Chapter 12

Games Theory

A two–person zero-sum game involves a competitive situation in which two players determine their optimal strategies to maximize gain or, equivalently, minimize loss, where the gain of one player is exactly the loss of the other player, and where gain and loss are given by a *payoff matrix*. We look at how to determine both the optimal strategies and the *value*, optimal gain/loss for both players, of this type of game. We look two cases, where the game is either strictly determined with optimal pure strategies that have saddlepoint values or the game has optimal mixed (probabilistic) strategies that have expected values.

12.1 Strictly Determined Games

Both the pure optimal strategies and the value in a strictly determined game are determined by the *saddlepoint* element in the payoff matrix: the element which is both the

1. smallest payoff element in its row, and the
2. largest payoff element in its column.

Row player R uses a pure optimal *maximin* strategy when choosing the saddlepoint row in the payoff matrix and column player C uses a pure optimal *minimax* strategy when choosing saddlepoint column. The saddlepoint itself is the *value*; in other words, the optimal gain/loss of the two players in the game.

Exercise 12.1 (Strictly Determined Games)

1. *Two Finger Morra.* Two–finger morra involves two players in which both can either show one or two fingers simultaneously, where either does not know what the other is going to show in advance. The amount that each player wins depends on the number of fingers each shows. The *payoff* matrix (given in

dollars) for one possible game of two–finger morra is given below. Each element represents the amount player R (row) gains and so the amount player C loses.

Player C → Player R ↓	column 1	column 2
row 1	-2	-3
row 2	-1	3

(a) If R shows 2 fingers and C shows 2 fingers, R's payoff is
(circle one) -$**3** / -$**2** / -$**1** / $**3**.
In other words, R **loses** / **wins** 3 dollars
and also C **loses** / **wins** 3 dollars
This is the best R can do in this game.

(b) If R shows 1 finger and C shows 2 fingers, R's payoff is
(circle one) -$**3** / -$**2** / -$**1** / $**3**.
In other words, R **loses** / **wins** 3 dollars
and also C **loses** / **wins** 3 dollars.
This is the best C can do in this game.

(c) Since R wants to win as much as possible, but taking into account C's possible strategies, it is reasonable s/he play a *maximin* strategy: identify maximum (worst) loss in each row and then choose the row which minimizes these losses (hope for the best but plan for the worse).

Player C → Player R ↓	column 1	column 2
row 1	-2	[-3]
row 2	[-1]*	3

So R chooses (circle one) **row 1** / **row 2**
since starred* maximin value is in this row.

(d) Since C also wants to win as much as possible, but taking into account R's possible strategies, it is reasonable s/he play a *minimax* strategy: identify minimum loss (for R, but maximum loss for C) in each column and then choose the column which maximizes these losses (for R, so minimizes for C).

Player C → Player R ↓	column 1	column 2
row 1	-2	-3
row 2	$\{-1\}*$	$\{3\}$

So C chooses (circle one) **column 1** / **column 2**
since starred* minimax value is in this column.

(e) **True** / **False** Both R and C choose starred* value -1, the *saddlepoint*, in row 2 and column 1 and so -1 is the *value* (outcome) of the game. The game favors player C since player R loses 1 dollar and so the game is not fair (the saddlepoint is not zero).

Player C → Player R ↓	column 1	column 2
row 1	-2	[-3]
row 2	{[-1]} *	{3}

(f) To repeat, since value -1 in row 2 and column 1 is the (circle one) **smallest** / **largest** value in its row, and the (circle one) **smallest** / **largest** value in its column, payoff element -1 (circle one) **is** / **is not** a saddlepoint.

(g) On the other hand, since payoff element -2 in row 1 and column 1 is the (circle one) **smallest** / **largest** value in its row, and the (circle one) **smallest** / **largest** value in its column, payoff element -2 (circle one) **is** / **is not** a saddlepoint.

(h) Can either player do better than the maximin or minimax strategies? Choose one or *more*.

i. No, for R. If R knew C always did minimax (always chose column 1), R would not deviate from maximin row 2 to row 1 because R would then lose 2 dollars instead of 1 dollar. Even if C chose column 2, R would win 3 dollars instead of losing 1 dollar.

ii. No, for C. If C knew R always did maximin (always chose row 2), C would not deviate from minimax column 1 to column 2 because C would then lose 3 dollars instead of winning 1 dollar. Even if R chose row 1, C would win 2 dollars instead of 1 dollar.

The saddlepoint does *not* allow both players to win (the game is *zero-sum*), but does identify a reasonable outcome for two rational players trying to do the best they can when they do not know the other person's strategy.

2. *Two finger morra, another case.* Identify value (saddlepoint), player strategies.

Player C → Player R ↓	one finger (1)	two fingers (2)
one finger (1)	\$1	\$6
two fingers (2)	\$2	\$5

(a) To identify saddlepoint, first box *minimum* in each *row* and pick optimal maximum, and bracket *maximum* in each *column* and pick optimal minimum, then identify if two optimals overlap, as shown below.

Player C → Player R ↓	one finger (1)	two fingers (2)
one finger (1)	[$1]	{$6}
two fingers (2)	{[$2]} *	$5

The saddlepoint element (value) is (circle one)

i. payoff element $1 in row 1 and column 1
ii. payoff element $6 in row 1 and column 2
iii. payoff element $2 in row 2 and column 1
iv. payoff element $5 in row 2 and column 2

(b) The maximin strategy for player R is (circle one) **row 1** / **row 2**.

(c) The minimax strategy for player C is (circle one) **column 1** / **column 2**.

3. *Options.* Identify value (saddlepoint), player strategies.

Player C → Player R ↓	option 1	option 2
option 1	1	-6
option 2	2	-10

(a) To identify saddlepoint, first box *minimum* in each *row* and pick optimal maximum, and bracket *maximum* in each *column* and pick optimal minimum, then identify if two optimals overlap. The saddlepoint element (value) is (circle one)

i. payoff 1 in row 1 and column 1
ii. payoff -6 in row 1 and column 2
iii. payoff 2 in row 2 and column 1
iv. payoff 10 in row 2 and column 2

(b) The maximin strategy for player R is (circle one) **row 1** / **row 2**.

(c) The minimax strategy for player C is (circle one) **column 1** / **column 2**.

4. *Three Options.*

Player C → Player R ↓	option 1	option 2	option 3
option 1	1	2	-3
option 2	-1	2	-2
option 3	2	3	-4

(a) To identify saddlepoint, first box *minimum* in each *row* and pick optimal maximum, and bracket *maximum* in each *column* and pick optimal minimum, then identify if two optimals overlap. The saddlepoint element (value) is (circle one)

   i. payoff 2 in row 2 and column 2
   ii. payoff -2 in row 2 and column 3
   iii. payoff 3 in row 3 and column 2
   iv. payoff -4 in row 3 and column 3

(b) The maximin strategy for player R is
(circle one) **option 1** / **option 2** / **option 3**
which is a (circle one) **pure** / **mixed** strategy since only option 2 is chosen.

(c) The minimax strategy for player C is
(circle one) **option 1** / **option 2** / **option 3**
which is a (circle one) **pure** / **mixed** strategy since only option 3 is chosen.

(d) This game favors (circle one) **R** / **C** / **neither R nor C**
and so is (circle one) **fair** / **unfair**

5. *Options, two saddlepoints.*

Player C → / Player R ↓	option 1	option 2	option 3
option 1	-2	2	-3
option 2	-1	2	-1

(a) To identify saddlepoint, first box *minimum* in each *row* and pick optimal maximum, and bracket *maximum* in each *column* and pick optimal minimum, then identify if two optimals overlap. The saddlepoints are (circle *two*)

   i. payoff -1 in row 2 and column 1
   ii. payoff 2 in row 2 and column 2
   iii. payoff -1 in row 2 and column 3

(b) The maximin strategy for player R is
(circle one) **option 1** / **option 2**

(c) The minimax strategy for player C is *either*
(circle *two*) **option 1** / **option 2** / **option 3**

(d) Game favors (circle one) **R** / **C** / **neither R nor C** / **both R and C**
and so is (circle one) **fair** / **unfair**

6. *Two finger morra, no saddlepoint.*

Player C → Player R ↓	one finger (1)	two fingers (2)
one finger (1)	\$1	-\$3
two fingers (2)	-\$2	\$0

(a) The saddlepoint (value) (circle one)

i. is \$1 in row 1 and column 1
ii. is -\$3 in row 1 and column 2
iii. is -\$2 in row 2 and column 1
iv. is \$0 in row 2 and column 2
v. does not exist

(b) The maximin (row with max of all mins) strategy for player R is (circle one) **row 1** / **row 2**.

(c) The minimax (column with min of all maxs) strategy for player C is (circle one) **column 1** / **column 2**.

(d) If maximin and minimax strategies used, then value of game would be (circle one) $\mathbf{-3}$ / $\mathbf{-2}$ / $\mathbf{0}$ / $\mathbf{1}$

(e) Can either player do better than the maximin or minimax strategies? Choose one or *more*.

i. Yes, for C. If C knew R always did maximin (always chose row 2), C could deviate from minimax column 2 to column 1 because C would then win 2 dollars instead of 0 dollars.
ii. No, for R. If R knew C always did minimax (always chose column 2), R would not deviate from maximin row 2 to row 1 because R would then lose 3 dollars instead of 0 dollars.

This game is no longer strictly determined, becomes unstable, since if C started to consistently choose column 1, instead of minimax column 2, R would then choose row 1 and so win 1 dollar instead of losing 2 dollars. But, then, C would choose column 2 and so on.

7. *Rock, Scissors and Paper (RSP), no saddlepoint.* Rock, scissors and paper involves two players in which both can either show either a "rock" (clenched fist), "scissors" (V–sign) or "paper" (open hand) simultaneously, where either does not know what the other is going to show in advance. Rock beats scissors (crushes it), scissors beats paper (cuts it) and paper beats rock (covers it). Whoever wins, receives a dollar (\$1). The RSP payoff matrix is given below. Each element represents the amount player R (row) gains and so the amount player C loses.

Player C → Player R ↓	rock (1)	scissors (2)	paper (3)
rock (1)	0	\$1	-\$1
scissors (2)	-\$1	0	\$1
paper (3)	\$1	-\$1	0

(a) The saddlepoint (value) (circle one)

   i. is payoff element \$1 in row 1 and column 1
   ii. is payoff element -\$3 in row 1 and column 2
   iii. is payoff element -\$2 in row 2 and column 1
   iv. is payoff element \$0 in row 2 and column 2
   v. does not exist

(b) The maximin strategy for player R is (circle one or *more*) **row 1** / **row 2** / **row 3** because there is no one minimum value.

(c) The minimax strategy for player C is (circle one or *more*) **column 1** / **column 2** / **column 3** because there is no one maximum value.

(d) **True** / **False** This game is *not* strictly determined and so requires a *mixed* strategy where players choose their options according to a probability distribution.

12.2 The Expected Value of Games with Mixed Strategies

In repeated plays of a *non*strictly determined game, a player will often switch between the various options in an attempt to gain an advantage over the opposing player. One way to view this is to say on any given play, the player has different probabilities of choosing between the various options. Given such a game with *mixed* strategies, we will first determine the *expected* value of the game.

Exercise 12.2 (The Expected Value of Games with Mixed Strategies)

1. *Two finger morra, pure and mixed strategies.* Recall the payoff matrix.

Player C → Player R ↓	one finger (1)	two fingers (2)
one finger (1)	-2	-3
two fingers (2)	-1	3

(a) *Solving using saddlepoint.* To identify saddlepoint, first box minimum in each row and bracket maximum in each column, then identify where they overlap, as shown below.

Player C → Player R ↓	column 1	column 2
row 1	-2	$\boxed{-3}$
row 2	$\{\boxed{-1}\}$ *	{3}

Saddlepoint value is **−1** in
(circle one) **row 1** / **row 2** and (circle one) **column 1** / **column 2**.

(b) *Mixed strategy notation.* Notice maximin strategy for player R is **row 2**; where the *chance* R chooses row 1 or row 2 is 0 and 1, respectively, or

$$P = \begin{bmatrix} p_1 & p_2 \end{bmatrix} =$$

(circle one) $\begin{bmatrix} \mathbf{0} & \mathbf{1} \end{bmatrix}$ / $\begin{bmatrix} \mathbf{1} & \mathbf{0} \end{bmatrix}$ / $\begin{bmatrix} \mathbf{0.5} & \mathbf{0.5} \end{bmatrix}$,
and where minimax strategy for player C is **column 1**; in other words, chance C chooses column 1 or column 2 is 1 and 0, respectively, or

$$Q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} =$$

(circle one) $\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}$ / $\begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix}$ / $\begin{bmatrix} \mathbf{0.5} \\ \mathbf{0.5} \end{bmatrix}$
and furthermore *expected* value of game (from R's point of view) is

$$E = PAQ = \begin{bmatrix} p_1 & p_2 \end{bmatrix} \times \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \times \begin{bmatrix} -2 & -3 \\ -1 & 3 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$$

(circle one) **−3** / **−2** / **−1** / **3**, the value of the saddlepoint in this case.
Calculator: Type numbers in [A], [B], [C], then [A] × [B] × [C].

(c) *Mixed strategies.* If R continued with maximin strategy,

$$P = \begin{bmatrix} p_1 & p_2 \end{bmatrix} =$$

(circle one) $\begin{bmatrix} \mathbf{0} & \mathbf{1} \end{bmatrix}$ / $\begin{bmatrix} \mathbf{0.6} & \mathbf{0.4} \end{bmatrix}$ / $\begin{bmatrix} \mathbf{0.5} & \mathbf{0.5} \end{bmatrix}$,
but C chose column 1 60% and column 2 40% of the time,

$$Q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} =$$

(circle one) $\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}$ / $\begin{bmatrix} \mathbf{0.6} \\ \mathbf{0.4} \end{bmatrix}$ / $\begin{bmatrix} \mathbf{0.5} \\ \mathbf{0.5} \end{bmatrix}$
then expected value of game (from R's point of view) is

$$E = PAQ = \begin{bmatrix} 0 & 1 \end{bmatrix} \times \begin{bmatrix} -2 & -3 \\ -1 & 3 \end{bmatrix} \times \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} =$$

(circle one) $\mathbf{-1.0}$ / $\mathbf{-0.3}$ / $\mathbf{0.1}$ / $\mathbf{0.6}$ which is larger than value $v = -1$.
Calculator: Type numbers in [A], [B], [C], then [A] × [B] × [C].
This indicates R, using optimal maximin strategy, expects to win more if C tries any strategy other than minimax in strictly determined games.

(d) **True** / **False**. In general

$$\begin{aligned} P_{\text{opt}}AQ &\geq v \text{ for any Q strategy} \\ PAQ_{\text{opt}} &\leq v \text{ for any P strategy} \end{aligned}$$

2. *Two finger morra, no saddlepoint.*

Player C → Player R ↓	one finger (1)	two fingers (2)
one finger (1)	\$1	-\$3
two fingers (2)	-\$2	\$0

(a) The saddlepoint (circle one) **does** / **does not** exist because

Player C → Player R ↓	one finger (1)	two fingers (2)
one finger (1)	{\$1}	[-\$3]
two fingers (2)	[-\$2]*	{\$0}**

(b) *Mixed strategy.* If R chose row 1 30% and row 2 70% of the time,

$$P = \begin{bmatrix} p_1 & p_2 \end{bmatrix} =$$

(circle one) $\begin{bmatrix} \mathbf{0.3} & \mathbf{0.7} \end{bmatrix}$ / $\begin{bmatrix} \mathbf{0.6} & \mathbf{0.4} \end{bmatrix}$ / $\begin{bmatrix} \mathbf{0.5} & \mathbf{0.5} \end{bmatrix}$,
but C chose column 1 60% and column 2 40% of the time,

$$Q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} =$$

(circle one) $\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}$ / $\begin{bmatrix} \mathbf{0.6} \\ \mathbf{0.4} \end{bmatrix}$ / $\begin{bmatrix} \mathbf{0.5} \\ \mathbf{0.5} \end{bmatrix}$
then expected value of game (from R's point of view) is

$$E = PAQ = \begin{bmatrix} 0.3 & 0.7 \end{bmatrix} \times \begin{bmatrix} 1 & -3 \\ -2 & 0 \end{bmatrix} \times \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} =$$

(circle one) **−1.01** / **−1.02** / **−1.03** / **−1.04**.
Calculator: Type numbers in [A], [B], [C], then [A] × [B] × [C].

3. *More two finger morra.*

Player C → Player R ↓	one finger (1)	two fingers (2)
one finger (1)	\$2	-\$3
two fingers (2)	-\$2	\$3

(a) The saddlepoint (circle one) **does** / **does not** exist because

Player C → Player R ↓	one finger (1)	two fingers (2)
one finger (1)	{\$2}	[-\$3]
two fingers (2)	[-\$2]*	{\$3}**

(b) *Game A.* If

$$P = \begin{bmatrix} 0.4 & 0.6 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix},$$

the *expected* value of game is

$$PAQ = \begin{bmatrix} 0.4 & 0.6 \end{bmatrix} \times \begin{bmatrix} 2 & -3 \\ -2 & 3 \end{bmatrix} \times \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix} =$$

(circle one) **0.4** / **0.6** / **1** / **2**
So game favors player (circle one) **R** / **C** since value of game is positive.

(c) *Game B.* If

$$P = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix},$$

the *expected* value of game is

$$PAQ = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix} \times \begin{bmatrix} 2 & -3 \\ -2 & 3 \end{bmatrix} \times \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} =$$

(circle one) **0** / **0.4** / **0.5** / **0.6**
So R prefers game (circle one) **A** / **B** since it has larger value, v.

(d) **True** / **False** Players R and C would like to know their optimal mixed strategies.

4. *Rock, Scissors, Paper.* Reconsider the following payoff matrix.

Player C → Player R ↓	option 1	option 2	option 3
option 1	0	1	-1
option 2	-1	0	1
option 3	1	-1	0

If the mixed strategies of R and C are

$$P = \begin{bmatrix} 0.1 & 0.8 & 0.1 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.3 \\ 0.5 \\ 0.2 \end{bmatrix},$$

then the expected value of the game is
$PAQ =$ (circle one) **−0.07 / −0.08 / −0.09 / −0.10**

12.3 Solving Mixed-Strategy Games

Given payoff matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where a, b, c, d are all positive (not zero or negative),
solving *minimization* linear program,

$$\begin{array}{lcccccc} \text{Minimize} & \text{X} & + & \text{Y} & & \\ \text{subject to} & \text{a X} & + & \text{b Y} & \geq & 1 \\ & \text{c X} & + & \text{d Y} & \geq & 1 \\ & \text{X} & & & \geq & 0 \\ & & & \text{Y} & \geq & 0 \end{array}$$

gives $v = \frac{1}{X+Y}$ and optimal *row* strategy $\begin{bmatrix} p_1 & p_2 \end{bmatrix}$, where $p_1 = vX$ and $p_2 = vY$,
whereas solving (dual) *maximization* linear program,

$$\begin{array}{lcccccc} \text{Maximize} & \text{U} & + & \text{V} & & \\ \text{subject to} & \text{a U} & + & \text{b V} & \leq & 1 \\ & \text{c U} & + & \text{d V} & \leq & 1 \\ & \text{U} & & & \geq & 0 \\ & & & \text{V} & \geq & 0 \end{array}$$

gives $v = \frac{1}{U+V}$ and optimal *column* strategy $\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$, where $q_1 = vU$ and $q_2 = vV$.
This works for both strictly determined (pure strategy) and nonstrictly determined (mixed strategy) games and generalizes to higher dimension payoff matrices.

Exercise 12.3 (Solving Mixed-Strategy Games)

1. *Two finger morra, pure strategies.* Recall the payoff matrix.

Player C → / Player R ↓	one finger (1)	two fingers (2)
one finger (1)	-2	-3
two fingers (2)	-1	3

   (a) *Solve using saddlepoint.* To identify saddlepoint, first box *minimum* in each *row* and pick optimal maximum, and bracket *maximum* in each *column* and pick optimal minimum, then identify if two optimals overlap.

Player C → / Player R ↓	column 1	column 2
row 1	-2	[-3]
row 2	{[-1]} *	{3}

   Saddlepoint value is **−1** in
   (circle one) **row 1** / **row 2** and (circle one) **column 1** / **column 2**.

   (b) *Solve using linear programming.*
   Since payoff matrix has negative numbers, add 4 to give

$$\begin{bmatrix} -2 & -3 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 7 \end{bmatrix}$$

   and so maximization problem is

$$\begin{array}{lcccccc} \text{Maximize} & U & + & V & & \\ \text{subject to} & 2U & + & V & \leq & 1 \\ & 3U & + & 7V & \leq & 1 \\ & U & & & \geq & 0 \\ & & & V & \geq & 0 \end{array}$$

   i. *Initial corner.*

U	V	s_1	s_2	f	
2	1	1	0	0	1
3	7	0	1	0	1
-1	-1	0	0	1	0

   Type this 3 by 6 table into MATRIX [A]; use 2nd MATRIX EDIT.
   Pivot column, most negative indicator, is ***U*** / ***V*** / **either *U* or *V***
   Using pivot column U, pivot row, smallest quotient, is row $\boldsymbol{R_1}$ / $\boldsymbol{R_2}$
   So pivot element: **2** / **3** / **7**

   ii. *First corner point.*

U	V	s_1	s_2	f		
2	1	1	0	0	1	Quotient: $\frac{1}{2}$
[3]	1	0	1	0	1	Quotient: $\frac{1}{3}$
-1	-1	0	0	1	0	

$\frac{1}{3}R_2 \to R_2,\quad R_1 - 2R_2 \to R_1,\ R_3 + R_2 \to R_3 \ \longrightarrow$

U	V	s_1	s_2	f	
0	-$\frac{11}{3}$	1	-$\frac{2}{3}$	0	$\frac{1}{3}$
1	$\frac{7}{3}$	0	$\frac{1}{3}$	0	$\frac{1}{3}$
0	$\frac{4}{3}$	0	$\frac{1}{3}$	1	$\frac{1}{3}$

2nd MATRIX MATH *row(1/3,[A],2) STO 2nd MATRIX [B]

2nd MATRIX MATH *row+(−3,[B],2,1) STO 2nd MATRIX [C]

2nd MATRIX MATH *row+(1,[C],2,3) STO 2nd MATRIX [D].

Optimal *primal* point $(U,V) = \left(\mathbf{\frac{1}{3}, 0}\right)$ / $\left(\mathbf{0, \frac{1}{3}}\right)$
Optimal *dual* point $(X,Y) = \left(\mathbf{\frac{1}{3}, 0}\right)$ / $\left(\mathbf{0, \frac{1}{3}}\right)$
with optimal (maximum) value $f = \mathbf{0}$ / $\mathbf{\frac{1}{3}}$
because all indicators **nonpositive** / **zero** / **nonnegative**

iii. value of (revised) game: $v = \frac{1}{X+Y} = \frac{1}{0+1/3} = \frac{1}{U+V} = \frac{1}{1/3+0} = 3$,
and $p_1 = vX = 3 \times 0 = 0$ and $p_2 = vY = 3 \times \frac{1}{3} = 1$,
and so optimal *row* strategy is $\begin{bmatrix} p_1 & p_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}$,
and also $q_1 = vU = 3 \times \frac{1}{3} = 1$ and $q_2 = vV = 3 \times 0 = 0$,
and so optimal *column* strategy $\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$,
and so value of (actual) game is

$$3 - 4 = PAQ = \begin{bmatrix} 0 & 1 \end{bmatrix} \times \begin{bmatrix} 2 & -3 \\ -1 & 3 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -1$$

which (circle one) **confirms** / **contradicts** saddlepoint result above.

(c) *Dominant options: row player.*

Player C → Player R ↓	column 1	column 3
row 1	-2	-7
row 2	-4	0
row 3	3	-5

Since *positive* games favor row player,
row 3 dominates row (circle one) **row 1** / **row 2**
and so payoff matrix reduces to

$$\begin{bmatrix} -4 & 0 \\ 3 & -5 \end{bmatrix}$$

(d) *Dominant options: column player.*

Player C → Player R ↓	column 1	column 2	column 3
row 1	-2	-5	-7
row 2	-1	2	0

Since *negative* games favor column player,
column 3 dominates column (circle one) **column 1** / **column 2**
and so payoff matrix reduces to

$$\begin{bmatrix} -2 & -7 \\ -1 & 0 \end{bmatrix}$$

2. *Two finger morra, mixed strategies.* Recall the payoff matrix.

Player C → Player R ↓	one finger (1)	two fingers (2)
one finger (1)	2	-3
two fingers (2)	-2	3

(a) The saddlepoint (circle one) **does** / **does not** exist because

Player C → Player R ↓	one finger (1)	two fingers (2)
one finger (1)	{\$2}	[-\$3]
two fingers (2)	[-\$2]*	{\$3}**

(b) *Solve using linear programming.*
Since payoff matrix has negative numbers, add 4 to give

$$\begin{bmatrix} 2 & -3 \\ -2 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 2 & 7 \end{bmatrix}$$

and so maximization problem is

$$\begin{array}{lrcrcl} \text{Maximize} & U & + & V & & \\ \text{subject to} & 6U & + & V & \leq & 1 \\ & 2U & + & 7V & \leq & 1 \\ & U & & & \geq & 0 \\ & & & V & \geq & 0 \end{array}$$

i. *Initial corner.*

U	V	s_1	s_2	f	
6	1	1	0	0	1
2	7	0	1	0	1
-1	-1	0	0	1	0

Type this 3 by 6 table into MATRIX [A]; use 2nd MATRIX EDIT.

Pivot column, most negative, column ***U*** / ***V*** / **either *U* or *V***
Using pivot column U, pivot row, smallest quotient, is row $\boldsymbol{R_1}$ / $\boldsymbol{R_2}$
So pivot element: **2** / **3** / **6**

ii. *First corner point.*

U	V	s_1	s_2	f		
[6]	1	1	0	0	1	Quotient: $\frac{1}{6}$
2	7	0	1	0	1	Quotient: $\frac{1}{2}$
-1	-1	0	0	1	0	

$\frac{1}{6}R_1 \to R_1,\quad R_2 - 2R_1 \to R_2,\ R_3 + R_1 \to R_3 \longrightarrow$

U	V	s_1	s_2	f	
1	$\frac{1}{6}$	$\frac{1}{6}$	0	0	$\frac{1}{6}$
0	$\frac{20}{3}$	$-\frac{1}{3}$	1	0	$\frac{2}{3}$
0	$-\frac{5}{3}$	$\frac{1}{6}$	0	1	$\frac{1}{6}$

2nd MATRIX MATH *row(1/6,[A],1) STO 2nd MATRIX [B]

2nd MATRIX MATH *row+(−2,[B],1,2) STO 2nd MATRIX [C]

2nd MATRIX MATH *row+(1,[C],1,3) STO 2nd MATRIX [D].

Pivot column, most negative, column ***U*** / ***V*** / **either *U* or *V***
Using pivot column V, pivot row, smallest quotient, is row $\boldsymbol{R_1}$ / $\boldsymbol{R_2}$
So pivot element: $\mathbf{\frac{20}{3}}$ / $\mathbf{\frac{1}{6}}$ / **6**

iii. *Second corner point.*

U	V	s_1	s_2	f	
1	$\frac{1}{6}$	$\frac{1}{6}$	0	0	$\frac{1}{6}$
0	$\frac{20}{3}$	$-\frac{1}{3}$	1	0	$\frac{2}{3}$
0	$-\frac{5}{3}$	$\frac{1}{6}$	0	1	$\frac{1}{6}$

$\frac{3}{20}R_2 \to R_2,\quad R_1 - \frac{1}{6}R_2 \to R_1,\ R_3 + \frac{5}{6}R_2 \to R_3 \longrightarrow$

U	V	s_1	s_2	f	
1	0	$\frac{7}{40}$	-$\frac{1}{40}$	0	$\frac{3}{20}$
0	1	$-\frac{1}{20}$	$\frac{3}{20}$	0	$\frac{1}{10}$
0	0	$\frac{1}{8}$	$\frac{1}{8}$	1	$\frac{1}{4}$

2nd MATRIX MATH *row(20/3,[D],2) STO 2nd MATRIX [E]

2nd MATRIX MATH *row+(−1/6,[E],2,1) STO 2nd MATRIX [F]

2nd MATRIX MATH *row+(5/6,[F],2,3) STO 2nd MATRIX [G].

Optimal primal point $(U, V) = \left(\mathbf{\frac{3}{20}, \frac{1}{10}}\right)$ / $\left(\mathbf{\frac{1}{8}, \frac{1}{8}}\right)$
Optimal dual point $(X, Y) = \left(\mathbf{\frac{3}{20}, \frac{1}{10}}\right)$ / $\left(\mathbf{\frac{1}{8}, \frac{1}{8}}\right)$
with optimal (maximum) value $f = \mathbf{0}$ / $\mathbf{\frac{1}{4}}$
because all indicators **nonpositive** / **zero** / **nonnegative**

iv. **True** / **False**
value of (revised) game: $v = \frac{1}{X+Y} = \frac{1}{\frac{1}{8}+\frac{1}{8}} = \frac{1}{U+V} = \frac{1}{\frac{3}{20}+\frac{1}{10}} = 4$,
and $p_1 = vX = 4 \times \frac{1}{8} = \frac{1}{2}$ and $p_2 = vY = 4 \times \frac{1}{8} = \frac{1}{2}$,
and so optimal *row* strategy is $\left[\begin{array}{cc} p_1 & p_2 \end{array}\right] = \left[\begin{array}{cc} \frac{1}{2} & \frac{1}{2} \end{array}\right]$,
and also $q_1 = vU = 4 \times \frac{3}{20} = \frac{3}{5}$ and $q_2 = vV = 4 \times \frac{1}{10} = \frac{2}{5}$,
and so optimal *column* strategy $\left[\begin{array}{c} q_1 \\ q_2 \end{array}\right] = \left[\begin{array}{c} \frac{3}{5} \\ \frac{2}{5} \end{array}\right]$,
and so value of (actual) game is

$$4 - 4 = PAQ = \left[\begin{array}{cc} \frac{1}{2} & \frac{1}{2} \end{array}\right] \times \left[\begin{array}{cc} 2 & -3 \\ -2 & 3 \end{array}\right] \times \left[\begin{array}{c} \frac{3}{5} \\ \frac{2}{5} \end{array}\right] = 0$$

(c) *Shortcut when both players' optimal strategies are mixed*[1].
Notice (sort of) *primal*

$$\begin{array}{ccccc} 6U & + & V & = & 1 \\ 2U & + & 7V & = & 1 \end{array}$$

can be written

$$\left[\begin{array}{cc|c} 6 & 1 & 1 \\ 2 & 7 & 1 \end{array}\right]$$

which has reduced row echelon form

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{3}{20} \\ 0 & 1 & \frac{1}{10} \end{array}\right]$$

in other words, $(U, V) =$ (choose one) $\left(\mathbf{\frac{3}{20}, \frac{1}{10}}\right)$ / $\left(\mathbf{\frac{1}{8}, \frac{1}{8}}\right)$
And also notice (sort of) *dual*

$$\begin{array}{ccccc} 6X & + & 2Y & = & 1 \\ X & + & 7Y & = & 1 \end{array}$$

can be written

$$\left[\begin{array}{cc|c} 6 & 2 & 1 \\ 1 & 7 & 1 \end{array}\right]$$

which has reduced row echelon form

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{1}{8} \\ 0 & 1 & \frac{1}{8} \end{array}\right]$$

in other words, $(X, Y) =$ (choose one) $\left(\mathbf{\frac{3}{20}, \frac{1}{10}}\right)$ / $\left(\mathbf{\frac{1}{8}, \frac{1}{8}}\right)$
and so, as above, value of (revised) game:
$v = \frac{1}{X+Y} = \frac{1}{\frac{1}{8}+\frac{1}{8}} = \frac{1}{U+V} = \frac{1}{\frac{3}{20}+\frac{1}{10}} = 4$,
and $p_1 = vX = 4 \times \frac{1}{8} = \frac{1}{2}$ and $p_2 = vY = 4 \times \frac{1}{8} = \frac{1}{2}$,
and so optimal *row* strategy is $\left[\begin{array}{cc} p_1 & p_2 \end{array}\right] = \left[\begin{array}{cc} \frac{1}{2} & \frac{1}{2} \end{array}\right]$,
and also $q_1 = vU = 4 \times \frac{3}{20} = \frac{3}{5}$ and $q_2 = vV = 4 \times \frac{1}{10} = \frac{2}{5}$,
and so optimal *column* strategy $\left[\begin{array}{c} q_1 \\ q_2 \end{array}\right] = \left[\begin{array}{c} \frac{3}{5} \\ \frac{2}{5} \end{array}\right]$,
and so value of (actual) game is

$$4 - 4 = PAQ = \left[\begin{array}{cc} \frac{1}{2} & \frac{1}{2} \end{array}\right] \times \left[\begin{array}{cc} 2 & -3 \\ -2 & 3 \end{array}\right] \times \left[\begin{array}{c} \frac{3}{5} \\ \frac{2}{5} \end{array}\right] = 0$$

[1]Works only if *both* players optimal strategies are mixed strategies. If one or the other or both have an optimal pure strategy, this shortcut does not work and either the mixed strategy method or strictly determined method discussed above must be used. It is risky to use this shortcut method because it is usually not know in advance whether the optimal strategies of both players are mixed or not. Geometrically, shortcut method assumes optimal corner point of feasible region does *not* occur at any of the intercepts.