability versus brightness*



(a) positive slope (b) negative slope

Figure 1.2 (Slope: reading ability versus brightness)

[2]This form of equation is often intermediate step towards slope-intercept form of equation.
[3]This form of equation is used later when we look at *systems* of equations.

(a) Slope, m, of line L_1 between points A and B in figure (a) above:

$$m = \frac{\triangle y}{\triangle x} = \frac{90 - 88}{9 - 4} = \frac{2}{5} =$$

(circle one) **0.1** / **0.2** / **0.4**.

(b) Slope of line L_1, $m = 0.4$, says

i. brightness increases by 0.4 units for unit increase reading ability.

ii. reading ability increases by 0.4 units for unit increase in brightness.

When $m > 0$, line **rises** / **falls**.

(c) Slope, m, of line L_2 between points C and D in figure (b) above:

$$m = \frac{\triangle y}{\triangle x} = \frac{90 - 100}{9 - 7} = -\frac{10}{2} =$$

(circle one) **−5** / **−1** / **5**.

(d) The slope of line L_2, $m = -5$, says (circle one)

i. brightness *de*creases 5 units for unit increase in reading ability.

ii. reading ability *de*creases 5 units for each unit increase in brightness.

When $m < 0$, line **rises** / **falls**.

2. *Equation of a line*

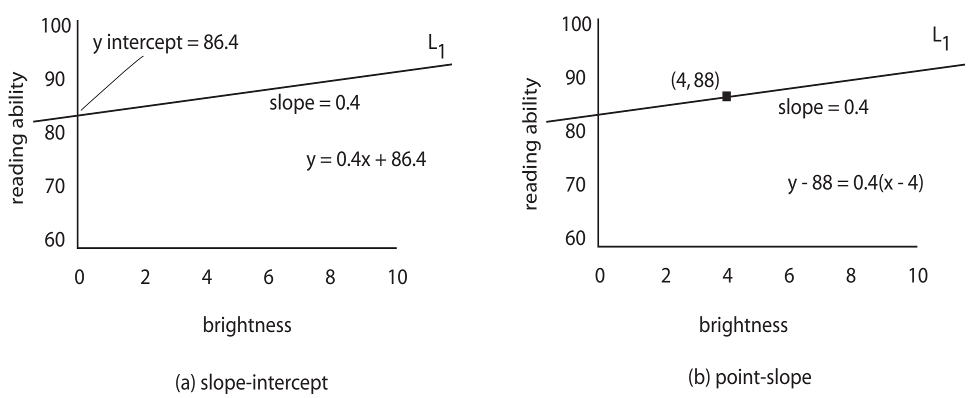


Figure 1.3 (Slope: reading ability versus brightness)

(a) *Slope–Intercept (figure (a)).*
If slope $m = 0.4$ and y–intercept $b = 86.4$, $y = mx + b =$

i. $y = 0.4x - 86.4$

ii. $y = 86.4x + 0.4$

iii. $y = 0.4x + 86.4$

(b) *Slope–Intercept.*
If slope $m = -0.2$ and y–intercept $b = 55$, $y = mx + b =$

i. $y = 55x - 0.2$
ii. $y = -0.2x + 55$
iii. $y = -0.2x - 0.2$

If $x = 5$, $y = -0.2x + 55 = -0.2(5) + 55 =$ **51** / **54** / **56**
If $x = -0.1$, $y = -0.2x + 55 = -0.2(-0.1) + 55 =$ **53.22** / **54.98** / **55.02**.
x is **dependent** / **independent** variable
and y is **dependent** / **independent** variable.

(c) *Point–Slope (figure (b)).*
Equation of line that passes point $(x_1, y_1) = (4, 88)$ with slope $m = 0.4$ is $y - y_1 = m(x - x_1) =$

i. $(x - 4) = 0.4\,(y - 88)$
ii. $y - 88 = 0.4\,(x - 88)$
iii. $y - 88 = 0.4(x - 4)$

or $y - 88 = 0.4x - 1.6$ or $y = 0.4x + 86.4$.

(d) *Point–Slope.*
Equation of line that passes through point $(4, 70)$ with slope $m = -0.2$ is $y - y_1 = m(x - x_1) =$ (circle none, one or more)

i. $y - 70 = -0.2(x - 4)$
ii. $y = -0.2x + 70.8$
iii. $y - 4 = -0.2(x - 70)$

(e) *Point–Slope.* Equation of line that passes through points (-2,3) and (5,8)

i. $y - 8 = \frac{5}{7}(x - 5)$
ii. $y + 8 = \frac{5}{7}(x - 5)$
iii. $y - 8 = \frac{6}{7}(x - 5)$
iv. $y - 8 = \frac{5}{7}(x + 5)$

[Hint: calculate slope first, $m = \frac{\triangle y}{\triangle x} = \frac{8-3}{5-(-2)} = \frac{5}{7}$ then use point–slope equation.]

3. *More equations of lines.*

(a) *Equation, Function*: Equation $y = 0.4x + 86.4$ is also function

i. $f(x) = 86.4x + 0.4$
ii. $f(x) = 0.4x + 86.4$
iii. $f(x) = 0.4x - 86.4$

(b) *General Function.*
Equation $y = 0.4x + 86.4$ is also general function

i. $Ax + By = C$, where $A = -0.4$, $B = -1$ and $C = 86.4$
ii. $Ax + By = C$, where $A = 0.4$, $B = 1$ and $C = 86.4$
iii. $Ax + By = C$, where $A = -0.4$, $B = 1$ and $C = 86.4$

(c) *General Function.*
Equation $y = \frac{3}{4}x - 2$ is also general function
i. $Ax + By = C$, where $A = \frac{3}{4}$, $B = 1$ and $C = 2$
ii. $Ax + By = C$, where $A = -\frac{3}{4}$, $B = -1$ and $C = 2$
iii. $Ax + By = C$, where $A = -\frac{3}{4}$, $B = 1$ and $C = -2$

(d) *Horizontal and Vertical Lines.*
Horizontal line that passes through point (0,6) is
$\boldsymbol{x = 6}$
$\boldsymbol{y = 6}$

(e) *Horizontal and Vertical Lines.*
Vertical line that passes through point (4,0) is
$\boldsymbol{x = 4}$
$\boldsymbol{y = 4}$

(f) *Horizontal and Vertical Lines.*
Equation $y = 2$ is **horizontal** / **vertical** line.
Equation $x = 2$ is **horizontal** / **vertical** line.

(g) *Horizontal and Vertical Lines.*
Slope of a vertical line **undefined** / **zero**.
Slope of a horizontal line **undefined** / **zero**.

1.3 Linear Modeling

Many real-world situations can be modeled by linear models. In business, short-run *total costs* are given by sum of *variable costs* (which vary according to production) and *fixed costs*. Total costs can often be represented by *linear cost function*:

$$C(x) = mx + b$$

where m is *marginal cost* (or *direct cost per item*), mx is variable cost and b is *fixed cost*. Related to this, *average cost* is given by $\frac{C(x)}{x}$. Also, if P is *purchase price*, $D = \frac{N}{n}$ is *annual depreciation* where N is *net cost of item* and n is number of years of useful life of item, then, after x years, remaining *depreciated value* is

$$V(x) = P - Dx$$

Exercise 1.3 (Applications of Linear Functions)

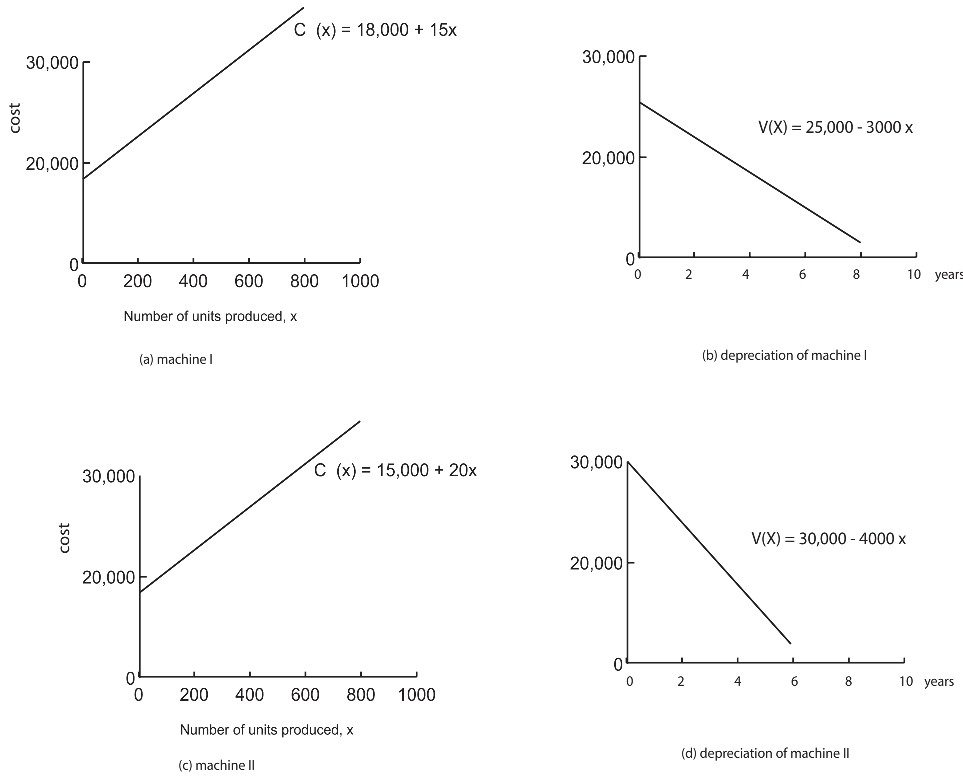


Figure 1.4 (Two cost functions and two depreciation functions)

1. *Machine costs (figure (a)).* Monthly fixed costs of using machine I are \$18,000. Marginal costs of manufacturing one widget using machine I is \$15.

   (a) Linear cost function in terms of x widgets is

      i. $C(x) = 15x + 15000$

      ii. $C(x) = 18000x + 15$

      iii. $C(x) = 15x + 18000$

   (b) Total cost of 300 widgets is
   $C(300) = 15(300) + 18000 = \mathbf{22,000 \ / \ 22,500 \ / \ 23,000}$

   (c) Additional cost of making 301st widget: **\$15 / \$16 / \$17**

   (d) Average cost per widget of making 300 widgets:
   $\frac{C(300)}{300} = \frac{15(300)+18000}{300} = \mathbf{50 \ / \ 75 \ / \ 100}$

2. *Depreciation of machine I (figure (b)).* Machine I is initial valued at \$25,000 and is depreciated by straight-line method over eight years, with a salvage of \$1,000.

   (a) How much is depreciated each year? $\frac{25000-1000}{8} = \mathbf{1,000 \ / \ 2,000 \ / \ 3,000}$

   (b) Linear depreciation function in terms of x years is

i. $V(x) = 15x + 15000$
ii. $V(x) = 18000x + 15$
iii. $V(x) = 25,000 - 3000x$

3. *More machine costs (figure (c)).* Monthly fixed costs of using machine II are \$15,000. Marginal costs of manufacturing one widget using machine II is \$20.

   (a) Linear cost function in terms of x widgets is

   i. $C(x) = 20x + 15000$
   ii. $C(x) = 18000x + 15$
   iii. $C(x) = 15x + 18000$

   (b) Total cost of 300 widgets is
   $C(300) = 20(300) + 15000 = \mathbf{21,000}$ / $\mathbf{22,500}$ / $\mathbf{23,000}$

   (c) Additional cost of making 301st widget: **\$15** / **\$17** / **\$20**

   (d) Average cost per widget of making 300 widgets:
   $\frac{C(300)}{300} = \frac{20(300)+15000}{300} = \mathbf{60}$ / $\mathbf{70}$ / $\mathbf{80}$

4. *Depreciation of machine II (figure (d)).* Machine II is initial valued at \$30,000 and is depreciated by straight-line method over six years, with a salvage of \$6,000.

   (a) How much is depreciated each year? $\frac{30000-6000}{6} = \mathbf{3,000}$ / $\mathbf{4,000}$ / $\mathbf{5,000}$

   (b) Linear depreciation function in terms of x years is

   i. $V(x) = 15x + 15000$
   ii. $V(x) = 30,000 - 4000x$
   iii. $V(x) = 25,000 - 3000x$

1.4 Two Lines: Relating the Geometry to the Equations

We look at solving a system of two linear equations in two unknowns. In particular, we look at economic applications where we look for an *equilibrium point* between revenue and cost, for an equilibrium quantity and price. Three possible solutions exist: one point, no point (inconsistent solution) or infinite point (dependent, identity solution) intersection. Methods of solution include methods of substitution and elimination, as well as Gauss-Jordan elimination method.

Exercise 1.4 (Two Lines: Relating the Geometry to the Equations)

1. *Cost and revenue function for machine.*

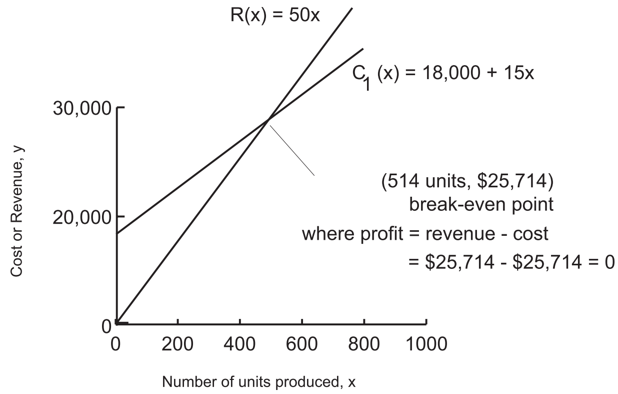


Figure 1.5 (Cost and revenue function for machine)

On one hand, monthly fixed cost of using a machine is $18,000, with variable costs of manufacturing one unit of product at $15. On other hand, each unit of product sells for $50. Determine equilibrium point, where revenue equals costs.

(a) *Using TI-84+ to geometrically find equilibrium (break-even) point.*
Quantity of units and corresponding cost/revenue where revenue equals costs is (quantity, cost/revenue)
$= (x, y) \approx$ **(514, $25, 714) / (515, $25, 714) / (516, $25, 714)**

First clear previous plots.

Enter $18000 + 15x$ beside $Y_1 =$ and $50X$ beside $Y_2 =$.

Enter domain: Press WINDOW, set 0, 1000, 1, 0, 50000, 1, 1.

Graph: Press ZOOM, ZoomFit.

Determine intersection: 2nd CALC, intersect, ENTER to First curve? and ENTER to Second curve?, arrow close to intersection, ENTER, and intersection is X = 514.28.., Y = 25714.28...

(b) *Using algebra to find equilibrium point.*

i. Cost and revenue functions for machine are
$\boldsymbol{C(x) = 18000 + 15x, R(x) = 20x}$
$\boldsymbol{C(x) = 15000 + 20x, R(x) = 15x}$
$\boldsymbol{C(x) = 18000 + 15x, R(x) = 50x}$

ii. Break-even occurs at intersection of cost and revenue

$$C(x) = R(x)$$

or,

$$18000 + 15x = 50x,$$

so $35x = 18000$ and $x = \frac{18000}{35} \approx$ **500 / 514.3 / 525.4** units
where $C(514.3) = 18000 + 15(514.3) \approx$ **$24, 714 / $25, 714**
so equilibrium is **(514, $25, 714) / (515, $25, 714)**

2. *Intersection: supply, demand and equilibrium for a market of vacuum cleaners.*

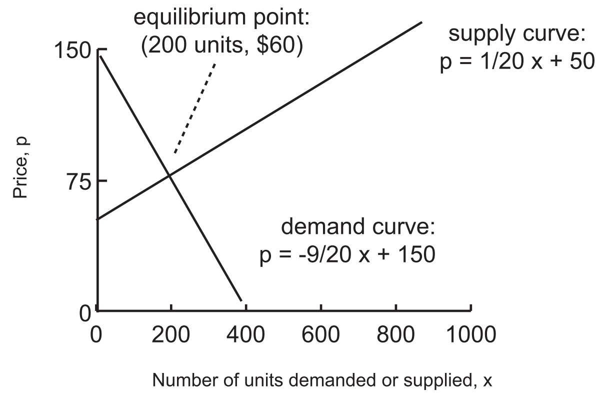


Figure 1.6 (Supply, demand and equilibrium for vacuum cleaners)

Determine equilibrium point, where supply and demand equal one another.

(a) *Using TI-84+ to geometrically find equilibrium point.*
Quantity, price where supply equals demand is
$(x, y) =$ **($60, 200)** / **(200, $60)** / **(260, $60)**

First clear previous plots.

Enter $\left(\frac{1}{20}\right) x + 50$ beside $Y_1 =$ and $-\left(\frac{9}{20}\right) x + 150$ beside $Y_2 =$.

Enter domain: Press WINDOW, set 0, 1000, 1, 0, 150, 1, 1.

Graph: Press Graph.

Determine intersection: 2nd CALC, intersect, ENTER to First curve? and ENTER to Second curve?, arrow close to intersection, ENTER, and intersection is X = 200, Y = 60.

(b) *Using algebra to find equilibrium point.*

i. Supply and demand functions for vacuum cleaners are
$\boldsymbol{p = \left(\frac{1}{20}\right) x + 30, \quad p = -\left(\frac{9}{20}\right) x + 150}$
$\boldsymbol{p = \left(\frac{1}{20}\right) x + 40, \quad p = -\left(\frac{9}{20}\right) x + 150}$
$\boldsymbol{p = \left(\frac{1}{20}\right) x + 50, \quad p = -\left(\frac{9}{20}\right) x + 150}$

ii. Break-even occurs at intersection of supply and demand

$$\left(\frac{1}{20}\right) x + 50 = -\left(\frac{9}{20}\right) x + 150,$$

so $0.5x = 100$ and $x = \frac{100}{0.5} =$ **100** / **200** / **300** units
where $p = \left(\frac{1}{20}\right)(200) + 50 =$ **$50** / **$60**
so equilibrium is **(200, $50)** / **(200, $60)**

3. *Equations and corner points of shaded regions.*

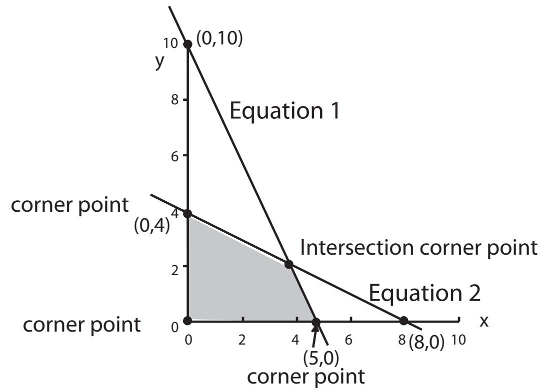


Figure 1.7 (Equations and corner points of shaded region)

(a) Equation 1 passes through y-intercept $(x, y) = (0, 10)$ and x-intercept $(x, y) = (5, 0)$ and so has slope $m = \frac{10-0}{0-5} = -2$ and so
$y - y_1 = m(x - x_1)$ or $y - 10 = -2(x - 0)$ or
$\mathbf{2x + y = 10}$
$\mathbf{-2x + y = 10}$
$\mathbf{2x + y = -10}$

(b) Equation 2 passes through y-intercept $(x, y) = (0, 4)$ and x-intercept $(x, y) = (8, 0)$ and so has slope $m = \frac{4-0}{0-8} = -0.5$ and so
$y - y_1 = m(x - x_1)$ or $y - 4 = -0.5(x - 0)$ or
$\mathbf{2x + y = 8}$
$\mathbf{x + 2y = 8}$
$\mathbf{2x + y = -8}$

(c) Corner point intersection of two equations,

$$\begin{aligned} 2x + y &= 10 \\ x + 2y &= 8 \end{aligned}$$

is, since $y = 10 - 2x$ and $2y = 8 - x$ or $y = 4 - 0.5x$, so

$$10 - 2x = 4 - 0.5x,$$

so $1.5x = 6$ and $x = \frac{6}{1.5} = \mathbf{2}\ /\ \mathbf{3}\ /\ \mathbf{4}$
where $y = 10 - 2x = 10 - 2(4) = \mathbf{0}\ /\ \mathbf{2}$
so $(x, y) = \mathbf{(0, 1)}\ /\ \mathbf{(2, 2)}\ /\ \mathbf{(4, 2)}$

4. *Intersection of lines: one, none or infinity of points.*

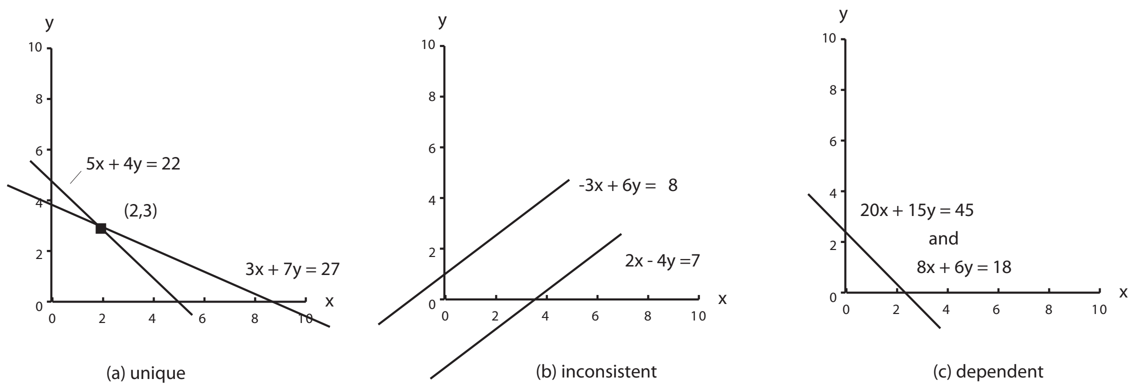


Figure 1.8 (Intersection of lines: one, none or infinity of points)

Two lines intersect

- at one point (they would *not* be parallel to one another)
- at no point (they are parallel and distinct)
- along a *line* (an infinity) of points (they are parallel and coincident)

(a) *Intersect at one point.*

$$\begin{aligned} 5x + 4y &= 22 \\ 3x + 7y &= 27 \end{aligned}$$

is, since $4y = 22 - 5x$, or $y = \frac{22}{4} - \frac{5}{4}x$ and $7y = 27 - 3x$ or $y = \frac{27}{7} - \frac{3}{7}x$, so

$$\frac{22}{4} - \frac{5}{4}x = \frac{27}{7} - \frac{3}{7}x,$$

so $\frac{23}{28}x = \frac{23}{14}$ and $x = \frac{28}{14} = \mathbf{2}$ / $\mathbf{3}$ / $\mathbf{4}$

Use calculator: $\frac{22}{4} - \frac{27}{7} = 1.64..$ then MATH ENTER ENTER for $\frac{23}{14}$; similar for $\frac{23}{28}$

where $y = \frac{22}{4} - \frac{5}{4}x = \frac{22}{4} - \frac{5}{4}(2) = \frac{27}{7} - \frac{3}{7}x = \frac{27}{7} - \frac{3}{7}(2) = \mathbf{1}$ / $\mathbf{3}$

so $(x, y) = \mathbf{(0, 1)}$ / $\mathbf{(2, 2)}$ / $\mathbf{(2, 3)}$

(b) *Intersect at no point.*

$$\begin{aligned} -3x + 6y &= 8 \\ 2x - 4y &= 7 \end{aligned}$$

is, since $6y = 8 + 3x$, or $y = \frac{8}{6} + \frac{3}{6}x$ and $4y = -7 + 2x$ or $y = -\frac{7}{4} + \frac{2}{4}x$, so

$$\frac{8}{6} + \frac{3}{6}x = -\frac{7}{4} + \frac{2}{4}x,$$

so $\frac{37}{12} = 0x$ and $x = \mathbf{0}$ / $\mathbf{3}$ / $\mathbf{4}$ / **huh?**

Use calculator: $\frac{8}{6} + \frac{7}{4} = 3.083...$ then MATH ENTER ENTER for $\frac{37}{12}$

so $(x, y) = \left(\mathbf{0, \frac{8}{6}}\right)$ / $\left(\mathbf{0, -\frac{7}{4}}\right)$ / **inconsistent (no solution)**

(c) *Intersect at infinity of points.*

$$\begin{aligned} 20x + 15y &= 45 \\ 8x + 6y &= 18 \end{aligned}$$

is, since $15y = 45 - 20x$, or $y = 3 - \frac{20}{15}x$ and $6y = 18 - 8x$ or $y = 3 - \frac{8}{6}x$, so

$$3 - \frac{20}{15}x = 3 - \frac{8}{6}x,$$

or $0x = 0$
so $(x, y) =$**(0, 1)** / **(2, 2)** / **identity (infinity of points)**

1.5 Regression and Correlation

Not covered