Chapter 11

Markov Chains

In this section, we look an application of mathematics which are based on a combination of matrix theory and probability theory: Markov Chains.

11.1 Markov Chains as Mathematical Models

Assume an experiment results in only one of a finite number of *states* (outcomes) and this experiment is repeated a number of times, resulting in a sequence of states over time. If each state depends only on a state one time period before (and not any previous state two or more time periods before), and where the chance, the *transition probability*, of going from one state in one time period to any state in the next time period is known, then the sequence of states is called a *Markov chain*.

Exercise 11.1 (Markov Chains as Mathematical Models)

1. *Actor's Performance.* A typical actor has found his performance on one night influences his performance on the next night.

   (a) This information can be displayed in the following *probability transition matrix*, T.

   $T =$

tomorrow → tonight ↓	good performance (1)	bad performance (2)
good performance (1)	0.85	0.15
bad performance (2)	0.60	0.40

   This transition matrix has dimension
   (circle one) **1 × 2 / 2 × 1 / 1 × 1 / 2 × 2**. It is square.

   (b) The *current time period, in time period 0*, is
   (circle one) **tonight's performance / tomorrow night's performance**,

whereas *next time period, in time period 1*, is
(circle one) **tonight's performance** / **tomorrow night's performance**.

(c) *If* the *current* state was state 1, the actor had a
(circle one) **good performance** / **bad performance** in time period 0,
whereas *if* the current state was state 2 , the actor had a
(circle one) **good performance** / **bad performance** in time period 0;
similarly, *if* the *next* state was state 1, the actor had a
(circle one) **good performance** / **bad performance** in time period 1,
whereas *if* the next state was state 2 , the actor had a
(circle one) **good performance** / **bad performance** in time period 1.

(d) **True** / **False** Both the current state and the next state could either be state 1 (good performance) or state 2 (bad performance). If this situation repeated itself for each time period; in other words, the actor could only have either a good or bad performance (and not, say, a mediocre performance) each night, this sequence of states would be an example of a *Markov* chain.

(e) A good performance followed by a good performance occurs
(circle one) **15%** / **40%** / **60%** / **85%** of the time.
whereas a good performance followed by a bad performance occurs
(circle one) **15%** / **40%** / **60%** / **85%** of the time.

(f) Notice, even though each *row* of transition matrix sums to one, for example,
sum of row 1 is (circle one) **0.15** / **0.85** / **1**
each column does not necessary add to one, for example,
sum of column 1 is (circle one) **0.15** / **0.85** / **1.45**.

(g) The transition probabilities along each row *must* sum to one because (circle none, one or more)

i. whatever tonight's performance, tomorrow's performance must be either good or bad (it cannot be mediocre, say).
ii. the transition from one of the current states must terminate in one (any one, but one) of the next states.
iii. in addition to either a good or bad performance, tomorrow's performance could also be lackluster or wooden.

(h) The transition matrix can also be displayed as a *transition diagram*.

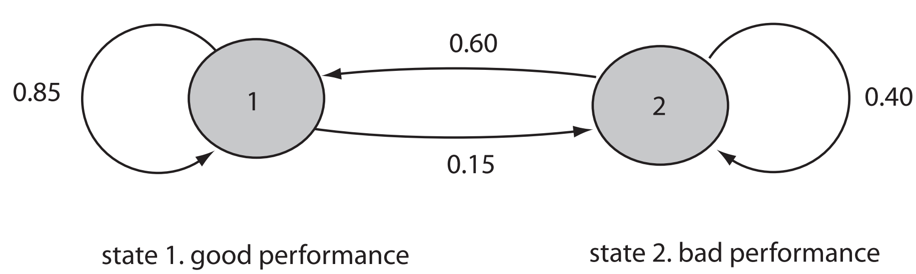

Figure 11.1 (Transition Diagram: Actor's Performance)

State 1 appears in the transition diagram as a (circle one)

i. shaded circle with a 1 in it.
ii. pointed line that starts and finishes at a shaded circle with a 1 in it.
iii. shaded circle with a 2 in it.
iv. pointed line that starts at the shaded circle with a 1 in it and finishes at the shaded circle with a 2 in it.

Transition probability 0.85 appears in the transition diagram as a (circle one)

i. shaded circle with a 1 in it.
ii. pointed line that starts and finishes at a shaded circle with a 1 in it.
iii. shaded circle with a 2 in it.
iv. pointed line that starts at the shaded circle with a 1 in it and finishes at the shaded circle with a 2 in it.

A good performance followed by a bad performance is (circle one)

i. shaded circle with a 1 in it.
ii. pointed line that starts and finishes at a shaded circle with a 1 in it.
iii. shaded circle with a 2 in it.
iv. pointed line that starts at the shaded circle with a 1 in it and finishes at the shaded circle with a 2 in it.

2. *Oatmeal or Cereal.* Fill in the transition matrix which corresponds to the following transition diagram of whether a student has oatmeal or cereal for breakfast each morning.

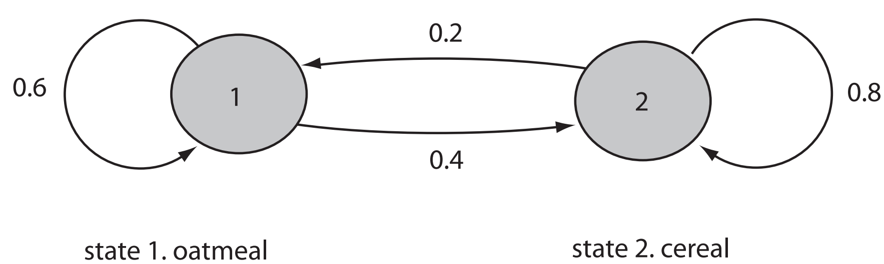

Figure 11.2 (Transition Diagram: Oatmeal or Cereal)

$T =$

tomorrow → / today ↓	oatmeal (1)	cereal (2)
oatmeal (1)	____	____
cereal (2)	____	____

3. *Car Insurance.* Although most drivers are insured with either Allstate or State Farm, some are insured with AutoSafe. Of those insured by Allstate in one year, 80% continued with Allstate in the following year, 20% switched to State Farm and, of course, 0% switched to AutoSafe. Of those insured by State Farm, 20% switched to Allstate and 65% stayed; of those insured by AutoSafe, 100% stayed with these AutoSafe.

   (a) Complete the following transition matrix.

$T =$

next year → / this year ↓	Allstate (1)	State Farm (2)	AutoSafe (3)
Allstate (1)	0.80	____	0
State Farm (2)	____	____	____
AutoSafe (3)	____	____	____

   (b) Complete the following transition diagram.

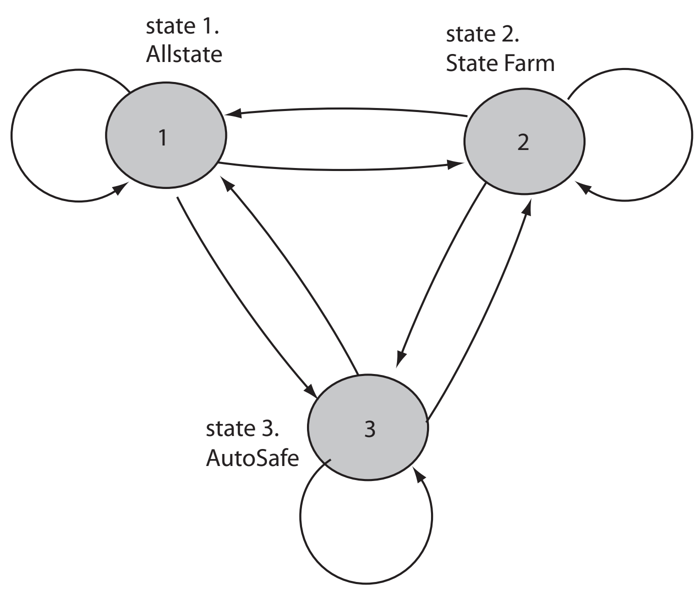

Figure 11.3 (Transition Diagram: Switching Insurance Companies)

Note: State 3 (AutoSafe) is an *absorbing* state because once in state 3, it is impossible to leave this state.

4. *Actor's performance, revisited.* Recall the following transition matrix.

$T =$

tomorrow → / tonight ↓	good performance (1)	bad performance (2)
good performance (1)	0.85	0.15
bad performance (2)	0.60	0.40

(a) What is the probability an actor gives a good performance on Tuesday night after a good performance on Monday night?
(circle one) **0.15** / **0.40** / **0.60** / **0.85**.
What is the probability of a bad performance on Tuesday night after a good performance on Monday night?
(circle one) **0.15** / **0.40** / **0.60** / **0.85**.

(b) What is the probability an actor gives a good (G) performance on *Wednesday* night, two nights after a good performance on Monday night (also let "B" be "bad")? Consider the following tree diagram.

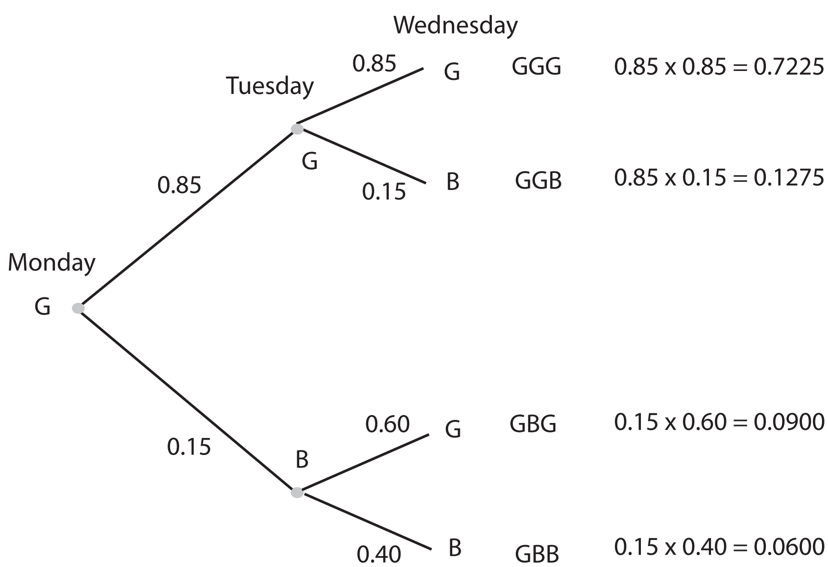

Figure 11.4 (Tree Diagram: two nights)

So chance is probability of GGG sequence plus GBG sequence
$0.7225 + 0.0900 =$ (circle one) **0.0900** / **0.7225** / **0.8125** / **0.8500**.

(c) What is the probability an actor gives a good performance on *Wednesday* night, two nights after a good performance on Monday night? *Square* the transition matrix.

$$T^2 = \begin{bmatrix} 0.85 & 0.15 \\ 0.60 & 0.40 \end{bmatrix}^2 = \begin{bmatrix} 0.8125 & 0.1875 \\ 0.75 & 0.25 \end{bmatrix}$$

=	good performance (1)	bad performance (2)
good performance (1)	0.8125	0.1875
bad performance (2)	0.75	0.25

(Calculator: Enter data in matrix [A], 2nd QUIT, then x^2 Enter.)

So chance appears in first row, first column position, or
(circle one) **0.0900** / **0.7225** / **0.8125** / **0.8500**.

(d) What is the probability an actor gives a good performance *three* nights after a good performance on Monday night? *Cube* the transition matrix.

$$T^3 = \begin{bmatrix} 0.85 & 0.15 \\ 0.60 & 0.40 \end{bmatrix}^3 = \begin{bmatrix} 0.803125 & 0.196875 \\ 0.7875 & 0.2125 \end{bmatrix}$$

(Calculator: $[A] \wedge 3$ Enter.)
Chance appears in first row, first column position, or
(circle one) **0.196875** / **0.2125** / **0.7875** / **0.803125**.

(e) What is the probability an actor gives a good performance *four* nights after a good performance on Monday night?

$$T^4 = \begin{bmatrix} 0.85 & 0.15 \\ 0.60 & 0.40 \end{bmatrix}^4 = \begin{bmatrix} 0.80078125 & 0.19921875 \\ 0.796875 & 0.203125 \end{bmatrix}$$

(Calculator: $[A] \wedge 4$ Enter.)
(circle one) **0.19921875** / **0.203125** / **0.796875** / **0.80078125**.

(f) What is the probability an actor gives a *bad* performance four nights after a good performance on Monday night?
(circle one) **0.19921875** / **0.203125** / **0.796875** / **0.80078125**.

(g) **True** / **False** Probability process starts in state S_i and ends in state S_j after n time periods is given in ith row and jth column of nth power of transition matrix, T^n.

5. *Oatmeal or cereal, regurgitated.* Recall the following transition matrix.

$T =$

tomorrow → today ↓	oatmeal (1)	cereal (2)
oatmeal (1)	0.6	0.4
cereal (2)	0.2	0.8

(a) Determine

$$T^3 = \begin{bmatrix} \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ \end{bmatrix}$$

(Calculator: Enter data in matrix [B], 2nd QUIT, then $[B] \wedge 3$ Enter.)

(b) Probability a student eats cereal three days after eating oatmeal is
(circle one) **0** / **0.216** / **0.624** / **1**.

6. *Car insurance, again.* Recall the following transition matrix.

$$T = \begin{array}{c|ccc|} \text{next year} \rightarrow & \text{Allstate (1)} & \text{State Farm (2)} & \text{AutoSafe (3)} \\ \text{this year} \downarrow & & & \\ \text{Allstate (1)} & 0.80 & 0.20 & 0 \\ \text{State Farm (2)} & 0.20 & 0.65 & 0.15 \\ \text{AutoSafe (3)} & 0 & 0 & 1 \end{array}$$

(a) Determine

$$T^3 = \begin{bmatrix} \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ & \_\_\_ \end{bmatrix}$$

(Calculator: Enter data in matrix [C], 2nd QUIT, then $[C] \wedge 3$ Enter.)

(b) Probability driver switches to AutoSafe three years after Allstate is (circle one) **0** / **0.0735** / **0.3245** / **1**.

(c) Notice

$$P_3 = P_0 \times T^3 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0.80 & 0.20 & 0 \\ 0.20 & 0.65 & 0.15 \\ 0 & 0 & 1 \end{bmatrix}^3 = \begin{bmatrix} \_\_\_ & \_\_\_ & \_\_\_ \end{bmatrix}$$

(Calculator: Enter 1, 0, 0 into matrix [D], multiple by matrix [C], [D] × [C].)

Notice probability driver switches to AutoSafe three years after Allstate is given in last position of row vector.

11.2 State Vectors

Focus in this section is on *state* vectors. Initial state vector with k states is given by

$$P_0 = \begin{bmatrix} p_1 & p_2 & \cdots & p_k \end{bmatrix}$$

where $p_1 + p_2 + \cdots + p_k = 1$.

Exercise 11.2 (State Vectors)

1. *Actor's performance.* Recall following transition matrix.

$$T = \begin{array}{c|cc|} \text{tomorrow} \rightarrow & \text{good performance (1)} & \text{bad performance (2)} \\ \text{tonight} \downarrow & & \\ \text{good performance (1)} & 0.85 & 0.15 \\ \text{bad performance (2)} & 0.60 & 0.40 \end{array}$$

(a) If there is a 70% chance of a typical actor giving a good performance Monday night (and so a 30% chance of giving a bad performance) on Monday night, then the *initial state vector* is,

$$P_0 = \begin{bmatrix} 0.7 & 0.3 \end{bmatrix}$$

and the state vector after one transition is

$$P_1 = P_0 \times T = \begin{bmatrix} 0.7 & 0.3 \end{bmatrix} \times \begin{bmatrix} 0.85 & 0.15 \\ 0.60 & 0.40 \end{bmatrix}$$

$$= \begin{bmatrix} 0.775 & 0.225 \end{bmatrix}$$

(Calculator: Enter 0.7, 0.3 into matrix [D], multiple by matrix [A], [D] × [A].)
and so the chance a typical actor gives a good performance on Tuesday night (one night after Monday night) is (circle one) **0.775 / 0.225**.

(b) Since state vector after *two* transitions is

$$P_2 = P_0 \times T^2 = \begin{bmatrix} 0.7 & 0.3 \end{bmatrix} \times \begin{bmatrix} 0.85 & 0.15 \\ 0.60 & 0.40 \end{bmatrix}^2$$

$$= \begin{bmatrix} \_\_\_\_ & \_\_\_\_ \end{bmatrix}$$

(Calculator: [D] × [A]^2.) the chance a typical actor gives a good performance *two* nights after Monday night is (circle one) **0.79375 / 0.20625**.

2. *Oatmeal or cereal, once again.* Recall the following transition matrix.

$$T = \begin{array}{lcc} \text{tomorrow} \rightarrow & \text{oatmeal (1)} & \text{cereal (2)} \\ \quad \text{today} \downarrow & & \\ \text{oatmeal (1)} & 0.6 & 0.4 \\ \text{cereal (2)} & 0.2 & 0.8 \end{array}$$

(a) If there is a 45% chance a student eats oatmeal (and so a 55% chance of eating cereal) today, then the initial state vector is,

$$P_0 = \begin{bmatrix} 0.45 & 0.55 \end{bmatrix}$$

and the state vector after one transition is

$$P_1 = P_0 \times T = \begin{bmatrix} 0.45 & 0.55 \end{bmatrix} \times \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}$$

$$= \begin{bmatrix} \_\_\_\_ & \_\_\_\_ \end{bmatrix}$$

(Calculator: Enter 0.45, 0.55 into matrix [D], multiple by matrix [B], [D] × [B].)
and so the chance a student eats oatmeal tomorrow is
(circle one) **0.38 / 0.73 / 0.162 / 0.838**.

(b) Since state vector after *two* transitions is

$$P_2 = P_0 \times T^2 = \begin{bmatrix} 0.45 & 0.55 \end{bmatrix} \times \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}^2$$

$$= \begin{bmatrix} \_\_\_\_\_ & \_\_\_\_\_ \end{bmatrix}$$

(Calculator: [D] × [B]^2.)

and so the chance a student eats oatmeal in two days time is (circle one) **0.352** / **0.73** / **0.162** / **0.838**.

(c) If there is a 25% chance a student eats oatmeal (and so a 75% chance of eating cereal) today, then the initial state vector is,

$$P_0 = \begin{bmatrix} \_\_\_\_\_ & \_\_\_\_\_ \end{bmatrix}$$

and the state vector after one transition is

$$P_1 = P_0 \times T = \begin{bmatrix} 0.25 & 0.75 \end{bmatrix} \times \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}$$

$$= \begin{bmatrix} \_\_\_\_\_ & \_\_\_\_\_ \end{bmatrix}$$

(Calculator: Enter 0.25, 0.75 into matrix [D], multiple by matrix [B], [D] × [B].)

and so the chance a student eats oatmeal tomorrow is (circle one) **0.30** / **0.85** / **0.27** / **0.73**.

3. *Car insurance, again.* Recall the following transition matrix.

$T =$

next year → this year ↓	Allstate (1)	State Farm (2)	AutoSafe (3)
Allstate (1)	0.80	0.20	0
State Farm (2)	0.20	0.65	0
AutoSafe (3)	0	0	1

(a) If there is a 15%, 45% and 40% chance a driver initially has Allstate, State Farm and AutoSafe, respectively, then the initial state vector is,

$$P_0 = \begin{bmatrix} 0.15 & 0.45 & 0.40 \end{bmatrix}$$

and the state vector after *three* transitions is

$$P_3 = P_0 \times T^3 = \begin{bmatrix} 0.15 & 0.45 & 0.40 \end{bmatrix} \times \begin{bmatrix} 0.80 & 0.20 & 0 \\ 0.20 & 0.65 & 0.15 \\ 0 & 0 & 1 \end{bmatrix}^3 = \begin{bmatrix} \_\_\_\_\_ & \_\_\_\_\_ & \_\_\_\_\_ \end{bmatrix}$$

(Calculator: Enter 0.15, 0.45, 0.40 into matrix [D], multiple by matrix [C] cubed, [D] × [C]^3.)

and so chance driver chooses *State Farm* (middle choice) in three years is (circle one) **0.21005625** / **0.236325** / **0.55361875** / **0.4675**.

(b) If the initial state vector is,

$$P_0 = \begin{bmatrix} 0.15 & 0.45 & 0.40 \end{bmatrix}$$

and the state vector after *100* transitions is

$$P_3 = P_0 \times T^{100} = \begin{bmatrix} 0.15 & 0.45 & 0.40 \end{bmatrix} \times \begin{bmatrix} 0.80 & 0.20 & 0 \\ 0.20 & 0.65 & 0.15 \\ 0 & 0 & 1 \end{bmatrix}^{100} \approx \begin{bmatrix} \underline{\qquad} & \underline{\qquad} & \underline{\qquad} \end{bmatrix}$$

(Calculator: [D] × $[C]^{100}$.)
and so ("long run") chance driver chooses *State Farm* (middle choice) is (circle one) **0** / **0.236325** / **0.55361875** / **1**.

(c) If, *next* year, there is a 15%, 35% and 50% chance a driver has Allstate, State Farm and AutoSafe, respectively, then *initial* state vector is,

$$P_0 = P_1 \times T^{-1} = \begin{bmatrix} 0.15 & 0.35 & 0.50 \end{bmatrix} \times \begin{bmatrix} 0.80 & 0.20 & 0 \\ 0.20 & 0.65 & 0.15 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \approx \begin{bmatrix} \underline{\qquad} & \underline{\qquad} & \underline{\qquad} \end{bmatrix}$$

(Calculator: Enter 0.15, 0.35, 0.50 into matrix [D], multiple by *inverse* of matrix [C], [D] × $[C]^{-1}$.)

4. **True** / **False** Since $P_1 = P_0 T$, then $P_2 = P_1 T = (P_0 T)T = P_0 T^2$ and so, in general,

$$P_n = P_{n-1} T, \quad \text{or} \quad P_n = P_0 T^n$$

11.3 Regular Markov Chains

Markov chains are said to be *regular* if all elements of *some* (not necessarily all) powers of the associated transition matrix, T, are *positive* (greater than zero). For regular Markov chains, the state vectors P_n have a *stable* ("long run") state vector S and the transition matrices T_n have a limiting matrix L (where each row is the stable state vector), where S and L are solved using

$$S = ST \qquad \text{and} \qquad p_1 + p_2 + \cdots + p_n = 1.$$

Exercise 11.3 (Regular Markov Chains)

1. *What is a regular Markov chain?*

   (a) *Actor's performance.* Recall the following transition matrix.

$T =$

tonight → / next night ↓	good prfmce (1)	bad prfmce (2)
good prfmce (1)	0.68	0.36
bad prfmce (2)	0.32	0.64

Transition matrix T is (circle one) **regular** / **not regular** because all of the elements in this matrix are positive.

(b) *Oatmeal or cereal.* Recall the following transition matrix.

$T =$

tomorrow → / today ↓	oatmeal (1)	cereal (2)
oatmeal (1)	0.6	0.4
cereal (2)	0.2	0.8

Transition matrix T is (circle one) **regular** / **not regular** because all of the elements in this matrix are positive.

(c) *Car insurance.* Recall the following transition matrix.

$T =$

next year → / this year ↓	Allstate (1)	State Farm (2)	AutoSafe (3)
Allstate (1)	0.80	0.20	0
State Farm (2)	0.20	0.65	0
AutoSafe (3)	0	0	1

Transition matrix T is (circle one) **regular** / **not regular** because not only are some of the elements in T zero, but also there are zeroes in *every* T^n matrix in this Markov chain.

(d) The matrix,

$$\begin{bmatrix} 0.3 & 1 \\ 0.7 & 0 \end{bmatrix}$$

(circle one) **is** / **is not** regular because

$$\begin{bmatrix} 0.3 & 1 \\ 0.7 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0.3 & 1 \\ 0.7 & 0 \end{bmatrix} \times \begin{bmatrix} 0.3 & 1 \\ 0.7 & 0 \end{bmatrix} = \begin{bmatrix} 0.79 & 0.3 \\ 0.21 & 0.7 \end{bmatrix}$$

(e) The matrix (notice 0 and 1 are flipped from previous example),

$$\begin{bmatrix} 0.3 & 0 \\ 0.7 & 1 \end{bmatrix}$$

(circle one) **is** / **is not** regular because, for $n = 1, 2, 3, \ldots$

$$\begin{bmatrix} 0.3 & 0 \\ 0.7 & 1 \end{bmatrix}^n = \begin{bmatrix} a & 0 \\ b & 1 \end{bmatrix}$$

where $a + b = 1$ and $a \geq 0$, $b \geq 0$. [Hint: Try $n = 3$, say.]

(f) The 2×2 identity matrix,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(circle one) **is** / **is not** regular because, for $n = 1, 2, 3, \ldots$,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(g) Transition matrix (circle one) **is** / **is not** regular.

$$\begin{bmatrix} 0.6 & 0.3 & 0.1 \\ 0.2 & 0.7 & 0.1 \\ 0.4 & 0.3 & 0.3 \end{bmatrix}$$

(h) **True** / **False** If T has all positive elements, then transition matrix and Markov chain are regular. If T has some zero elements, Markov chain may still be regular since T^n may have all positive elements for some n.

2. *Actor's performance.* Recall following transition matrix.

$T =$

tomorrow → tonight ↓	good performance (1)	bad performance (2)
good performance (1)	0.85	0.15
bad performance (2)	0.60	0.40

(a) *Steady state vector.* Since transition matrix is regular, solve $S = ST$,

$$S = \begin{bmatrix} u & v \end{bmatrix} = S \times T = \begin{bmatrix} u & v \end{bmatrix} \times \begin{bmatrix} 0.85 & 0.15 \\ 0.60 & 0.40 \end{bmatrix}$$

which can be rewritten as

$$\begin{aligned} 0.85u + 0.60v &= u \\ 0.15u + 0.40v &= v \end{aligned}$$

or *subtracting u from both sides of the first equation* and *subtracting v from both sides of the second equation*,

$$\begin{aligned} -0.15u + 0.60v &= 0 \\ 0.15u - 0.60v &= 0 \end{aligned}$$

and which combined with the condition $u + v = 1$, becomes

$$\begin{aligned} -0.15u + 0.60v &= 0 \\ 0.15u - 0.60v &= 0 \\ u + v &= 1 \end{aligned}$$

Solving this system of equations gives

$$\begin{aligned} u + 0v &= 0.8 \\ 0u + v &= 0.2 \\ 0u + 0v &= 0 \end{aligned}$$

(Calculator: Type into [E], then rref([E]).)
in other words, $u =$ (circle one) **0** / **0.2** / **0.8** / **1**,
and $v =$ (circle one) (circle one) **0** / **0.2** / **0.8** / **1**,
so steady state vector is

$$S = \begin{bmatrix} u & v \end{bmatrix} = \begin{bmatrix} 0.8 & 0.2 \end{bmatrix},$$

which means, in the long run, actor's performance is good 80% of the time (and so poor 20% of the time). By the way, to *verify* $S = ST$, notice

$$S = \begin{bmatrix} 0.8 & 0.2 \end{bmatrix} = S \times T = \begin{bmatrix} 0.8 & 0.2 \end{bmatrix} \times \begin{bmatrix} 0.85 & 0.15 \\ 0.60 & 0.40 \end{bmatrix}.$$

(b) *Limiting transition matrix.* **True** / **False** Notice,

$$T^3 = \begin{bmatrix} 0.85 & 0.15 \\ 0.60 & 0.40 \end{bmatrix}^3 = \begin{bmatrix} 0.803125 & 0.196875 \\ 0.7875 & 0.2125 \end{bmatrix}$$

(Calculator: $[A] \wedge 3$ Enter.)
and

$$T^4 = \begin{bmatrix} 0.85 & 0.15 \\ 0.60 & 0.40 \end{bmatrix}^4 = \begin{bmatrix} 0.80078125 & 0.19921875 \\ 0.796875 & 0.203125 \end{bmatrix}$$

(Calculator: $[A] \wedge 4$ Enter.)
and, in fact, as $n \to \infty$, T^n approaches limiting matrix L,

$$T^n = \begin{bmatrix} 0.85 & 0.15 \\ 0.60 & 0.40 \end{bmatrix}^n \to L = \begin{bmatrix} 0.8 & 0.2 \\ 0.8 & 0.2 \end{bmatrix},$$

where each row of L is the steady state vector S.

3. *Oatmeal or cereal.* Recall the following transition matrix.

$T =$

tomorrow → today ↓	oatmeal (1)	cereal (2)
oatmeal (1)	0.6	0.4
cereal (2)	0.2	0.8

(a) *Steady state vector.* Since transition matrix is regular, solve $S = ST$,

$$S = \begin{bmatrix} u & v \end{bmatrix} = S \times T = \begin{bmatrix} u & v \end{bmatrix} \times \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}$$

which can be rewritten as

$$\begin{aligned} \underline{\qquad} u + \underline{\qquad} v &= u \\ \underline{\qquad} u + \underline{\qquad} v &= v \end{aligned}$$

or subtracting u from both sides of the first equation and subtracting v from both sides of the second equation,

$$\begin{aligned} \underline{\qquad} u + \underline{\qquad} v &= 0 \\ \underline{\qquad} u + \underline{\qquad} v &= 0 \end{aligned}$$

and which combined with the condition $u + v = 1$, becomes

$$\begin{aligned} -0.4u + 0.2v &= 0 \\ 0.4u - 0.2v &= 0 \\ u + v &= 1 \end{aligned}$$

Solving this system of equations gives

$$\begin{aligned} u + 0v &= \frac{1}{3} \\ 0u + v &= \frac{2}{3} \\ 0u + 0v &= 0 \end{aligned}$$

(Calculator: Type into [E], then rref([E]).)
so steady state vector is

$$S = \begin{bmatrix} u & v \end{bmatrix} = \begin{bmatrix} \underline{\qquad} & \underline{\qquad} \end{bmatrix},$$

which means, in the long run, a student eats oatmeal $\frac{1}{3}$ of the time (and so cereal $\frac{2}{3}$s of the time).

(b) *Limiting transition matrix.*

$$L = \begin{bmatrix} \underline{\qquad} & \underline{\qquad} \\ \underline{\qquad} & \underline{\qquad} \end{bmatrix}$$

4. *Car insurance.* Recall the following transition matrix.

$$T = \begin{array}{l} \text{next year} \rightarrow \\ \text{this year} \downarrow \\ \text{Allstate (1)} \\ \text{State Farm (2)} \\ \text{AutoSafe (3)} \end{array} \begin{array}{c} \text{Allstate (1)} \\ \\ \left| \begin{array}{ccc} 0.80 & 0.20 & 0 \\ 0.20 & 0.65 & 0 \\ 0 & 0 & 1 \end{array} \right| \end{array}$$

next year → / this year ↓	Allstate (1)	State Farm (2)	AutoSafe (3)
Allstate (1)	0.80	0.20	0
State Farm (2)	0.20	0.65	0
AutoSafe (3)	0	0	1

It (circle one) **is** / **is not** advisable to attempt to solve for either the steady state vector S or limiting matrix L using the methods here in this case because the transition matrix here is not regular.

5. *One last example.* Determine the steady state vector for the following transition matrix,

$$T = \begin{bmatrix} 0.6 & 0.2 & 0.4 \\ 0.3 & 0.7 & 0.3 \\ 0.1 & 0.1 & 0.3 \end{bmatrix}$$

Since T is regular,

$$S = \begin{bmatrix} u & v & w \end{bmatrix} = S \times T = \begin{bmatrix} u & v & w \end{bmatrix} \times \begin{bmatrix} 0.6 & 0.2 & 0.4 \\ 0.3 & 0.7 & 0.3 \\ 0.1 & 0.1 & 0.3 \end{bmatrix}$$

which can be rewritten as

$$\begin{aligned} \_\_\_\_ u + \_\_\_\_ v + \_\_\_\_ w &= u \\ \_\_\_\_ u + \_\_\_\_ v + \_\_\_\_ w &= v \\ \_\_\_\_ u + \_\_\_\_ v + \_\_\_\_ w &= w \end{aligned}$$

or subtracting u from both sides of the first equation and subtracting v from both sides of the second equation, and subtracting w from both sides of the third equation,

$$\begin{aligned} \_\_\_\_ u + \_\_\_\_ v + \_\_\_\_ w &= 0 \\ \_\_\_\_ u + \_\_\_\_ v + \_\_\_\_ w &= 0 \\ \_\_\_\_ u + \_\_\_\_ v + \_\_\_\_ w &= 0 \end{aligned}$$

and which combined with the condition $u + v + w = 1$, becomes

$$\begin{aligned} -0.4u + 0.3v + 0.1w &= 0 \\ 0.2u - 0.3v + 0.1w &= 0 \\ 0.4u + 0.3v - 0.7w &= 0 \\ u + v + w &= 1 \end{aligned}$$

(Calculator: Type into [F], then rref([F]).) Solving this system of equations gives steady state vector

$$S = \begin{bmatrix} u & v & w \end{bmatrix} = \begin{bmatrix} \_\_\_\_\_ & \_\_\_\_\_ & \_\_\_\_\_ \end{bmatrix}.$$