Chapter 8

Basic Concepts of Probability

In this chapter, we look at the basic properties of probability.

8.1 Sample Spaces with Equally Likely Outcomes

Definitions related to sample spaces with equally likely outcomes listed below.

- *Relative frequency*[1]:
  Repeatedly sampling, sample relative frequency (proportion) will approach and stay close to expected population probability.

- *Empirical probability*:
  Probability event (E) in experiment approximated by

  $$P(E) \approx \text{frequency of } E \div \text{number of trials of experiment.}$$

- *Theoretical probability*:
  Probability event (E) in experiment, assuming equally likely outcomes,

  $$P(E) = \text{number of ways } E \text{ occurs} \div \text{number outcomes in experiment.}$$

- *Sample space, S.*
  List of all possible outcomes (or simple events) of experiment.

- *(Compound) event, E; simple event.*
  (Compound) event is subset of all possible outcomes of experiment $E \subseteq S$; simple event if subset consists of one outcome.

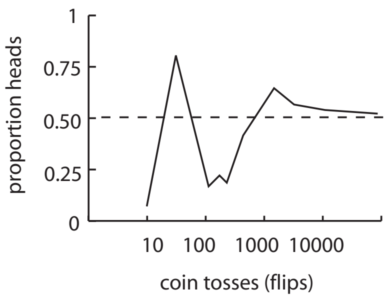
- *Probability experiment.*
  Process which results in sample space where each outcome is assigned a chance of occurrence.

[1] Often called law of large numbers.

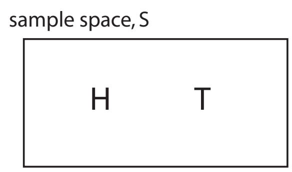
Exercise 8.1 (Sample Spaces with Equally Likely Outcomes)

1. *Terminology.*

   (a) *Coin tossing.*



coin	frequency	empirical probability
heads	5065	5065/10000 = 0.5065
tails	4935	4935/10000 = 0.4935



coin	number outcomes	classical probability
heads	1	1/2 = 0.5
tails	1	1/2 = 0.5

Figure 8.1 (Terminology: coin tossing)

i. **True** / **False**. As number of coin tosses increase, proportion of total tosses which are heads will approach and stay close to expected *probability* of tossing a head.

ii. *Approximating probability with empirical approach*[2].
Since 5065 tosses of 10000 tosses are heads we approximate probability of tossing a head by $P(H) \approx \frac{5065}{10000} = \mathbf{0.4935}$ / $\mathbf{0.5}$ / $\mathbf{0.5065}$.

iii. *Calculating probability with theoretical method*[3].
Since a coin can be tossed only as a head (H) or tail (T) and *assuming equally* likely outcomes, $P(H) = \frac{1}{2} = \mathbf{0.4935}$ / $\mathbf{0.5}$ / $\mathbf{0.5065}$.

iv. *Theoretical method: experiment.*
Flipping a coin is a *probability experiment*. It is generally *un*known, when flipping a coin, whether coin comes up heads (Hs) or tails (Ts). However, it is known there are only two possible outcomes (choose one) **{H, T}** / **{H, H}** / **{T, T}** .

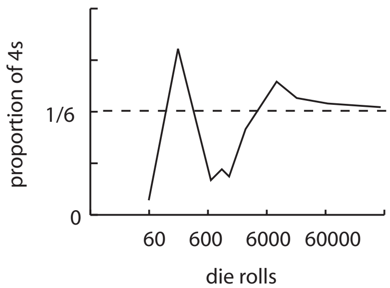
v. *Theoretical method: sample space*[4].
Sample space for flipping a coin is $S = \{\text{H, T}\}$. Sample space is (choose one) **set** / **subset** / **element** of all possible outcomes.

vi. *Theoretical method: event, simple event.*
Flipping a head {H} is an example of an *event*, E. An event is a

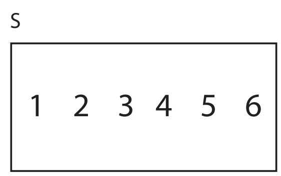
[2] Accuracy of empirical method depends on how well we can consistently flip the coin.
[3] Accuracy of theoretical method depends on accuracy of equally likely outcomes assumption.
[4] Displayed as a *Venn diagram* in figure.

(choose one) **set** / **subset** / **element** of all possible outcomes. Since only one outcome, {H}, this event is a *simple* event, $E = e_1 = \{H\}$

(b) *Die rolling.*



die	frequency	empirical probability
1	9965	9965/60000 = 0.166
2	10035	10035/60000 = 0.167
3	10100	10100/60000 = 0.168
4	9900	9900/60000 = 0.165
5	10200	10200/60000 = 0.170
6	9800	9800/60000 = 0.163



die	number outcomes	classical probability
1	1	1/6
2	1	1/6
3	1	1/6
4	1	1/6
5	1	1/6
6	1	1/6

Figure 8.2 (Terminology: die rolling)

i. **True** / **False**. As number of die rolls increase, proportion of total rolls which are 4s will approach and stay close to expected probability of rolling a 4.

ii. *Approximating probability with empirical approach.*
Since 9900 tosses of 60000 rolls are 4s we approximate probability of rolling a 4 by $P(4) \approx \frac{9900}{60000} =$ (choose one) **0.163** / **0.164** / **0.165**.

iii. *Calculating probability with theoretical method.*
Since a die can be rolled either 1, 2, 3, 4, 5 or 6 and assuming equally likely outcomes, $P(4) = \frac{1}{6} \approx$ (choose one) **0.163** / **0.165** / **0.167**.

iv. *Theoretical method: experiment.*
Die rolling is *probability experiment*. Value to be rolled unknown, but six possible outcomes (choose one) **known** / **unknown**.

v. *Theoretical method: sample space.*
Sample space is (choose one) **{1, 2, 3, 4, 5}** / **{1, 2, 3, 4, 5, 6}** .

vi. *Theoretical method: event, simple event.*
Examples of events are
(choose one or *more*!) **{1}** / **{1, 2}** / **{1, 2, 3, 6}** .

2. *Sample space: rolling two dice.*

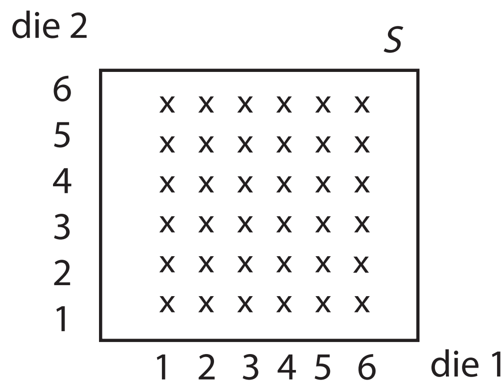


Figure 8.3 (Rolling dice)

(a) If outcomes equally likely, chance any one outcome occurs: $\mathbf{\frac{1}{10}}$ / $\mathbf{\frac{1}{30}}$ / $\mathbf{\frac{1}{36}}$

(b) Outcome of rolling two ones, (1,1), has **no** / **one** / **two** sixes.
Outcome (6,1) has **no** / **one** / **two** sixes.
Number of outcomes in event E, "no sixes": **1** / **6** / **11** / **25**.
Probability event E occurs, $P(E) = \mathbf{\frac{1}{36}}$ / $\mathbf{\frac{6}{36}}$ / $\mathbf{\frac{11}{36}}$ / $\mathbf{\frac{25}{36}}$.

(c) *Sum* of outcome (5,2) is **1** / **2** / **7**. Sum of outcome (6,1) is **1** / **2** / **7**.
Number of outcomes in event F, "sum of dice is 7": **1** / **6** / **11** / **25**.
Probability event F occurs, $P(F) = \mathbf{\frac{1}{36}}$ / $\mathbf{\frac{6}{36}}$ / $\mathbf{\frac{11}{36}}$ / $\mathbf{\frac{25}{36}}$.

3. *Counting and probability: Personal Identification Number (PIN).*

(a) If equally likely, chance 4-digit PIN drawn from barrel begins with 3

$$\frac{1 \times 10 \times 10 \times 10}{10 \times 10 \times 10 \times 10} =$$

(circle one or more) $\mathbf{\frac{1000}{10000}}$ / **0.1** / **10%**.

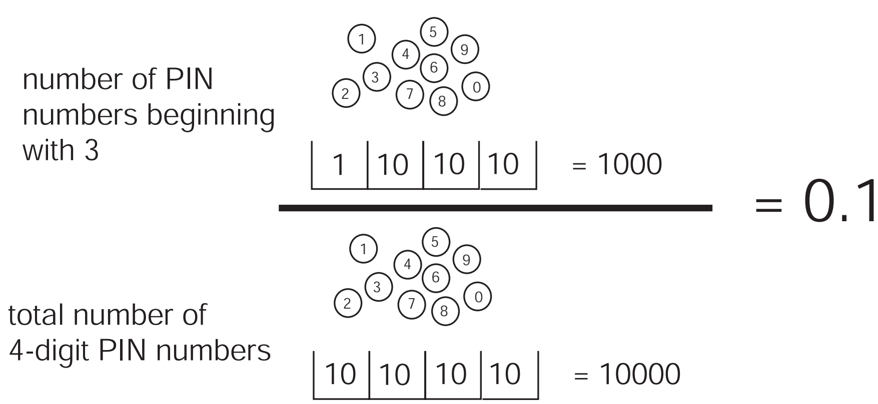


Figure 8.4 (Probability and counting: PIN)

(b) Chance 4–digit PIN drawn from a barrel has exactly three 3s is
$P(3\ threes) = \frac{\mathbf{36}}{\mathbf{10000}}\ /\ \frac{\mathbf{2916}}{\mathbf{10000}}\ /\ \frac{\mathbf{1}}{\mathbf{10000}} = 0.0036$ or 0.36%
($1 \times 1 \times 1 \times 9 = 9$, or $1 \times 1 \times 9 \times 1 = 9$ or $1 \times 9 \times 1 \times 1 = 9$ or $9 \times 1 \times 1 \times 1 = 9$ then sum.)

(c) Chance 4–digit PIN drawn from a barrel has exactly one 3 is
$P(one\ three) = \frac{\mathbf{36}}{\mathbf{10000}}\ /\ \frac{\mathbf{2916}}{\mathbf{10000}}\ /\ \frac{\mathbf{1}}{\mathbf{10000}} = 0.2916$ or 29.16%
($9 \times 9 \times 9 \times 1 = 729$, or $9 \times 9 \times 1 \times 9 = 729$ or $9 \times 1 \times 9 \times 9 = 729$ or $1 \times 9 \times 9 \times 9 = 729$ then sum.)

(d) Chance 4–digit PIN drawn from a barrel is number 1234 is
$P(1234) = \frac{\mathbf{36}}{\mathbf{10000}}\ /\ \frac{\mathbf{2916}}{\mathbf{10000}}\ /\ \frac{\mathbf{1}}{\mathbf{10000}} = 0.0001$ or 0.01%
This is *unusual* because $P(1234) = 0.0001$ is small, smaller than 0.05.

4. *Permutations in probability: parking cars.*

(a) Since $5 \times 4 \times 3 = 60$ permutations for 3 of five cars, A, B, C, D and E, to park in 3 parking spots, and, furthermore, $1 \times 4 \times 3 = 12$ ways for car A to park in first spot (and two other cars park in other two spots), chance car A parks in first spot is $\frac{12}{60} = \mathbf{0.1}\ /\ \mathbf{0.2}\ /\ \mathbf{0.3}$ or 20%.

(b) Chance car A in first spot if 4 of 5 cars park in 4 parking spots, is
$\frac{\mathbf{1\times4\times3\times2}}{\mathbf{5\times4\times3\times2}}\ /\ \frac{\mathbf{5\times4\times3\times2}}{\mathbf{5\times4\times3\times2}}\ /\ \frac{\mathbf{1\times3\times2\times1}}{\mathbf{5\times4\times3\times2}} = \frac{1\times P(4,3)}{P(5,4)} = 0.20$ or 20%
Calculator: (4 nPr 3) / (5 nPr 4) where type MATH PRB for nPr.

(c) Chance car A in first spot if 4 of 7 cars park in 4 parking spots, is
$\frac{\mathbf{1\times7\times6\times5}}{\mathbf{5\times4\times3\times2}}\ /\ \frac{\mathbf{1\times6\times5\times4}}{\mathbf{7\times6\times5\times4}}\ /\ \frac{\mathbf{1\times4\times3\times2}}{\mathbf{7\times6\times5\times4}} = \frac{1\times P(6,3)}{P(7,4)} \approx 0.1429$ or 14.29%
Calculator: (6 nPr 3) / (7 nPr 4) where type MATH PRB for nPr.

(d) Chance A in 1st spot and B in 2nd spot if 4 of 6 cars park in 4 spots
$\frac{\mathbf{1\times1\times6\times5}}{\mathbf{6\times5\times4\times3}}\ /\ \frac{\mathbf{1\times1\times4\times3}}{\mathbf{6\times5\times4\times3}}\ /\ \frac{\mathbf{1\times5\times4\times3}}{\mathbf{6\times5\times4\times3}} = \frac{1\times1\times P(4,2)}{P(6,4)} \approx 0.0333$ or 3.33%
Calculator: (4 nPr 2) / (6 nPr 4) where type MATH PRB for nPr.
This is *unusual* because 3.33% is small, smaller than 5%.

5. *Combinations in probability.*

(a) *Cards.* Determine chance of dealing 4 (of 52) playing cards with 3 jacks.

i. Total number of ways of dealing four cards is
$\boldsymbol{C(49,3)}\ /\ \boldsymbol{C(52,4)}\ /\ \boldsymbol{C(52,2)}\ /\ \boldsymbol{C(52,1)} = 270,725$
Calculator: 52 nCr 4 where type MATH PRB for nCr.

ii. Ways of dealing 4 cards with 3 jacks (and 1 non-jack card) is
$\boldsymbol{C(4,3) \times C(48,1)}\ /\ \boldsymbol{C(4,0) \times C(48,1)} = 192$
Calculator: (4 nCr 3) × (48 nCr 1) where type MATH PRB for nCr.

iii. So, probability of dealing four cards with three jacks is
$\frac{\boldsymbol{C(4,3)\times C(48,1)}}{\boldsymbol{C(52,2)}}\ /\ \frac{\boldsymbol{C(4,3)\times C(48,1)}}{\boldsymbol{C(52,4)}}\ /\ \frac{\boldsymbol{C(4,0)\times C(48,1)}}{\boldsymbol{C(52,4)}} =$ teensy weensy
Calculator: (4 nCr 3) × (48 nCr 1) / (52 nCr 4) where type MATH PRB for nCr.

(b) *Televisions.* Seven television tubes chosen from shipment of 240 television tubes (of which 15 defective). Determine chance 4 of chosen 7 defective.

i. Total number of ways of choosing seven televisions is
$\boldsymbol{C(240,4)}$ / $\boldsymbol{C(240,5)}$ / $\boldsymbol{C(240,6)}$ / $\boldsymbol{C(240,7)}$
Calculator: 240 nCr 7 where type MATH PRB for nCr.

ii. Ways 4 defective (and 3 nondefective) televisions chosen is
$\boldsymbol{C(15,4) \times C(225,2)}$ / $\boldsymbol{C(15,4) \times C(225,3)}$
Calculator: (15 nCr 4) × (225 nCr 3) where type MATH PRB for nCr.

iii. So probability 4 of chosen 7 defective
$\boldsymbol{\frac{C(15,4)\times C(225,3)}{C(240,4)}}$ / $\boldsymbol{\frac{C(15,3)\times C(225,3)}{C(240,7)}}$ / $\boldsymbol{\frac{C(15,4)\times C(225,3)}{C(240,7)}} \approx 0.0003069$
Calculator: (15 nCr 4) × (225 nCr 3) / (240 nCr 7) where type MATH PRB for nCr.

(c) *Poker Hand.* Determine probability dealt two pair in a 5–card poker hand.

i. The total number of ways of dealing 5 cards is
$\boldsymbol{C(52,2)}$ / $\boldsymbol{C(52,3)}$ / $\boldsymbol{C(52,4)}$ / $\boldsymbol{C(52,5)}$
Calculator: 52 nCr 5 where type MATH PRB for nCr.

ii. Ways of dealing *particular* pair of *twos* and then pair of *threes* is
$\boldsymbol{C(4,2)C(4,2)C(44,1)}$
$\boldsymbol{C(4,3)C(4,3)C(44,1)}$
$\boldsymbol{C(4,4)C(4,4)C(44,1)}$
Calculator: (4 nCr 2) × (4 nCr 2) × (44 nCr 1) where type MATH PRB for nCr.

iii. Since $C(13,2)$ pair choices, ways of picking *any* two pair is
$\boldsymbol{C(13,0) \times C(4,2)C(4,2)C(44,1)}$
$\boldsymbol{C(13,1) \times C(4,2)C(4,2)C(44,1)}$
$\boldsymbol{C(13,2) \times C(4,2)C(4,2)C(44,1)}$
Calculator: (13 nCr 2) × (4 nCr 2) × (4 nCr 2) × (44 nCr 1) where type MATH PRB for nCr.

iv. The probability that two pair is dealt is
$\boldsymbol{\frac{C(13,0)\times C(4,2)C(4,2)C(44,1)}{C(52,5)}}$
$\boldsymbol{\frac{C(13,1)\times C(4,2)C(4,2)C(44,1)}{C(52,5)}}$
$\boldsymbol{\frac{C(13,2)\times C(4,2)C(4,2)C(44,1)}{C(52,5)}}$
Calculator: (13 nCr 2) × (4 nCr 2) × (4 nCr 2) × (44 nCr 1) / (52 nCr 5).

6. *Married couples sitting in line.* What is chance five married couples are arranged in row of ten seats where each married couple is seated together?

(a) Total arrangements of 10 people is
7! / **8!** / **9!** / **10!**
Calculator: 10! where type MATH PRB for "!".

(b) Arrangements of 5 couples is
4! / **5!** / **6!** / **7!**
Calculator: 5! where type MATH PRB for "!".

(c) Man/wife arrangements within five couples: $\mathbf{2}$ / $\mathbf{2^2}$ / $\mathbf{2^4}$ / $\mathbf{2^5} = 32$
For example, MW,MW,MW,MW,MW or MW,MW,MW,MW,WM or ... WM,WM,WM,WM,WM.

(d) So, arrangements where five married couples seated together
$2 \times 5!$ / $2^2 \times 5!$ / $2^4 \times 5!$ / $2^5 \times 5!$

(e) So, chance where five married couples seated together
$\frac{2\times 5!}{10!}$ / $\frac{2^2\times 5!}{10!}$ / $\frac{2^3\times 5!}{10!}$ / $\frac{2^5\times 5!}{10!}$

8.2 Outcomes With Unequal Probability; Odds

Rules of probability, outcomes with unequally probability, are discussed.

- Probability of any event, E, must be between 0 and 1: $0 \leq P(E) \leq 1$.
- Sum of probability of all outcomes equals 1; for sample space $S = \{O_1, O_2, \ldots, O_n\}$,

$$P(O_1) + P(O_2) + \cdots + P(O_n) = p_1 + p_2 + \cdots + p_n = 1.$$

- $P(\emptyset) = 0$

On the one hand, *odds "a to b" for event E* converted into probabilities

$$P(E) = \frac{a}{a+b}, \quad P(E') = \frac{b}{a+b}.$$

On the other hand, probability $P(E)$ converted into *odds "a to b" for event E*

$$\frac{P(E)}{P(E')} = \frac{a}{b}, \quad \text{or "}a\text{ to }b\text{"}.$$

and *odds against event E* by $\frac{P(E')}{P(E)}$.

Exercise 8.2 (Outcomes With Unequal Probability; Odds)

1. *Rules of probability: outcomes with unequal probability.*

   (a) *Flipping coin three times*

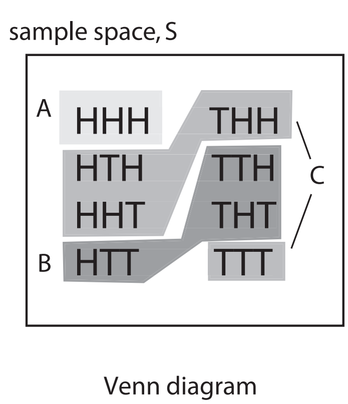


Figure 8.5 (Probability: flipping coin three times)

i. Assuming each outcome equally likely,
$P(A) = P(HHH) = \mathbf{\frac{1}{8}}$ / $\mathbf{\frac{3}{8}}$ / $\mathbf{\frac{4}{8}}$ / $\mathbf{\frac{8}{1}}$

ii. Probability flipping *exactly one head*,
$P(B) = P(HTT, THT, TTH) = \mathbf{\frac{1}{8}}$ / $\mathbf{\frac{3}{8}}$ / $\mathbf{\frac{4}{8}}$ / $\mathbf{\frac{5}{8}}$.

iii. $P(C) = \mathbf{\frac{1}{8}}$ / $\mathbf{\frac{4}{8}}$ / $\mathbf{\frac{6}{8}}$ / $\mathbf{\frac{8}{8}}$.

iv. Events A, B and C
overlap or intersect
do not overlap, are disjoint (mutually exclusive)
from one another.

v. Taken together, events A, B and C
cover or shade
do not cover or do not completely shade
entire sample space.

vi. Rules of probabilities **violated** / **obeyed**
since $0 \leq P(E) \leq 1$, for $E = A, B, C$ and $P(A) + P(B) + P(C) = 1$.

(b) *Box of coins.*
Box contains ten coins, with following denominations and years: five cents (Cs) (1974, 1976, 1978, 1978 and 1980), three nickels (Ns) (1974, 1978 and 1980) and two dimes (Ds) (1978 and 1981).

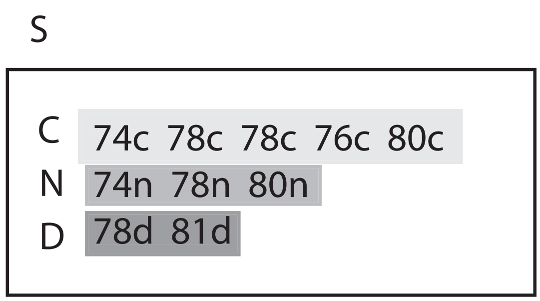


Figure 8.6 (Probability: choosing coins from a box)

i. Assuming each outcome equally likely, chance choosing a *cent* (*penny*)
$\mathbf{\frac{2}{10}}$ / $\mathbf{\frac{3}{10}}$ / $\mathbf{\frac{5}{10}}$.

ii. Probability of choosing nickel, $P(N) = \mathbf{\frac{2}{10}}$ / $\mathbf{\frac{3}{10}}$ / $\mathbf{\frac{5}{10}}$.

iii. Probability of choosing dime, $P(D) = \mathbf{\frac{2}{10}}$ / $\mathbf{\frac{3}{10}}$ / $\mathbf{\frac{5}{10}}$.

iv. Rules of probabilities **violated** / **obeyed**
since $0 \leq P(E) \leq 1$, for $E = C, N, D$ and $P(C) + P(N) + P(D) = 1$.

v. Probability choosing *quarter* is *impossible* since no quarters in box, so
$P(\text{quarter}) = \mathbf{0}$ / $\mathbf{\frac{1}{10}}$ /

vi. Probability choosing cent, nickel or dime is *certain*, so
$P(C \text{ or } N \text{ or } D) = \mathbf{0}$ / $\mathbf{\frac{1}{10}}$ / $\mathbf{\frac{2}{10}}$ / $\mathbf{1}$.

(c) *Fathers, sons and college.*
Data from a sample of 80 families in a midwestern city gives record of college attendance by fathers (F) and their oldest sons (S).

	son attended college	son did not attend college	
father attended college	18	7	25
father did not attend college	22	33	55
	40	40	80

i. Probability son, in a randomly chosen family, attended college, is
$P(S) = \mathbf{\frac{18}{40}}$ / $\mathbf{\frac{18}{25}}$ / $\mathbf{\frac{40}{80}}$ / $\mathbf{\frac{25}{80}}$.

ii. Probability father, in a randomly chosen family, attended college, is
$P(F) = \mathbf{\frac{18}{40}}$ / $\mathbf{\frac{18}{25}}$ / $\mathbf{\frac{40}{80}}$ / $\mathbf{\frac{25}{80}}$.

iii. Probability son *and* father both attended college is
$P(S \text{ and } F) = \mathbf{\frac{18}{40}}$ / $\mathbf{\frac{18}{25}}$ / $\mathbf{\frac{40}{80}}$ / $\mathbf{\frac{18}{80}}$.

iv. Probability son *or* father both attended college is
$P(S \text{ or } F) = \mathbf{\frac{45}{80}}$ / $\mathbf{\frac{46}{80}}$ / $\mathbf{\frac{47}{80}}$ / $\mathbf{\frac{48}{80}}$.

(d) *Relative-frequency distribution: epileptic seizures.*
Data from 100 epileptic people sampled at random in one year was

number seizures	number people
0	17
2	21
4	18
6	11
8	16
10	17

i. *Approximate probability distribution* given by

number seizures, x	p
0	$\frac{17}{100} = 0.17$
2	0.21
4	0.18
6	0.11
8	0.16
10	0.17

Chance of *at least* 8 seizures is **0.16** / **0.17** / **0.33**

ii. *Probability histogram (graph) of distribution.*

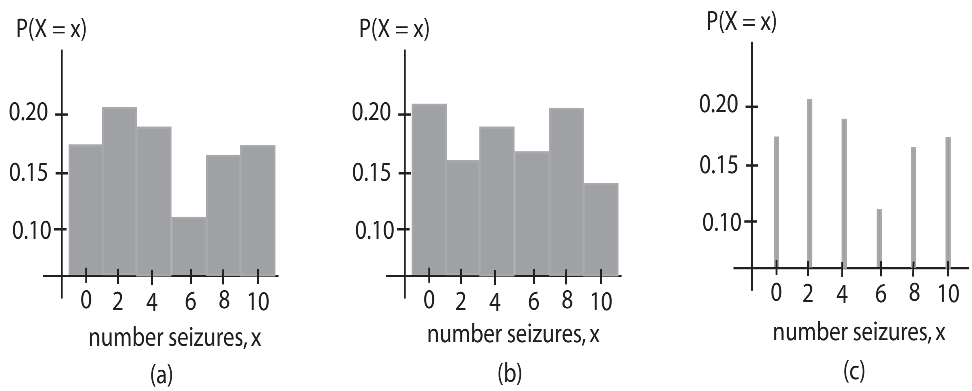


Figure 8.7 (Probability histogram: seizures)

Which of three probability histograms describes probability distribution of number of seizures? Choose *two*. **(a)** / **(b)** / **(c)**

2. *Odds and probability.*

   (a) *Odds and Tennis.*

      i. *Odds To Probability.*
      Odds 1 to 3 for Federer winning a game of tennis same as $\frac{1}{1+3} = \mathbf{\frac{1}{4}}$ / $\mathbf{\frac{1}{5}}$ / $\mathbf{\frac{1}{6}}$ chance Federer wins.

      ii. *Odds To Probability.*
      Odds 7 to 3 for Federer winning a game of tennis same as $\frac{7}{7+3} = \mathbf{\frac{3}{7}}$ / $\mathbf{\frac{7}{3}}$ / $\mathbf{\frac{7}{10}}$ / $\mathbf{\frac{3}{10}}$ chance Federer wins.

      iii. *Odds To Probability.*
      Odds 7:100 for Federer winning a game of tennis same as $\mathbf{\frac{7}{99}}$ / $\mathbf{\frac{7}{100}}$ / $\mathbf{\frac{7}{103}}$ / $\mathbf{\frac{7}{107}}$ chance Federer wins.

      iv. *Probability to Odds.*
      Chance $\frac{3}{4}$ Federer wins same as odds $\frac{3/4}{1-3/4} = \frac{3}{1}$ or

         A. 1 to 3 odds for Federer winning
         B. 3 to 1 odds for Federer winning
         C. 4 to 1 odds for Federer winning
         D. 1 to 4 odds for Federer winning

      v. *Probability to Odds.*
      Chance $\frac{2}{11}$ Federer wins same as odds $\frac{2/11}{1-2/11} = \frac{2}{9}$ or

         A. 2 to 11 odds for Federer winning
         B. 11 to 9 odds for Federer winning
         C. 2 to 9 odds for Federer winning
         D. 9 to 11 odds for Federer winning

vi. *Probability to Odds.*
Chance $\frac{9}{13}$ Federer wins same as
A. 2 to 11 odds for Federer winning
B. 9 to 4 odds for Federer winning
C. 6 to 9 odds for Federer winning
D. 9 to 11 odds for Federer winning

(b) *Odds and Rental Car Agencies.*

i. *Probability to Odds.*
60% chance of renting from car agency A same as odds $\frac{0.6}{1-0.6} = \frac{6}{4}$ or
A. 2 to 11 odds for renting from A
B. 9 to 4 odds for renting from A
C. 3 to 2 odds for renting from A
D. 4 to 6 odds for renting from A

ii. *Probability to Odds.*
37% chance of renting from agency A same as odds $\frac{0.37}{1-0.37} = \frac{0.37}{0.63}$ or
A. 2 to 11 odds for renting from A
B. 37 to 63 odds for renting from A
C. 63 to 37 odds for renting from A
D. 4 to 6 odds for renting from A

iii. *Probability to Odds.*
37% chance of renting from A same as odds $\frac{1-0.37}{0.37} = \frac{0.63}{0.37}$ or
A. 2 to 11 odds *against* renting from A
B. 37 to 63 odds *against* renting from A
C. 63 to 37 odds *against* renting from A
D. 4 to 6 odds *against* renting from A

iv. *Odds To Probability.*
Odds 7 to 93 for renting from agency A same as
$\frac{7}{7+93} = \mathbf{\frac{7}{99}}$ / $\mathbf{\frac{7}{100}}$ / $\mathbf{\frac{7}{101}}$ / $\mathbf{\frac{7}{107}}$ chance of renting from A.

8.3 Discrete Random Variables and Expected Value

Discrete distributions of random variables and their expected (mean) values discussed.

- $\sum P(x) = 1, \quad 0 \leq P(x) \leq 1$
- *expected value, mean*: $E(X) = x_1p_1 + x_2p_2 + \cdots + x_np_n$

Exercise 8.3 (Discrete Random Variables and Expected Value)

1. *Discrete or Continuous?*

   (a) **discrete** / **continuous**. Number of seizures in a year.

   (b) **discrete** / **continuous**. Waiting time at Burger King.

   (c) **discrete** / **continuous**. Temperature in Michigan City.

   (d) **discrete** / **continuous**. Number of bikes on bike rack.

   (e) **discrete** / **continuous**. Number of heads in three coin tosses.

   (f) **discrete** / **continuous**. Number of pips in roll of dice.

   (g) **discrete** / **continuous**. Patient's ages.

2. *Probability distribution, expected value $E(X)$: number of seizures, X*

number seizures, x	p
0	0.17
2	0.21
4	0.18
6	0.11
8	0.16
10	0.17

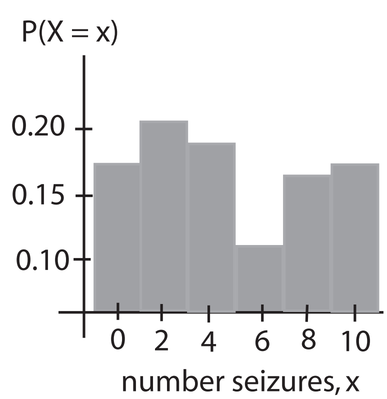


Figure 8.8 (Probability histogram: seizures)

   (a) *Various probabilities associated with number of seizures.*

      i. Chance a person has 8 epileptic seizures is
      $P(8) = P(X = 8) =$ (circle one) **0.14** / **0.15** / **0.16** / **0.17**.

      ii. Chance a person has *at most* 4 seizures is
      $P(X \leq 4) = P(0) + P(2) + P(4) =$
      (circle one) **0.17** / **0.21** / **0.56** / **0.67**.

iii. Chance a person has *at least* 4 seizures is
$P(X \geq 4) = P(4) + P(6) + P(8) + P(10) = 1 - P(X \leq 3) =$
(circle one) **0.21** / **0.38** / **0.56** / **0.62**.

iv. $P(0) + P(2) + P(4) + P(6) + P(8) + P(10) =$ **0.97** / **0.98** / **1**.

v. $P(2.1) =$ (circle one) **0** / **0.21** / **0.56** / **0.67**.

(b) *Expected value (mean) number of seizures.*

$$\begin{aligned} E(X) &= x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4 + x_5p_5 + x_6p_6 \\ &= 0(0.17) + 2(0.21) + 4(0.18) + 6(0.11) + 8(0.16) + 10(0.17) = \end{aligned}$$

(circle one) **4.32** / **4.78** / **5.50** / **5.75**.

(Type 0,2,4,6,8,10 into L_1, 17,21,18,11,16,17 into L_2; STAT CALC ENTER 2nd $L1$,$L2$; $\bar{x} = 4.78$.

Understanding mean (expected value): point of balance.

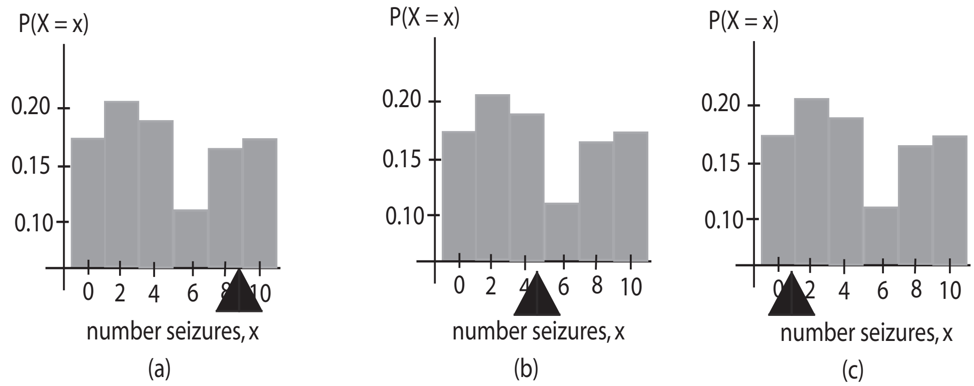


Figure 8.9 (Mean number of seizures: point of balance.)

Mean balances "weight" of probability in graph **(a)** / **(b)** / **(c)**.
In other words, mean (expected value) close to (circle one) **1** / **5** / **9**.

3. *Probability distribution, mean $E(X)$: number of bikes on bike rack, X*

number bikes, x	p
5	$\frac{1}{5} = 0.2$
6	0.2
7	0.2
8	0.2
9	0.2

For example, there is a 20% chance 6 bikes are on bike rack.

(a) *Probability histogram, number of bikes.*

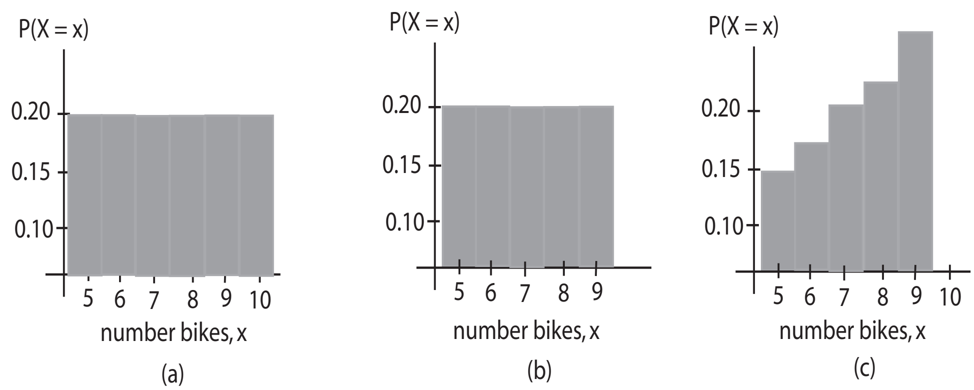


Figure 8.10 (Probability histogram of number of bikes.)

Probability histogram, number of bikes: (choose one) **(a)** / **(b)** / **(c)**.

(b) *Various probabilities associated with number of bikes.*

i. Chance bike rack has 8 bicycles is
$P(8) =$ (circle one) $\mathbf{\frac{1}{5}}$ / $\mathbf{\frac{2}{5}}$ / $\mathbf{\frac{3}{5}}$ / $\mathbf{\frac{4}{5}}$.

ii. Chance bike rack has *at most* 6 bicycles is
$P(X \leq 6) = P(5) + P(6) =$ (circle one) $\mathbf{\frac{1}{5}}$ / $\mathbf{\frac{2}{5}}$ / $\mathbf{\frac{3}{5}}$ / $\mathbf{\frac{4}{5}}$.

iii. Chance bike rack has *at least* 6 bicycles is
$P(X \geq 6) = P(6) + P(7) + P(8) + P(9) = 1 - P(X \leq 5) =$
(circle one) $\mathbf{\frac{1}{5}}$ / $\mathbf{\frac{2}{5}}$ / $\mathbf{\frac{3}{5}}$ / $\mathbf{\frac{4}{5}}$.

iv. Chance bike rack has *more than* 6 bicycles is
$P(X > 6) = P(X \geq 7) = P(7) + P(8) + P(9) = 1 - P(X \leq 6) =$
(circle one) $\mathbf{\frac{1}{5}}$ / $\mathbf{\frac{2}{5}}$ / $\mathbf{\frac{3}{5}}$ / $\mathbf{\frac{4}{5}}$.

(c) *Expected value (mean)number of bikes.*

$$\begin{aligned} E(X) &= x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4 + x_5p_5 \\ &= 5\times\frac{1}{5} + 6\times\frac{1}{5} + 7\times\frac{1}{5} + 8\times\frac{1}{5} + 9\times\frac{1}{5} \ = \end{aligned}$$

5 / **6** / **7** / **8**.

(Type x into L_1, p into L_2; STAT CALC ENTER 2nd L_1, L_2 ENTER; read $\bar{x} = 7$ or notice 7 is the balance point of probability histogram.)

4. *Probability distribution, expected value $E(X)$: roulette payoff, X*
Roulette table has 38 numbers: numbers are 1 to 36, 0 and 00. A ball is spun on a roulette wheel. After a time, ball drops into one of 38 slots which correspond to 38 numbers on roulette table.

(a) *Betting on even.* Let random variable X be payoff from \$1 bet on even: \$1 lost if ball drops on odd or 0 or 00, \$1 won (added to \$1 bet) if even.

payoff, x	p
-\$1	$\frac{20}{38}$
\$1	$\frac{18}{38}$

So expected earnings per game is

$$E(X) = -1 \times \frac{20}{38} + 1 \times \frac{18}{38}$$

which is equal to $-\mathbf{\frac{20}{38}}$ / $-\mathbf{\frac{2}{38}}$ / $\mathbf{\frac{2}{38}}$ / $\mathbf{\frac{20}{38}}$ ≈ -0.05.

(b) *Betting on a section.* Let random variable Y be payoff from a \$1 bet on a section (with 12 numbers): \$1 lost if ball drops on one of 24 numbers not in section or 0 or 00, \$2 won (added to \$1 bet) if number in section.

payoff, x	$P(x)$
-\$1	$\frac{26}{38}$
\$2	$\frac{12}{38}$

So expected earnings per game is

$$E(X) = -1 \times \frac{26}{38} + 2 \times \frac{12}{38}$$

which is equal to $-\mathbf{\frac{20}{36}}$ / $-\mathbf{\frac{2}{38}}$ / $\mathbf{\frac{2}{38}}$ / $\mathbf{\frac{20}{38}}$ ≈ -0.05.

(c) *Fair game.* A *fair game* is one where $E(X) = 0$.
Betting on an even is **unfair** / **fair**
Betting on a section is **unfair** / **fair**

5. *Flipping until head comes up.* Coin (weighted) has probability $p = 0.7$ coming up heads (and so probability $1 - p = 0.3$ coming up tails). Coin flipped until head appears or until total of 4 flips made. Let X be number of flips.

(a) $P\{X = 1\} = P\{H\} =$ **0.7** / **0.3** / **0.3(0.7)**

(b) $P\{X = 2\} = P\{TH\} =$ **0.7** / **0.3** / **0.3(0.7)**

(c) $P\{X = 3\} = P\{TTH\} =$ **0.7** / **0.3(0.7)** / $\mathbf{0.3^2(0.7)}$

(d) $P\{X = 4\} = P\{TTTT, TTTH\} =$ $\mathbf{0.3^3(0.7)}$ / $\mathbf{0.3^3(0.7 + 0.3) = 0.3^3}$

(e) *Expected number of flips until a head appears.*

$$\begin{aligned} E(X) &= x_1p_1 + x_2p_2 + x_3p_3 + x_4p_4 \\ &= 1(0.7) + 2(0.3 \cdot 0.7) + 3(0.3^2 \cdot 0.7) + 4(0.3^3 \cdot 0.7 + 0.3^4) = \end{aligned}$$

1.317 / **1.417** / **1.517**.

6. *TI–84+: Probability distributions, expected value.*

- Calculate all values of following discrete probability distribution:

$$P(X = x) = \frac{x^2 + 5}{50}, \quad x = 1, 2, 3, 4.$$

First type values of random variable in first list of STAT/EDIT:

  - STAT ENTER 1 ENTER 2 ENTER 3 ENTER 4 ENTER

Define list L_2 as equal to $\frac{x^2+5}{50}$: (push cursor up and over the line to on top of L_2!!!!)

  - (2nd L_1 x^2 + 5) / 50 ENTER

The values 0.12, 0.18, 0.28 and 0.42 will appear in list L_2. These are the four values of the discrete probability distribution. Draw a histogram of this density by pressing ZOOM 9:ZoomStat; adjust the histogram, if necessary, by altering the options given in WINDOW.

- Determine expected value of

$$P(X = x) = \frac{x^2 + 5}{50}, \quad x = 1, 2, 3, 4.$$

  (a) Type 1, 2, 3 and 4 into L_1
  (b) *Define* (push cursor up and over line to on top of L_2) $L_2 = \frac{L_1^2+5}{50}$
  (c) STAT CALC ENTER 2nd L_1, L_2 ENTER
  (d) expected value is $\mu = 3$

- Calculate all the (three) values of the following discrete probability distribution:

$$P(X = x) = \frac{C_{2,x} C_{4,3-x}}{C_{6,3}}, \quad x = 0, 1, 2.$$

First type the values of the random variable in the first list, L_1, of STAT/EDIT; in other words, type 0, 1 and 2 into L_1. (Remember to clear any previous numbers from L_1 by pushing cursor on top of L_1, then CLEAR ENTER.)

Define list L_2 (push cursor up and over the line to on top of L_2!!!!) as equal to $\frac{C_{L_1,2} C_{3-L_1,4}}{C_{3,6}}$:

  - (2 MATH PRB NCR ENTER 2nd L_1)
  - * (4 MATH PRB NCR ENTER (3 - 2nd L_1))
  - ÷ (6 MATH PRB NCR ENTER 3)

The values 0.2, 0.6 and 0.2 will appear in list L_2. These are the three values of the discrete probability distribution. Draw a histogram of this density by pressing ZOOM 9:ZoomStat; adjust the histogram by altering the options given in WINDOW.

- Determine the expected value of

$$P(X = x) = \frac{C_{2,x} C_{4,3-x}}{C_{6,3}}, \quad x = 0, 1, 2.$$

  (a) Type 0, 1 and 2 into L_1
  (b) *Define* $L_2 = \frac{(2\ nCr\ L_1) \times (4\ nCr\ (3-L_1))}{6\ nCr\ 3}$
  (c) STAT CALC ENTER 2nd L_1, L_2 ENTER
  (d) expected value is $\mu = 1$