Chapter 9

Additional Topics in Probability

In this chapter, we look at additional properties of probability.

9.1 Addition Rules for Probability; Mutually Exclusive Events

We discuss set operations and their relationship to probability calculations:

- *And*: outcomes common to events; in intersection of events
- *Or*: outcomes in union of events
- *Not (Complement)*: outcomes not in event, but in sample space
- *Complement rule*: $P(A') = 1 - P(A)$

Then we focus on "or"; in particular, we discuss the *addition rules* for probability:

- Addition rule for events A, B.

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- Addition rule for *(mutually exclusive) disjoint* events A, B, when $A \cap B = \emptyset$.

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B)$$

- Addition rule for mutually exclusive events A_1, A_2, $\ldots A_n$,

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$$

Exercise 9.1 (Addition Rules for Probability; Mutually Exclusive Events)

1. *And, Or and Not (Complement).*

   (a) *Box of tickets.*
   Box has six tickets. Each ticket has 1, 2 or 3 with one of three subscripts: a, b or c. One ticket drawn from box at random.

   $\boxed{1_a}\ \boxed{2_a}\ \boxed{1_b}\ \boxed{3_b}\ \boxed{2_c}\ \boxed{3_c}$

   Probability ticket is

   i. "1" is $P(1) = \mathbf{\frac{1}{6}}$ / $\mathbf{\frac{2}{6}}$ / $\mathbf{\frac{3}{6}}$.
   ii. "a" is $P(a) = \mathbf{\frac{1}{6}}$ / $\mathbf{\frac{2}{6}}$ / $\mathbf{\frac{3}{6}}$.
   iii. "1" *and* an "a" is $P(1 \textit{ and } a) = P(1 \cap a) = \mathbf{\frac{1}{6}}$ / $\mathbf{\frac{2}{6}}$ / $\mathbf{\frac{3}{6}}$.
   iv. "1" *or* an "a" is $P(1 \textit{ or } a) = P(1 \cup a) = \mathbf{\frac{1}{6}}$ / $\mathbf{\frac{2}{6}}$ / $\mathbf{\frac{3}{6}}$.
   v. "1" *and* a "2" is $P(1 \textit{ and } 2) = P(1 \cap 2) = \mathbf{\frac{0}{6}}$ / $\mathbf{\frac{1}{6}}$ / $\mathbf{\frac{2}{6}}$.
   vi. *not* an "a" is $P(a') = 1 - P(a) = 1 - \frac{2}{6} = \mathbf{\frac{2}{6}}$ / $\mathbf{\frac{3}{6}}$ / $\mathbf{\frac{4}{6}}$.

   (b) *Box of coins.*
   Coins are sampled at random from box.

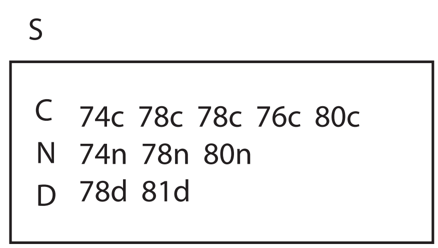

Figure 9.1 (And, or, not: box of coins)

   Chance coin is a

   i. cent is $P(C) = \mathbf{\frac{3}{10}}$ / $\mathbf{\frac{4}{10}}$ / $\mathbf{\frac{5}{10}}$ / $\mathbf{\frac{6}{10}}$.
   ii. 1978 is $P(1978) = \mathbf{\frac{3}{10}}$ / $\mathbf{\frac{4}{10}}$ / $\mathbf{\frac{5}{10}}$ / $\mathbf{\frac{6}{10}}$.
   iii. cent *and* a 1978 is $P(C \textit{ and } 1978) = P(C \cap 1978) = \mathbf{\frac{1}{10}}$ / $\mathbf{\frac{2}{10}}$ / $\mathbf{\frac{3}{10}}$ / $\mathbf{\frac{4}{10}}$.
   iv. cent *or* a nickel is $P(C \textit{ or } N) = P(C \cup N) = \mathbf{\frac{5}{10}}$ / $\mathbf{\frac{6}{10}}$ / $\mathbf{\frac{7}{10}}$ / $\mathbf{\frac{8}{10}}$.
   v. cent *or* a 1978 is $P(C \textit{ or } 1978) = P(C \cup 1978) = \mathbf{\frac{5}{10}}$ / $\mathbf{\frac{6}{10}}$ / $\mathbf{\frac{7}{10}}$ / $\mathbf{\frac{8}{10}}$.
   vi. *not* a dime is $P(D') = \mathbf{\frac{5}{10}}$ / $\mathbf{\frac{6}{10}}$ / $\mathbf{\frac{7}{10}}$ / $\mathbf{\frac{8}{10}}$.

2. *Addition rule: two events.*

   (a) *Box of coins.* Refer to box of coins figure above.

   i. Since not possible to choose a single coin that is both a cent and a nickel; in other words, choosing a cent and nickel are mutually exclusive (disjoint) events, probability of choosing cent *or* a nickel is

$$P(C \text{ or } N) = P(C \cup N) = P(C) + P(N) = \frac{5}{10} + \frac{3}{10} =$$

$\frac{\mathbf{5}}{\mathbf{10}}$ / $\frac{\mathbf{6}}{\mathbf{10}}$ / $\frac{\mathbf{7}}{\mathbf{10}}$ / $\frac{\mathbf{8}}{\mathbf{10}}$.

ii. Since it is possible to choose a single coin that is both a cent and a 1978; in other words, choosing a cent and 1978 coin is *not* mutually exclusive, probability of choosing a cent *or* a 1978 is

$$\begin{aligned} P(C \text{ or } 1978) &= P(C \cup 1978) \\ &= P(C) + P(1978) - P(C \text{ and } 1978) \\ &= \frac{5}{10} + \frac{4}{10} - \frac{2}{10} = \end{aligned}$$

$\frac{\mathbf{5}}{\mathbf{10}}$ / $\frac{\mathbf{6}}{\mathbf{10}}$ / $\frac{\mathbf{7}}{\mathbf{10}}$ / $\frac{\mathbf{8}}{\mathbf{10}}$.

iii. Since C and D mutually exclusive,

$$\begin{aligned} P(C \cup D) &= P(C) + P(D) \\ &= \frac{5}{10} + \frac{2}{10} = \end{aligned}$$

$\frac{\mathbf{5}}{\mathbf{10}}$ / $\frac{\mathbf{6}}{\mathbf{10}}$ / $\frac{\mathbf{7}}{\mathbf{10}}$ / $\frac{\mathbf{8}}{\mathbf{10}}$.

iv. Since 1978 and D *not* mutually exclusive,

$$\begin{aligned} P(1978 \cup D) &= P(1978) + P(D) - P(1978 \cap D) = \\ &= \frac{4}{10} + \frac{2}{10} - \frac{1}{10} = \end{aligned}$$

$\frac{\mathbf{5}}{\mathbf{10}}$ / $\frac{\mathbf{6}}{\mathbf{10}}$ / $\frac{\mathbf{7}}{\mathbf{10}}$ / $\frac{\mathbf{8}}{\mathbf{10}}$.

(b) *Addition rule: fathers, sons and college.*

Data from a sample of 80 families in a midwestern city gives record of college attendance by fathers (F) and their oldest sons (S).

	son attended college	son did not attend college	
father attended college	18	7	25
father did not attend college	22	33	55
	40	40	80

i. Probability son, in a randomly chosen family, attended college, is $P(S) =$ $\frac{\mathbf{18}}{\mathbf{40}}$ / $\frac{\mathbf{18}}{\mathbf{25}}$ / $\frac{\mathbf{40}}{\mathbf{80}}$ / $\frac{\mathbf{25}}{\mathbf{80}}$.

ii. Probability father, in a randomly chosen family, attended college, is $P(F) =$ $\frac{\mathbf{18}}{\mathbf{40}}$ / $\frac{\mathbf{18}}{\mathbf{25}}$ / $\frac{\mathbf{25}}{\mathbf{80}}$ / $\frac{\mathbf{55}}{\mathbf{80}}$.

iii. Probability father does *not* attend college, is $P(F') = 1 - P(F) =$ $\frac{\mathbf{18}}{\mathbf{40}}$ / $\frac{\mathbf{18}}{\mathbf{25}}$ / $\frac{\mathbf{40}}{\mathbf{80}}$ / $\frac{\mathbf{55}}{\mathbf{80}}$.

iv. Probability son attends *and* father does not attend college is $P(S \textit{ and } F') =$ $\frac{\mathbf{22}}{\mathbf{40}}$ / $\frac{\mathbf{22}}{\mathbf{55}}$ / $\frac{\mathbf{22}}{\mathbf{80}}$ / $\frac{\mathbf{18}}{\mathbf{80}}$.

v. Probability son attends *or* father does not attend college is
$P(S \text{ or } F') = P(S) + P(F') - P(S \text{ and } F') = \frac{40}{80} + \frac{55}{80} - \frac{22}{80} =$
$\mathbf{\frac{73}{80}}$ / $\mathbf{\frac{74}{80}}$ / $\mathbf{\frac{75}{80}}$ / $\mathbf{\frac{76}{80}}$.

(c) *Additive rule: dice.* In two rolls of fair die, let event A be "sum of dice is five". Let event B be event "no fours are rolled".

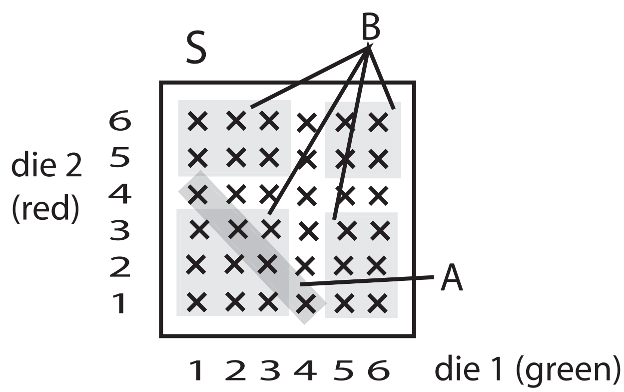

Figure 9.2 (Venn diagram for tossing two dice)

i. $P(A) =$ (i) $\mathbf{\frac{1}{36}}$ (ii) $\mathbf{\frac{2}{36}}$ (iii) $\mathbf{\frac{3}{36}}$ (iv) $\mathbf{\frac{4}{36}}$.

ii. $P(B) =$ (i) $\mathbf{\frac{24}{36}}$ (ii) $\mathbf{\frac{25}{36}}$ (iii) $\mathbf{\frac{26}{36}}$ (iv) $\mathbf{\frac{27}{36}}$.

iii. $P(A \cap B) =$ (i) $\mathbf{\frac{1}{36}}$ (ii) $\mathbf{\frac{2}{36}}$ (iii) $\mathbf{\frac{3}{36}}$ (iv) $\mathbf{\frac{4}{36}}$.

iv. So $P(A \cup B) = P(A) + P(B) - P(A \cap B) =$
(i) $\mathbf{\frac{26}{36}}$ (ii) $\mathbf{\frac{27}{36}}$ (iii) $\mathbf{\frac{28}{36}}$ (iv) $\mathbf{\frac{29}{36}}$.

v. Event A and event B
are mutually exclusive / **are not mutually exclusive**.

(d) *More addition rule.*

i. If $P(E) = \frac{3}{36}$, $P(F) = \frac{9}{36}$ and $P(E \text{ and } F) = \frac{2}{36}$.
$P(E \cup F) = P(E) + P(F) - P(E \cap F) =$ $\mathbf{\frac{10}{36}}$ / $\mathbf{\frac{11}{36}}$ / $\mathbf{\frac{12}{36}}$.

ii. Since $P(E) = 0.25$, $P(F) = 0.10$ and $P(E \text{ and } F) = 0.03$, then
$P(E \cup F) = P(E) + P(F) - P(E \cap F) =$ **0.30** / **0.32** / **0.33**.

iii. **True** / **False**. Addition rule determines chance of E "or" F.

iv. **True** / **False**. Events E and F are mutually exclusive if $P(E \cap F) = 0$.

(e) *Waiting in line.* Assuming all arrangements equally likely, what is chance line of 5 females and 4 males waiting to buy books at bookstore be arranged where females are grouped together and males are grouped together?

i. Total arrangements of 9 individuals:
$9! =$ **128, 800** / **257, 600** / **362, 880**

ii. Arrangements of 5 females and 4 males OR 4 males and 5 females:
$P((5F \cap 4M) \cup (4M \cap 5F)) = P(5F \cap 4M) + P(4M \cap 5F) = 5!4! + 4!5! =$
2880 / **5760** / **7620**

iii. So chance females grouped together and males are grouped together:
$\mathbf{\frac{2880}{128800}}$ / $\mathbf{\frac{5760}{257600}}$ / $\mathbf{\frac{5760}{362880}}$

3. *Addition rule: two or more mutually exclusive events.*

(a) *Number of seizures, X*

number seizures, x	p
0	0.17
2	0.21
4	0.18
6	0.11
8	0.16
10	0.17

i. Chance a person has *at most* 4 seizures is
$P(X \leq 4) = P(0 \cup 2 \cup 4) = P(0) + P(2) + P(4) =$
0.17 / 0.21 / 0.56 / 0.67
since number of seizures per year mutually exclusive of one another.

ii. Chance a person has *at least* 4 seizures is
$P(X \geq 4) = P(4) + P(6) + P(8) + P(10) = 1 - P(X \leq 3) =$
0.21 / 0.38 / 0.56 / 0.62.

(b) *Sum of dice, X.*

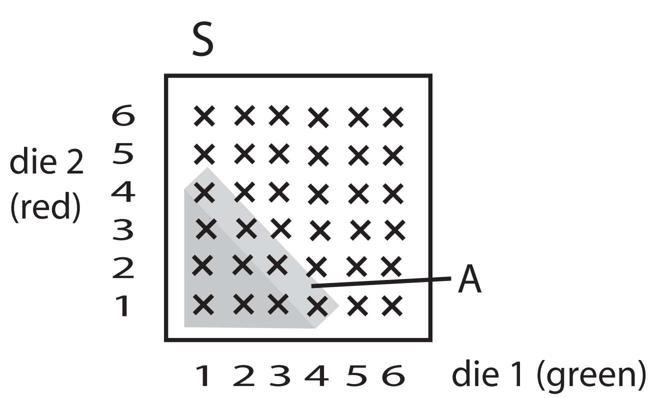

Figure 9.3 (Venn diagram for tossing two dice)

Let X be "sum of dice", so chance "sum of dice is *at most* five"
$P(X \leq 5) = P(2 \cup 3 \cup 4 \cup 5) = P(2)+P(3)+P(4)+P(5) = \frac{1}{36}+\frac{2}{36}+\frac{3}{36}+\frac{4}{36} =$
$\mathbf{\frac{8}{36}}$ / $\mathbf{\frac{9}{36}}$ / $\mathbf{\frac{10}{36}}$ / $\mathbf{\frac{11}{36}}$ since sums mutually exclusive of one another.

(c) *Committees.* From a group of 6 females and 9 males, what is chance of forming committees of 5 individuals consisting of *at most* 2 females?

i. Total combinations of choosing 5 of $6 + 9 = 15$ individuals:
$\boldsymbol{C(15, 4)}$ / $\boldsymbol{C(15, 5)}$ / $\boldsymbol{C(15, 6)}$

ii. Combinations at most 2 females from 5 individuals is either 0F and 5M OR 1F and 4M OR 2F and 3M

$P((0F \cap 5M) \cup (1F \cap 4M) \cup (2F \cap 3M)) = P(0F \cap 5M) + P(1F \cap 4M) + P(2F \cap 3M) =$

$\boldsymbol{C(6,0)C(9,5) + C(6,1)C(9,4) + C(6,2)C(9,3)}$
$\boldsymbol{C(6,1)C(9,5) + C(6,2)C(9,4) + C(6,3)C(9,3)}$

iii. So chance at most two females on committee:
$\boldsymbol{\frac{C(6,0)C(9,5)+C(6,1)C(9,4)+C(6,2)C(9,3)}{C(15,5)}}$ / $\boldsymbol{\frac{C(6,1)C(9,5)+C(6,2)C(9,4)+C(6,3)C(9,3)}{C(15,5)}}$

9.2 Conditional Probability

We discuss conditional probability:

$$P(A \mid B) = \frac{P(B \cap A)}{P(B)} = \frac{P(A \cap B)}{P(B)}.$$

Exercise 9.2 (Conditional Probability)

1. *Conditional probability and dependence: box of coins.*

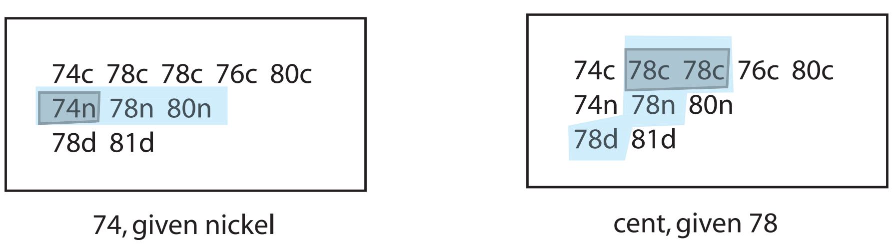

Figure 9.4 (Conditional probability: box of coins)

(a) *Choosing 1974.*
Chance a coin chosen at random from box is a 1974 coin is
$P(1974) = \boldsymbol{\frac{1}{10}}$ / $\boldsymbol{\frac{2}{10}}$ / $\boldsymbol{\frac{3}{10}}$ / $\boldsymbol{\frac{4}{10}}$.

(b) *Choosing 1974, given nickel.*
Of three coins that are nickels, **1** / **2** / **3** are 1974 coins. Given coin taken from box is a nickel, chance this coin is a 1974 nickel is
$P(1974 \mid N) = \boldsymbol{\frac{1}{3}}$ / $\boldsymbol{\frac{2}{3}}$ / $\boldsymbol{\frac{3}{3}}$ / $\boldsymbol{\frac{4}{3}}$.

(c) *Choosing 1974 depends on choosing nickel.*
Unconditional chance coin is "1974", $P(1974) = \frac{2}{10}$, is **equal** / **not equal** to conditional chance coin is "1974, given a nickel", $P(1974 \mid N) = \frac{1}{3}$. Choosing a "1974" and choosing a "nickel" are *dependent*.

(d) *Choosing cent.*
Chance of choosing a cent is $P(C) = \mathbf{\frac{2}{5}}$ / $\mathbf{\frac{5}{10}}$ / $\mathbf{\frac{2}{10}}$ / $\mathbf{\frac{2}{4}}$.

(e) *Choosing cent, given 1978.*
Of coins that are 1978s, **2** / **4** / **5** are cent coins. Given a coin is a 1978, chance this coin is a cent is $P(C \mid 1978) = \mathbf{\frac{2}{5}}$ / $\mathbf{\frac{5}{10}}$ / $\mathbf{\frac{2}{10}}$ / $\mathbf{\frac{2}{4}}$.

(f) *Choosing cent independent of choosing 1978.*
Since $P(C) = \frac{5}{10} = P(C \mid 1978) = \frac{2}{4}$, choosing a "cent" and choosing a "1978" are **independent** / **dependent** events.

(g) *In general.*
If $P(E) = P(E \mid F)$, E and F **dependent** / **independent**; otherwise, *dependent*. This is one method to determine independence/dependence.

2. *More conditional chance: fathers, sons and college.*

	son attends college, S	son does not attend college, S'	
father attended college, F	18	7	25
father did not attend college, F'	22	33	55
	40	40	80

(a) Probability son attends college given father attended college
$P(S \mid F) = \mathbf{\frac{18}{40}}$ / $\mathbf{\frac{18}{25}}$ / $\mathbf{\frac{40}{80}}$ / $\mathbf{\frac{25}{80}}$.

(b) Probability son does not attend college given a father attended college
$P(S' \mid F) = \mathbf{\frac{7}{25}}$ / $\mathbf{\frac{18}{25}}$ / $\mathbf{\frac{7}{18}}$ / $\mathbf{\frac{25}{7}}$.

(c) $P(S \mid F') = \mathbf{\frac{55}{22}}$ / $\mathbf{\frac{33}{55}}$ / $\mathbf{\frac{22}{55}}$ / $\mathbf{\frac{22}{80}}$.

(d) $P(S' \mid F') = \mathbf{\frac{22}{55}}$ / $\mathbf{\frac{33}{80}}$ / $\mathbf{\frac{22}{33}}$ / $\mathbf{\frac{33}{55}}$.

(e) $P(F \mid S) = \mathbf{\frac{18}{40}}$ / $\mathbf{\frac{18}{25}}$ / $\mathbf{\frac{18}{22}}$ / $\mathbf{\frac{25}{80}}$

(f) $P(F \mid S) = \frac{18}{40}$ **equals** / **does not equal** $P(S \mid F) = \frac{18}{25}$.

(g) *Using the formula.*

$$P(S \mid F) = \frac{P(S \cap F)}{P(F)} = \frac{P(F \cap S)}{P(F)} = \frac{18/80}{25/80} =$$

$\mathbf{\frac{18}{40}}$ / $\mathbf{\frac{18}{25}}$ / $\mathbf{\frac{40}{80}}$ / $\mathbf{\frac{25}{80}}$.

3. *And more conditional probability: coins.*

HHH	THH
HTH	TTH
HHT	THT
HTT	TTT

Figure 9.5 (Flipping three coins)

(a) Chance one head appear given at least one head appears $\frac{1}{7}$ / $\frac{3}{7}$ / $\frac{3}{8}$ / $\frac{4}{8}$.

(b) Chance one head appear given one tail appears $\frac{0}{3}$ / $\frac{1}{3}$ / $\frac{2}{3}$ / $\frac{3}{3}$.

(c) Chance at least two heads given three heads appears $\frac{1}{1}$ / $\frac{1}{2}$ / $\frac{1}{3}$ / $\frac{1}{4}$.

4. *And more conditional probability: cards.* Cards are taken out of deck at random. Let E_i represent event ith card taken from deck.

   (a) Chance first card dealt is an ace
   $P(E_1) = \frac{1}{52}$ / $\frac{4}{52}$ / $\frac{3}{51}$ / $\frac{1}{51}$.

   (b) Chance *second* card dealt is a jack, given first card dealt is an ace
   $P(E_2 \mid E_1) = \frac{1}{52}$ / $\frac{4}{50}$ / $\frac{4}{51}$ / $\frac{1}{51}$.

   (c) Chance *third* card dealt is jack, given first two are jack and ace
   $P(E_3 \mid E_1 \cap E_2) = \frac{1}{50}$ / $\frac{4}{52}$ / $\frac{3}{51}$ / $\frac{3}{50}$.

5. *Conditional chance versus unconditional chance: interviews.*
   Conditional probability calculated for event, *depends* on occurrence of another event. Seven candidates, three are females (Kathy, Susan and Jamie), interviewed for two identical jobs. Candidates are interviewed at random.

	Kathy	Susan	Jamie	Tom	Tim	Tyler	Toothy
Kathy		x	x	x	x	x	x
Susan			x	x	x	x	x
Jamie				x	x	x	x
Tom					x	x	x
Tim						x	x
Tyler							x
Toothy							

Figure 9.6 (Venn diagram for two candidates chosen)

   (a) One candidate is chosen. Chance Tom chosen given that male chosen is **conditional** / **unconditional** probability
   $P(Tom \mid male) = \frac{1}{7}$ / $\frac{2}{7}$ / $\frac{1}{4}$ / $\frac{1}{3}$

(b) One candidate is chosen. Chance Tom chosen given that female chosen is **conditional** / **unconditional** probability
$P(Tom \mid female) = \frac{1}{7}$ / $\frac{3}{7}$ / $\frac{1}{4}$ / $\frac{0}{3}$.

(c) One candidate is chosen. Chance Tom chosen is **conditional** / **unconditional** probability $P(Tom) = \frac{1}{7}$ / $\frac{3}{7}$ / $\frac{1}{4}$ / $\frac{0}{3}$.

(d) One candidate is chosen. Chance Tom *not* chosen is **conditional** / **unconditional** probability $P(Tom') = \frac{7}{6}$ / $\frac{3}{7}$ / $\frac{6}{7}$ / $\frac{0}{3}$.

(e) Two candidates are chosen. Chance Susan *and* Tom chosen is **conditional** / **unconditional** probability. Since only 21 couples[1] possible (Venn diagram), $P(Susan \cap Tom) = \frac{1}{7}$ / $\frac{2}{7}$ / $\frac{1}{4}$ / $\frac{1}{21}$.

(f) Two candidates are chosen. Chance Susan *or* Tom (or both) are chosen is **conditional** / **unconditional** probability. Since eleven (Venn diagram) couples from 21 couples possible,
$P(Susan \cup Tom) = \frac{21}{11}$ / $\frac{2}{7}$ / $\frac{1}{4}$ / $\frac{11}{21}$.

(g) Two candidates are chosen. Chance Susan and Jamie are chosen, given two females are chosen is **conditional** / **unconditional** probability, where
$P(Susan \cap Jamie \mid two\ females) = \frac{2}{7}$ / $\frac{3}{7}$ / $\frac{2}{3}$ / $\frac{1}{3}$.

(h) **True** / **False** Conditional probability essentially involves taking a subset, defined by the conditional event, of the original sample space and then calculating the probability within this subset.

9.3 Multiplication Rules for Probability: Independent Events

We discuss *multiplication rule*

$$P(A \cap B) = P(A \mid B)P(B) = P(B \mid A)P(A)$$

or, more generally,

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2 \mid A_1) \cdots P(A_n \mid A_1 \cap \cdots \cap A_{n-1}).$$

Two events are independent if

$$P(A \cap B) = P(A)P(B)$$

[1]Notice we count "Susan and Tom", but not "Tom and Susan", in our 21 couples. The *order* of couples does *not* matter to us. Also, we do *not* count "Susan and Susan" as one of our 21 couples. *If* order mattered because, for example, the first interview was for president and second for secretary, then we would count $2 \times 21 = 42$ ways of choosing our two candidates.

or, more generally, if for every subset $A_{i_1}, \ldots, A_{i_r}$ of them,

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_r})$$

and so, in this case, multiplication rule simplifies to

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2) \cdots P(A_n)$$

Exercise 9.3 (Multiplication Rules for Probability: Independent Events)

1. *Independence (sampling with replacement)*
   versus dependence (sampling without replacement): box of tickets.
   Two things are *independent* if chance for second given first are the same, no matter how first turns out; otherwise, two things are *dependent*.

1_a	2_a	1_b	3_b	2_c	3_c

   (a) *Sample with replacement: independence.*
   Two tickets are sampled *with* replacement at random from box. All *six* tickets remain in box when second ticket is drawn. Chance second ticket is a "2" *given* first ticket is a "1" is $\mathbf{\frac{1}{6}}$ / $\mathbf{\frac{2}{6}}$ / $\mathbf{\frac{3}{6}}$.

   (b) When sampling at random with replacement, chance second ticket of two drawn from box is "2", no matter what the first, is *always* $\mathbf{\frac{1}{6}}$ / $\mathbf{\frac{2}{6}}$ / $\mathbf{\frac{3}{6}}$.

   (c) *Sample without replacement: dependence.*
   Two tickets are sampled with*out* replacement at random from box. Only five tickets remain in box when second ticket is drawn. Chance second ticket is a "2" *given* first ticket is a "1" is $\mathbf{\frac{1}{5}}$ / $\mathbf{\frac{2}{5}}$ / $\mathbf{\frac{3}{5}}$.

   (d) **True** / **False** When sampling at random with*out* replacement, chance second ticket of two drawn from box is any given number *depends* on number drawn on first ticket.

   (e) When sampling at random with*out* replacement, draws are
   independent / **dependent** of one another;
   with replacement, draws are independent.

2. *Multiplication rule: box of tickets.*

1_a	2_a	1_b	3_b	2_c	3_c

   (a) Two tickets are sampled with replacement from box: tickets are independent of one another. Chance first ticket is "1" *and* second ticket is "a" is[2]
   $P(1 \textit{ and } a) = P(1) \cdot P(a) = \mathbf{\frac{1}{6}} \times \mathbf{\frac{1}{6}}$ / $\mathbf{\frac{2}{6}} \times \mathbf{\frac{1}{6}}$ / $\mathbf{\frac{2}{6}} \times \mathbf{\frac{2}{6}} = \frac{1}{9}$.
   This is an example of (a special case of) the multiplication rule.

[2] *Order* of tickets matters here. *If* you did not see how two tickets were picked from box, it would not be clear whether "1" or "a" was first or second ticket and so, in this case,
$P(1 \textit{ and } a) + P(a \textit{ and } 1) = P(1) \cdot P(a) + P(a) \cdot P(1) = 2 \times \frac{2}{6} \times \frac{2}{6}$.

(b) **True / False**
Since "1" and "a" are *in*dependent, *then* $P(1 \textit{ and } a) = P(1) \cdot P(a)$.
If "1" and "a" had been *dependent*, *then* $P(1 \textit{ and } a) \neq P(1) \cdot P(a)$.

(c) Three tickets are sampled with replacement from box. Chance first ticket is "1" *and* second ticket is "3" *and* third ticket is "3" is
$P(1 \textit{ and } 3 \textit{ and } 3) = P(1) \cdot P(3) \cdot P(3) = \mathbf{\frac{2}{6}}$ / $\mathbf{\frac{2}{6} \times \frac{2}{6}}$ / $\mathbf{\frac{2}{6} \times \frac{2}{6} \times \frac{2}{6}} = \frac{1}{27}$.

(d) Three tickets are sampled with replacement from box. Chance all three tickets are "3"s is[3] $\mathbf{\frac{2}{6}}$ / $\mathbf{\frac{2}{6} \times \frac{2}{6}}$ / $\mathbf{\frac{2}{6} \times \frac{2}{6} \times \frac{2}{6}} = \frac{1}{27}$.

(e) Three tickets are sampled with replacement from box. Chance *at least* one of the three tickets is a "3" is either

- chance one a "3" *or* two are "3"s *or* three are "3"s, **OR**
- one minus chance *none* of three tickets are "3"s,

$1 - P(\text{no } 3s) = \mathbf{1 - \frac{4}{6}}$ / $\mathbf{1 - \frac{4}{6} \times \frac{4}{6}}$ / $\mathbf{1 - \frac{4}{6} \times \frac{4}{6} \times \frac{4}{6}} = \frac{19}{27}$.

3. *Independence versus dependence: fathers, sons and college.*

	son attends college	son does not attend college	
father attended college	18	7	25
father did not attend college	22	33	55
	40	40	80

(a) Probability son, in a randomly chosen family, attends college, is
$P(S) = \mathbf{\frac{18}{40}}$ / $\mathbf{\frac{18}{25}}$ / $\mathbf{\frac{40}{80}}$ / $\mathbf{\frac{25}{80}}$.

(b) Probability father, in a randomly chosen family, attended college, is
$P(F) = \mathbf{\frac{18}{40}}$ / $\mathbf{\frac{18}{25}}$ / $\mathbf{\frac{40}{80}}$ / $\mathbf{\frac{25}{80}}$.

(c) Probability son attends college *and* father attended college is
$P(S \cap F) = \mathbf{\frac{18}{40}}$ / $\mathbf{\frac{18}{25}}$ / $\mathbf{\frac{40}{80}}$ / $\mathbf{\frac{18}{80}}$.

(d) Since $\frac{18}{80} \neq \left(\frac{40}{80}\right) \times \left(\frac{25}{80}\right)$ or $0.225 \neq 0.15625$; in other words,
$P(S \cap F) \neq P(S) \times P(F)$,
event "son attends college"
is independent of / depends on event
"father attended college".

(e) **True / False** If events A and B are *in*dependent then
$P(A \cap B) = P(A) \cdot P(B)$.
If events A and B are dependent then

[3]Order of the tickets does *not* matter here, whether you saw in what order the tickets were chosen or not, since all three tickets are the same: all "3"s. The answer will always be $\frac{2}{6} \times \frac{2}{6} \times \frac{2}{6}$ in this case.

$P(A \cap B) \neq P(A) \cdot P(B)$.
This is another method to determine independence/dependence.

4. *Multiplication rule.* A deck is shuffled and three cards are dealt.

   (a) Chance first card dealt is an ace is
   $P(ace) = \mathbf{\frac{1}{52}}$ / $\mathbf{\frac{4}{52}}$ / $\mathbf{\frac{3}{51}}$ / $\mathbf{\frac{1}{51}}$.

   (b) Chance *second* card dealt is a jack, given first card dealt is an ace, is
   $P(jack \mid ace) = \mathbf{\frac{1}{52}}$ / $\mathbf{\frac{4}{50}}$ / $\mathbf{\frac{4}{51}}$ / $\mathbf{\frac{1}{51}}$.

   (c) Probability first card is an ace and second card is a jack is
   $P(ace \cap jack) = P(ace) \cdot P(jack \mid ace) = \mathbf{\frac{1}{52}} \times \mathbf{\frac{3}{51}}$ / $\mathbf{\frac{4}{52}} \times \mathbf{\frac{4}{51}}$ / $\mathbf{\frac{4}{51}}$ / $\mathbf{\frac{1}{51}}$.
   This is an example of multiplication rule.

   (d) Probability *third* card dealt is a jack, conditional on first two cards dealt are a jack and an ace, is $P(jack \mid (ace \cap jack)) = \mathbf{\frac{1}{50}}$ / $\mathbf{\frac{4}{52}}$ / $\mathbf{\frac{3}{51}}$ / $\mathbf{\frac{3}{50}}$.

   (e) Probability of an ace, jack and another jack is
   $P(ace \cap jack \cap jack) = P(ace) \cdot P(jack \mid ace) \cdot P(jack \mid (ace \cap jack)) =$
   $\mathbf{\frac{4}{52}} \times \mathbf{\frac{3}{51}} \times \mathbf{\frac{2}{50}}$
   $\mathbf{\frac{4}{52}} \times \mathbf{\frac{4}{51}} \times \mathbf{\frac{3}{50}}$
   $\mathbf{\frac{4}{50}} \times \mathbf{\frac{3}{49}} \times \mathbf{\frac{2}{48}}$
   This is another example of general multiplication rule[4].

5. *More multiplication rule: renting cars.* A firm rents 60% of cars from A and 40% from B. Of cars from A, 9% needed a tune–up; of cars from B, 20% needed a tune–up. A car is chosen at random.

[4] The order of the cards *matters* here. *If* you did not see how the three cards were picked from the deck, it would not be clear which one of three possibilities occurred: ace, jack, jack or jack, ace, jack or jack, jack, ace. In this case, the answer would be $3 \times \frac{4}{52} \times \frac{4}{51} \times \frac{3}{50}$.

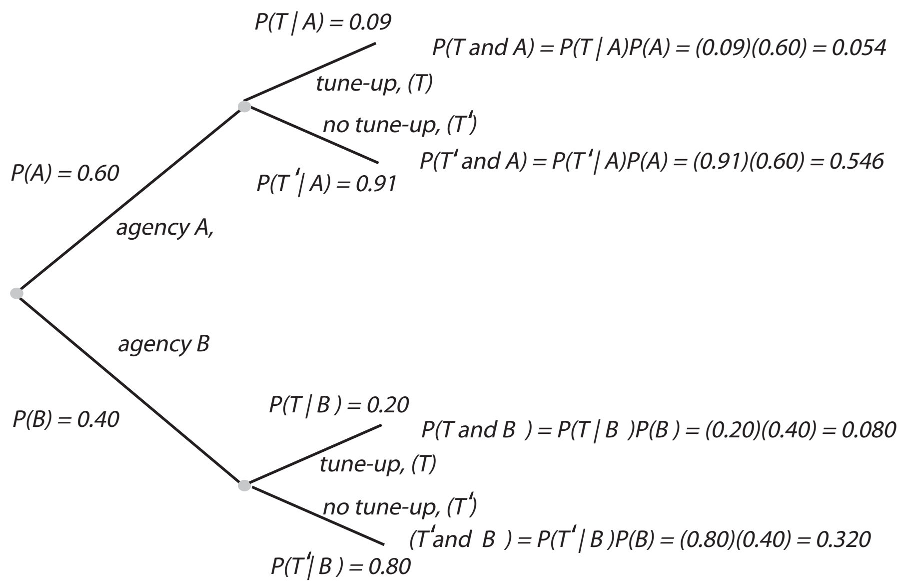

Figure 9.7 (Multiplication rule: renting cars)

(a) *Multiplication rule.*
Chance car from A: $P(A) =$ **0.40 / 0.60 / 0.80**
chance car needs tune–up, given from A: $P(T \mid A) =$ **0.09 / 0.60 / 0.91**
so chance car comes from A *and* needs tune–up:
$P(T \cap A) = P(T \mid A)P(A) = (0.09)(0.60) =$ **0.054 / 0.60 / 0.91**
Notice, uppermost path tree diagram gives $P(T \cap A) = 0.054$.

(b) *More multiplication rule.*
Chance car comes from B: $P(B) =$ **0.40 / 0.60 / 0.80**
chance car needs tune–up, given from B: $P(T \mid B) =$ **0.20 / 0.60 / 0.91**
so chance car comes from B *and* needs tune–up:
$P(T \cap B) = P(T \mid B)P(B) = (0.20)(0.40) =$ **0.080 / 0.60 / 0.91**
Notice, third path of tree diagram gives $P(T \cap B) = 0.080$.

(c) *Total probability.*
Chance car needs tune–up:

$$\begin{aligned} P(T) &= P(T \cap A) + P(T \cap B) \\ &= P(T \mid A)P(A) + P(T \mid B)P(B) \\ &= (0.09)(0.60) + (0.20)(0.40) = \end{aligned}$$

0.080 / 0.134 / 0.280

6. *Multiplication rule: gender and majors.*

	% of sophomores in this major	% who are females	% who are males
Liberal Arts	0.55	0.75	0.25
Education	0.25	0.85	0.15
Technology	0.10	0.25	0.75
Sciences	0.05	0.50	0.50
Other	0.05	0.65	0.35

(a) *Multiplication rule.*
Chance student in Technology: $P(T) =$ **0.10 / 0.20 / 0.30**
chance student female, given Technology: $P(F \mid T) =$ **0.10 / 0.25 / 0.75**
so chance student female and in Technology:
$P(F \cap T) = P(F \mid T)P(T) = (0.25)(0.10) =$ **0.025 / 0.060 / 0.075**

(b) *More multiplication rule.*
Chance student in Technology: $P(T) =$ **0.10 / 0.20 / 0.30**
chance student male, given Technology: $P(M \mid T) =$ **0.10 / 0.25 / 0.75**
so chance student male and in Technology:
$P(M \cap T) = P(M \mid T)P(T) = (0.75)(0.10) =$ **0.025 / 0.060 / 0.075**

7. *Multiplication rule when events independent: basketball.* Basketball player has $n = 10$ free throws and sinks each throw with probability $p = 0.3$. Assume each throw independent and identical to each other throw. Let event S_i represent when basketball player sinks ith throw.

(a) Chance of sinking ten throws in a row:
$P(S_1 \cap S_2 \cap \cdots \cap S_{10}) = (0.3)^{10} =$ **0.059 / 0.00059 / 0.0000059**

(b) Chance of missing ten throws in a row:
$P(S_1' \cap S_2' \cap \cdots \cap S_{10}') = (0.7)^{10} =$ **0.028 / 0.059 / 0.064**

(c) Chance of sinking first (and only first) throw of ten:
$P(S_1 \cap S_2' \cap \cdots \cap S_{10}') = (0.3)(0.7)^9 =$ **0.012 / 0.059 / 0.064**

(d) Chance of sinking second (only second) throw of ten:
$P(S_1' \cap S_2 \cap S_3' \cap \cdots \cap S_{10}') = (0.3)(0.7)^9 =$ **0.012 / 0.059 / 0.064**

(e) Number of ways of sinking only one of ten throws: $C(10, 1) =$ **1 / 5 / 10**

(f) Probability of sinking only one of ten throws:
$C(10, 1)(0.3)(0.7)^9 =$ **0.12 / 0.59 / 0.64**

8. *Independence versus disjoint events: cards.*
Independent events are different from disjoint events.
Events A and B are **disjoint / independent** if $P(A \cap F) = P(A) \cdot P(B)$.
Events A and B are **disjoint / independent** if $P(A \cap B) = 0$.

9. *Addition and multiplication rules.* **True** / **False**.
   When "and" is involved, "multiply": $P(E \cap F) = P(E) \cdot P(F|E)$.
   When "or" is involved, "add": $P(E \cup F) = P(E) + P(F) - P(E \cap F)$;