9.4 Bayes's Theorem

Bayes Theorem[5] is

$$P(A \mid E) = \frac{P(E \mid A)P(A)}{P(E \mid A)P(A) + P(E \mid B)P(B)}$$

Let $A_1, \ldots, A_n$ be mutually exclusive and exhaustive events with $P(A_i) > 0$ for all i. Then for new evidence E, where $P(E) > 0$, general form of *Bayes Theorem* is

$$P(A_i \mid E) = \frac{P(E \mid A_i)P(A_i)}{P(E \mid A_1)P(A_1) + \cdots + P(E \mid A_n)P(A_n)}$$

Exercise 9.4 (Bayes's Theorem)

1. *Renting cars.* Firm rents 60% of cars from A, 40% from B. Of cars from A, 9% need a tune–up; of cars from B, 20% need a tune–up. Cars chosen at random.

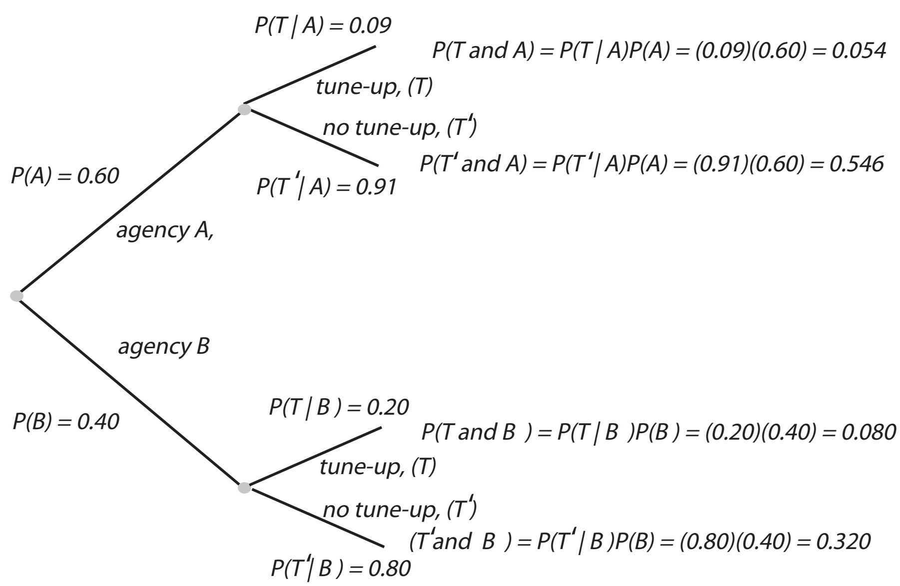

Figure 9.8 (Bayes theorem: car rental)

(a) *Chance car needs a tune-up, if car from agency A?*
$P(T \mid A) =$ **0.09 / 0.20 / 0.4**
Given in problem, look on upper branch of tree diagram.

(b) *Chance car from agency A, if car needs a tune-up?*
Chance car comes from A *and* needs tune–up:
$P(T \cap A) = P(T \mid A)P(A) = (0.09)(0.60) =$ **0.054 / 0.60 / 0.91**

[5] Sometimes called the "reverse probability" formula.

Notice, uppermost path tree diagram gives $P(T \cap A) = 0.054$.

Chance car comes from B *and* needs tune–up:
$P(T \cap B) = P(T \mid B)P(B) = (0.20)(0.40) =$ **0.080 / 0.60 / 0.91**

Notice, third path of tree diagram gives $P(T \cap B) = 0.080$.

Chance car needs tune–up:

$$\begin{aligned} P(T) &= P(T \cap A) + P(T \cap B) \\ &= P(T \mid A)P(A) + P(T \mid B)P(B) \\ &= (0.09)(0.60) + (0.20)(0.40) = \end{aligned}$$

0.080 / 0.134 / 0.280

So, chance car from agency A, if car needs a tune-up

$$\begin{aligned} P(A \mid T) &= \frac{P(T \mid A)P(A)}{P(T \mid A)P(A) + P(T \mid B)P(B)} \\ &= \frac{(0.09)(0.60)}{(0.09)(0.60) + (0.20)(0.40)} = \end{aligned}$$

0.403 / 0.597 / 0.880

(c) *Chance car from agency B, if car needs a tune-up?*

$$\begin{aligned} P(B \mid T) &= \frac{P(T \mid B)P(B)}{P(T \mid A)P(A) + P(T \mid B)P(B)} \\ &= \frac{(0.20)(0.40)}{(0.09)(0.60) + (0.20)(0.40)} = \end{aligned}$$

0.403 / 0.597 / 0.880

(d) *Chance car from agency A, if car does not need a tune-up?*

Chance car comes from A and does *not* need a tune–up:
$P(T' \cap A) = P(T' \mid A)P(A) = (0.91)(0.60) =$ **0.512 / 0.546 / 0.645**

Notice, second path tree diagram gives $P(T' \cap A) = 0.546$.

Chance car comes from B and does *not* need a tune–up:
$P(T' \cap B) = P(T' \mid B)P(B) = (0.80)(0.40) =$ **0.320 / 0.460 / 0.951**

Notice, last path of tree diagram gives $P(T' \cap B) = 0.320$.

Chance car does not need a tune–up:

$$\begin{aligned} P(T') &= P(T' \cap A) + P(T' \cap B) \\ &= P(T' \mid A)P(A) + P(T' \mid B)P(B) \\ &= (0.91)(0.60) + (0.80)(0.40) = \end{aligned}$$

0.680 / 0.734 / 0.866

So, chance car from agency A, if car does not need a tune-up

$$P(A \mid T') = \frac{P(T' \mid A)P(A)}{P(T' \mid A)P(A) + P(T' \mid B)P(B)}$$

$$= \frac{(0.91)(0.60)}{(0.91)(0.60)+(0.80)(0.40)} =$$

0.403 / 0.597 / 0.630

2. *Gender and majors.*

	% of sophomores in this major	% who are females	% who are males
Liberal Arts	0.55	0.75	0.25
Education	0.25	0.85	0.15
Technology	0.10	0.25	0.75
Sciences	0.05	0.50	0.50
Other	0.05	0.65	0.35

(a) *Chance student female, if in Technology?*
$P(F|T) =$ **0.25 / 0.55 / 0.75**
Given in table.

(b) *Chance student in Technology, if female?*
Chance student female and in Liberal Arts:
$P(F \cap L) = P(F \mid L)P(L) = (0.75)(0.55) =$ **0.025 / 0.2125 / 0.4125**
Chance student female and in Education:
$P(F \cap E) = P(F \mid E)P(E) = (0.85)(0.25) =$ **0.0325 / 0.2125 / 0.4125**
Chance student female and in Technology:
$P(F \cap T) = P(F \mid T)P(T) = (0.25)(0.10) =$ **0.025 / 0.0325 / 0.2125**
Chance student female and in Sciences:
$P(F \cap S) = P(F \mid S)P(S) = (0.50)(0.05) =$ **0.025 / 0.0325 / 0.2125**
Chance student female and in Other:
$P(F \cap O) = P(F \mid O)P(O) = (0.65)(0.05) =$ **0.025 / 0.0325 / 0.4125**
Chance student female:

$$\begin{aligned} P(F) &= P(F\cap L)+P(F\cap E)+P(F\cap T)+P(F\cap S)+P(F\cap O) \\ &= P(F|L)P(L)+P(F|E)P(E)+P(F|T)P(T)+P(F|S)P(S)+P(F|O)P(O) \\ &= (0.75)(0.55)+(0.85)(0.25)+(0.25)(0.10)+(0.50)(0.05)+(0.65)(0.05) = \end{aligned}$$

0.6045 / 0.7075 / 0.8280
So, chance student in Technology, if female

$$\begin{aligned} P(T|F) &= \frac{P(F|T)P(T)}{P(F|L)P(L)+P(F|E)P(E)+P(F|T)P(T)+P(F|S)P(S)+P(F|O)P(O)} \\ &= \frac{(0.25)(0.10)}{(0.75)(0.55)+(0.85)(0.25)+(0.25)(0.10)+(0.50)(0.05)+(0.65)(0.05)} \approx \end{aligned}$$

0.035 / 0.056 / 0.180

(c) *Chance student in Education, if female?*

$$\begin{aligned} P(E|F) &= \frac{P(F|E)P(E)}{P(F|L)P(L) + P(F|E)P(E) + P(F|T)P(T) + P(F|S)P(S) + P(F|O)P(O)} \\ &= \frac{(0.85)(0.25)}{(0.75)(0.55) + (0.85)(0.25) + (0.25)(0.10) + (0.50)(0.05) + (0.65)(0.05)} \approx \end{aligned}$$

0.30 / 0.56 / 0.68

3. *Renting cars.* Firm rents 40% of cars from A_1, 40% from A_2 and 20% from A_3. Of cars from A_1, 9% need a tune–up; of cars from A_2, 20% need a tune–up; of cars from A_3, 13% need a tune–up. Cars chosen at random.

(a) *Chance car from agency A_1, if car needs a tune-up?*
Chance car comes from A_1 and needs tune–up:
$P(T \cap A_1) = P(T \mid A_1)P(A_1) = (0.09)(0.40) =$ **0.026 / 0.036 / 0.08**
Chance car comes from A_2 and needs tune–up:
$P(T \cap A_2) = P(T \mid A_2)P(A_2) = (0.20)(0.40) =$ **0.026 / 0.036 / 0.08**
Chance car comes from A_3 and needs tune–up:
$P(T \cap A_3) = P(T \mid A_3)P(A_3) = (0.13)(0.20) =$ **0.026 / 0.036 / 0.08**
Chance car needs tune–up:

$$\begin{aligned} P(T) &= P(T \cap A_1) + P(T \cap A_2) + P(T \cap A_3) \\ &= P(T \mid A_1)P(A_1) + P(T \mid A_2)P(A_2) + P(T \mid A_3)P(A_3) \\ &= (0.09)(0.40) + (0.20)(0.40) + (0.13)(0.20) = \end{aligned}$$

0.080 / 0.142 / 0.280
So, chance car from agency A_1, if car needs a tune-up

$$\begin{aligned} P(A_1 \mid T) &= \frac{P(T \mid A_1)P(A_1)}{P(T \mid A_1)P(A_1) + P(T \mid A_2)P(A_2) + P(T \mid A_3)P(A_3)} \\ &= \frac{(0.09)(0.40)}{(0.09)(0.40) + (0.20)(0.40) + (0.13)(0.20)} = \end{aligned}$$

0.254 / 0.397 / 0.480

(b) *Chance car from agency A_2, if car needs a tune-up?*

$$\begin{aligned} P(A_2 \mid T) &= \frac{P(T \mid A_2)P(A_2)}{P(T \mid A)P(A) + P(T \mid B)P(B)} \\ &= \frac{(0.20)(0.40)}{(0.09)(0.40) + (0.20)(0.40) + (0.13)(0.20)} = \end{aligned}$$

0.303 / 0.497 / 0.563

9.5 Binomial Experiment; A Guide to Probability

Probability function for *binomial* is

$$P(x) = P(X = x) = C(n, x)p^x(1 - p)^{n-x},\ x = 0, 1, 2 \ldots, n$$

with expected value,

$$E(X) = np.$$

Probability distribution is binomial if

- n trials, where n is fixed in advance,
- trials independent of one another,
- trials have two possible outcomes: success or failure; probability of success same for each trial.

Exercise 9.5 (Binomial Experiment; A Guide to Probability)

1. *Binomial probability distribution, mean $E(X) = np$: number of cases won, X*
   Lawyer estimates she wins 40% ($p = 0.4$) of her cases. Assume each trial is independent of one another and, in general, this problem obeys conditions of a binomial experiment. Lawyer is currently involved in 10 ($n = 10$) cases. For example, there is a 4.3% chance she wins 7 of her 10 cases.

number cases won, x	$P(x)$
0	0.006
1	0.040
2	0.121
3	0.215
4	0.251
5	0.201
6	0.111
7	0.043
8	0.011
9	0.002
10	0.000

   (a) *Probability histogram, number of wins.*

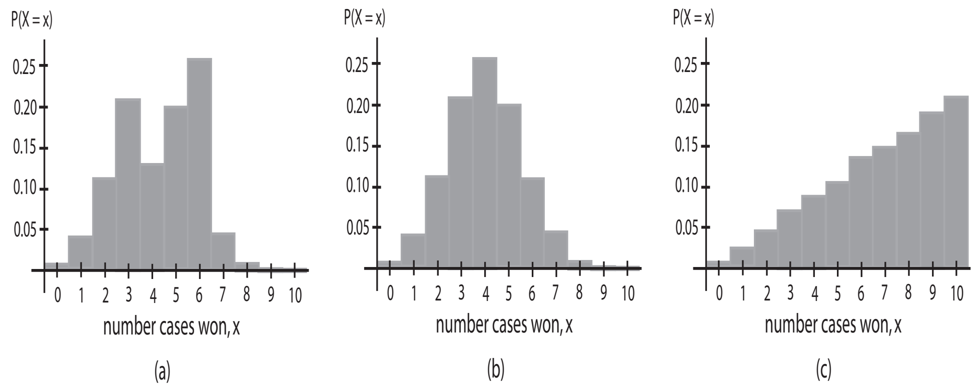

Figure 9.9 (Probability histogram of number of wins.)

Probability histogram, number of wins: (choose one) **(a)** / **(b)** / **(c)**.

(b) *Probability function, number of wins.* Choose *two!*

i. Function (a).

$$P(X = x) = \frac{x}{46}, \quad x = 0, 1, 2, \ldots, 10.$$

(Type 0,1,2 . . . , 10 into L_1. Define $L_2 = L_1 \div 46$ ENTER. Probabilities in L_2 equal to $P(x)$?)

ii. Function (b).

$$P(X = x) = \begin{cases} 0.006, & \text{if } x = 0, \\ 0.040, & \text{if } x = 1, \\ 0.121, & \text{if } x = 2, \\ \vdots & \vdots \\ 0.002, & \text{if } x = 9, \\ 0.000, & \text{if } x = 10. \end{cases}$$

iii. Function (c).

$$P(X = x) = C(n, x)0.4^x(1 - 0.4)^{10-x}, \; x = 0, 1, 2 \ldots, n$$

(Type 0,1,2 . . . 10 into L_1. Define L_2 = (10 MATH PRB nCr L_1) × 0.4L_1 × 0.6(10 − L_1) ENTER. Probabilities in L_2 equal to $P(x)$?)

(c) *Various probabilities associated with number of wins.*

i. Chance lawyer wins 8 cases is

$$P(8) = C(10, 8)p^8(1-p)^{10-8} = \frac{10!}{8!(10-8)!}(0.4)^8(1-0.4)^2 = 45{\cdot}0.4^8 0.6^2 \approx$$

0.006 / **0.011** / **0.040** / **0.121**.

(Either 45 × 0.48 × 0.62 OR
2nd DISTR binompdf(10, 0.4, 8) ENTER.)

ii. Chance lawyer has *at most* 6 wins is

$$\begin{aligned}P(X \leq 6) &= P(0)+P(1)+P(2)+P(3)+P(4)+P(5)+P(6)\\ &= C(10,0)p^0(1-p)^{10-0}+C(10,1)p^1(1-p)^{10-1}+\cdots+C(10,6)p^6(1-p)^{10-6}\\ &= \frac{10!}{0!(10-0)!}(0.4)^0(1-0.4)^{10}+\cdots+\frac{10!}{6!(10-6)!}(0.4)^6(1-0.4)^4\\ &= 1\cdot 0.4^0 0.6^{10}+10\cdot 0.4^1 0.6^9+\cdots+210\cdot 0.4^6 0.6^4 \approx\end{aligned}$$

0.834 / **0.934** / **0.945** / **0.993**.

(Long way: $1 \times 0.4\hat{}(0) \times 0.6\hat{}(10) + 10 \times 0.4\hat{}(1) \times 0.6\hat{}(9) + \cdots + 210 \times 0.4\hat{}(6) \times 0.6\hat{}(4)$ OR

Short way: 2nd DISTR binomcdf(10, 0.4, 6) ENTER; notice: *binomCdf*, not *binomPdf*!)

iii. Chance lawyer has *less than* 6 wins is
$P(X < 6) = P(X \leq 5) \approx$ **0.834** / **0.934** / **0.945** / **0.993**.

(2nd DISTR binomcdf(10, 0.4, 5) ENTER; notice: *binomCdf*, not *binomPdf*!)

(d) *Mean (expected value) number of wins.*
long way:

$$\begin{aligned}E(X) &= x_1p_1+x_2p_2+\cdots+x_{10}p_{10}\\ &= 0\times 0.006+1\times 0.040+\cdots+10\times 0.000 \approx\end{aligned}$$

3 / **4** / **5** / **6**.

(Type x into L_1, $P(x)$ into L_2; STAT CALC ENTER 2nd L_1, L_2 ENTER; read $\bar{x} = 4$.)

short way:

$$E(X) = np = 10 \times 0.4 =$$

(circle one) **3** / **4** / **5** / **6**.

(e) *Binomial experiment?* Match columns.

binomial conditions	lawyer example
(a) n trials, n is fixed in advance of experiment .	(A) $p = 0.4$ chance lawyer wins each trial.
(b) Trials have possible outcomes: success or failure.	(B) Each trial is independent of one another.
(c) Trials are independent of one another.	(C) There are $n = 10$ trials.
	(D) Trials can only be won or lost.

binomial conditions	(a)	(b)	(c)
lawyer example	(C)		

2. *Binomial: number of airplane engine failures.*
Each engine of four ($n = 4$) on an airplane fails 11% ($p = 0.11$) of the time. Assume this problem obeys conditions of a binomial experiment.

(a) Chance two engines fail is
$P(2) = C(4,2) \times (0.11)^2 \times (1-0.11)^{4-2} \approx$ **0.005** / **0.011** / **0.058**.

(2nd DISTR binompdf(4,0.11,2).)

(b) Chance three engines fail is
$P(3) = C(4,3) \times (0.11)^3 \times (1-0.11)^{4-3} \approx$ **0.005** / **0.011** / **0.040**.
(2nd DISTR binompdf(4,0.11,3).)

(c) Chance *at most* two engines fail is
$P(X \leq 2) = P(0) + P(1) + P(2) \approx$ **0.995** / **0.997** / **0.999**.
(2nd DISTR binomcdf(4,0.11,2): notice *binomcdf*, not *binompdf*!)

(d) Chance *less than* two engines fail is
$P(X < 2) = P(X \leq 1) = P(0) + P(1) \approx$ **0.938** / **0.997** / **0.999**.
(2nd DISTR binomcdf(4,0.11,1).)

(e) *Expected number of failures*
$\mu_X = np = 4(0.11) =$ **0.44** / **0.51** / **0.62**.

3. *Binomial: number of widget defects.*
Each of fourteen randomly chosen widgets are defective 21% of the time.

(a) *Binomial experiment?* Match columns.

binomial conditions	widget example
(a) n trials, n is fixed in advance of experiment.	(A) $p = 0.21$ chance widget is defective.
(b) Trials have possible outcomes: success or failure.	(B) $n = 14$ widgets chosen.
(c) Trials are independent of one another.	(C) each widget is defective or not.
	(D) each widget chosen independent of another.

binomial conditions	(a)	(b)	(c)
widget example			

(b) Chance seven widgets defective
$P(7) = C(14,7) \times (0.21)^7 \times (1-0.21)^{14-7} \approx$ **0.005** / **0.012** / **0.040**.
(2nd DISTR binompdf(14,0.21,7).)

(c) Chance *at most* ten widgets defective
$P(X \leq 10) = P(0) + P(1) + \cdots + P(10) \approx$ **0.995** / **0.997** / **0.999**.
(2nd DISTR binomcdf(14,0.21,10).)

(d) Chance *at least* ten widgets defective
$P(X \geq 10) = P(10) + P(11) + P(12) + P(13) + P(14) = 1 - P(X \leq 9) \approx$
0.000072 / **0.00072** / **0.0072**.
($1 -$ 2nd DISTR binomcdf(14,0.21,9), notice 1 - 2nd DISTR binomcdf(14,0.21,9),
not 1 - 2nd DISTR binomcdf(14,0.21,10)!)

(e) Chance *between* 7 and 10 widgets defective, *inclusive*
$P(7 \leq X \leq 10) = P(7)+P(8)+P(9)+P(10) = P(X \leq 10) - P(X \leq 6) \approx$
0.005 / **0.015** / **0.034**.
(2nd DISTR binomcdf(14,0.21,10) subtract 2nd DISTR binomcdf(14,0.21,6))

(f) *Expected number of defectives*
$E(X) = np = 14(0.21) =$
2.44 / **2.51** / **2.94**.

(g) *Probability histograms, for different chance each widget defective, p.* Identify different p for different probability histograms.

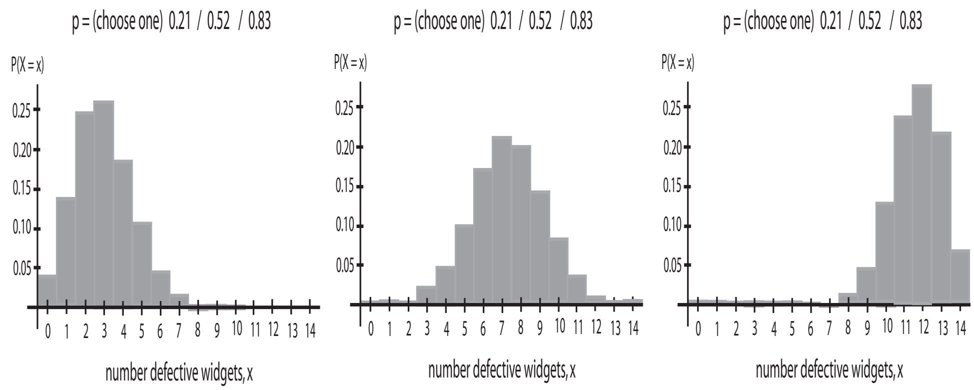

Figure 9.10 (Probability histograms: number of defective, different p.)

Notice, probability histograms for $p = 0.21$, $p = 0.52$ and $p = 0.83$ are right-skewed, (more or less) symmetric and left-skewed, respectively.

4. *Binomial: number of correct multiple choice answers.*
On a multiple–choice exam with 4 possible choices for each of 5 questions, what is probability a student gets 3 or more correct answers just by guessing?

(a) Since there are five questions,
$n =$ **3 / 4 / 5**

(b) Student wants 3 or more correct answers,
$x =$ **3 / 3, 4 / 3, 4, 5**

(c) Each question has 4 possible choices and student is choosing at random,
$p = \mathbf{\frac{1}{3}}$ / $\mathbf{\frac{1}{4}}$ / $\mathbf{\frac{1}{5}}$

(d) Chance student gets *at least* 3 correct answers just by guessing is
$P(X \geq 3) = P(3) + P(4) + P(5) = 1 - P(X \leq 2) \approx$
(choose one) **0.097 / 0.104 / 0.112**
(1 - 2nd DISTR binomcdf(5,$\frac{1}{4}$,2), not 2nd DISTR binomcdf(5,$\frac{1}{4}$,2)!)

(e) *Expected number of correct answers*
$E(X) = np = 5 \times \frac{1}{4} = \mathbf{\frac{1}{4}}$ / $\mathbf{\frac{3}{4}}$ / $\mathbf{\frac{5}{4}} = 1.25$.

5. *TI–84+: Binomial probability distribution.*

- Calculate all the values of the following binomial probability distribution:

$$C_{16,x}\left(\frac{2}{7}\right)^x\left(\frac{5}{7}\right)^{16-x}, \quad x = 0, 1, \ldots, 16$$

- One way to do this is to first type the values of the random variable, 0, 1, ..., 16, in the first list, L_1, of STAT/EDIT and then *define* (push cursor up and on top of) L_2 = (16 MATH PRB nCr ENTER L_1) $\left(\frac{2}{7}\right)^x\left(\frac{5}{7}\right)^{16-x}$ (ENTER). The values 0.00459, 0.02939, ... will appear in list L_2. These are the values of the binomial probability distribution.

- A second way to do this is to use the "binompdf" function directly. Go to the home screen, then type
  - 2nd DISTR 0:binompdf(16 , (2 ÷ 7))

  The values 0.00459, 0.02939, . . . will appear after pushing the ENTER button. You can see all of them by repeatedly pushing the "arrow right" key. These are the values of the binomial probability distribution.

  If you wanted to see just the binomial probability for $x = 2$ and $x = 5$, say, you would type:
  - 2nd DISTR 0:binompdf(16 , (2 ÷ 7) , 2nd { 2 , 5 2nd })

  The values 0.0881156 (for $P(X = 2)$) and 0.20536 (for $P(X = 5)$) will appear.

  If you wanted to see just the binomial probability for $x = 2$, you would type:
  - 2nd DISTR 0:binompdf(16 , (2 ÷ 7) , 2)

  The value 0.0881156 (for $P(X = 2)$) will appear.

- *Graphing The Binomial Distribution.* To graph the binomial distribution with $n = 10$, $p = 0.2$, type
  - WINDOW 0 10 1 -0.1 0.4 0.1 1
  - STAT ENTER, then enter 1 2 . . . in L_1 and define L_2 as 2nd DISTR 0:binompdf(10, 0.2 , L_1) ENTER
  - 2nd STAT PLOT ENTER ON ENTER pick histogram plot ENTER GRAPH

- *Binomial Cumulative Distribution Function.*
  - Calculate all the *cumulative* probability values of the following binomial probability distribution:

    $$C_{16,x}\left(\frac{2}{7}\right)^x\left(\frac{5}{7}\right)^{16-x}, \quad x = 0, 1, \ldots, 16$$

  - One way to do this is to first type the values of the random variable in the first list of STAT/EDIT and then define list L_2 as equal to the *cumulative* function for the density function above:
    * 2nd DISTR A:binomcdf(16 , (2 ÷ 7) , 2nd L_1)

    The values 0.00459, 0.03398, . . . will appear in list L_2. These are the values of the cumulative binomial distribution.
  - A second way to do this is to use the "binomcdf" function directly. Go to the home screen and type
    * 2nd DISTR A:binomcdf(16 , (2 ÷ 7))

    The values 0.00459, 0.03398, . . . will appear after pushing the ENTER button. You can see all of them by repeatedly pushing the "arrow right" key. These are the values of the binomial probability distribution. If you wanted to see just, say the cumulative binomial values for $x = 2$ and $x = 5$, say, you would type:
    * 2nd DISTR A:binomcdf(16 , (2 ÷ 7) , 2nd { 2 , 5 2nd })

    The values 0.12213 (for $P(X \leq 2)$) and 0.70598 (for $P(X \leq 5)$) will appear.