Chapter 2

Systems of Linear Equations

We learn how to translate a written description into a system of linear equations. We then solve these systems with none, one or many solutions, using *Gauss-Jordan elimination method*.

2.1 Linear Systems as Mathematical Models

We translate a written description into a system of linear equations, then into an augmented matrix, with coefficient matrix and constant matrix parts.

Exercise 2.1 (Linear Systems as Mathematical Models)

1. *Acme's Party Service*[1]. A birthday party requires 25 balloons and 20 noisemakers, while a Halloween party requires 15 balloons and 10 noisemakers. Only 250 balloons and 175 noisemakers are available. How many birthday parties, x, and Halloween parties, y, can Acme schedule? Set up system but do not solve.

   (a) Identify system of linear equations which describes this situation.

      i. System A
      $$\begin{aligned} 25x &+ 250y = 15 \\ 20x &+ 175y = 10 \end{aligned}$$

      ii. System B
      $$\begin{aligned} 25x &+ 20y = 250 \\ 15x &+ 10y = 175 \end{aligned}$$

      iii. System C
      $$\begin{aligned} 25x &+ 15y = 250 \\ 20x &+ 10y = 175 \end{aligned}$$

   (b) Identify associated augmented matrix for system.

[1] Spence and Vanden Eynden, 4, p 124, 1990.

i. Augmented matrix A

$$\left[\begin{array}{cc|c} 25 & 250 & 15 \\ 20 & 175 & 10 \end{array}\right]$$

ii. Augmented matrix B

$$\left[\begin{array}{cc|c} 25 & 20 & 250 \\ 15 & 10 & 175 \end{array}\right]$$

iii. Augmented matrix C

$$\left[\begin{array}{cc|c} 25 & 15 & 250 \\ 20 & 10 & 175 \end{array}\right]$$

(c) Identify associated coefficient and constant matrices for system.

i. Coefficient and constant matrices A

$$\begin{bmatrix} 25 & 250 \\ 20 & 175 \end{bmatrix}, \quad \begin{bmatrix} 15 \\ 10 \end{bmatrix}$$

ii. Coefficient and constant matrices B

$$\begin{bmatrix} 25 & 20 \\ 20 & 10 \end{bmatrix}, \quad \begin{bmatrix} 250 \\ 175 \end{bmatrix}$$

iii. Coefficient and constant matrices C

$$\begin{bmatrix} 25 & 15 \\ 20 & 10 \end{bmatrix}, \quad \begin{bmatrix} 250 \\ 175 \end{bmatrix}$$

(d) *Aside.* Since negative birthday or Halloween parties impossible,

i. $x < 0$ and $y \geq 0$
ii. $x \geq 0$ and $y < 0$
iii. $x \leq 0$ and $y \leq 0$
iv. $x \geq 0$ and $y \geq 0$

2. *Trout and Perch*[2]. Trout consume 6 units of food A and 2 units of food B per day, and perch consume 8 units of food A and 4 units of food B per day. Food A and B grow in lake at rates of 6000 units and 2400 units per day, respectively. How many trout and perch can be stocked in lake? Set up but do not solve.

(a) What are x and y?

i. (x, y) are two types of food, A and B
ii. (x, y) are consumption rates of two types of food
iii. (x, y) are number of trout and perch

[2]Grosnick, Chapter 3 Exam, March 1996.

(b) Identify system of linear equations which describes this situation.

i. System A

$$\begin{array}{rcrcr} 6x & + & 2y & = & 6000 \\ 8x & + & 4y & = & 2400 \end{array}$$

ii. System B

$$\begin{array}{rcrcr} 6x & + & 8y & = & 6000 \\ 2x & + & 4y & = & 2400 \end{array}$$

iii. System C

$$\begin{array}{rcrcr} 6x & + & 8y & = & 2400 \\ 2x & + & 4y & = & 6000 \end{array}$$

(c) Identify associated augmented matrix for system.

i. Augmented matrix A

$$\left[\begin{array}{cc|c} 6 & 2 & 6000 \\ 8 & 4 & 2400 \end{array}\right]$$

ii. Augmented matrix B

$$\left[\begin{array}{cc|c} 6 & 8 & 6000 \\ 2 & 4 & 2400 \end{array}\right]$$

iii. Augmented matrix C

$$\left[\begin{array}{cc|c} 6 & 8 & 2400 \\ 2 & 4 & 6000 \end{array}\right]$$

(d) Identify associated coefficient and constant matrices for system.

i. Coefficient and constant matrices A

$$\begin{bmatrix} 6 & 2 \\ 8 & 4 \end{bmatrix}, \quad \begin{bmatrix} 6000 \\ 2400 \end{bmatrix}$$

ii. Coefficient and constant matrices B

$$\begin{bmatrix} 6 & 8 \\ 2 & 4 \end{bmatrix}, \quad \begin{bmatrix} 6000 \\ 2400 \end{bmatrix}$$

iii. Coefficient and constant matrices C

$$\begin{bmatrix} 6 & 18 \\ 2 & 4 \end{bmatrix}, \quad \begin{bmatrix} 2400 \\ 6000 \end{bmatrix}$$

(e) Since impossible to have negative number of trout or perch,

i. $x < 0$ and $y \geq 0$

ii. $x \geq 0$ and $y < 0$

iii. $x \leq 0$ and $y \leq 0$

iv. $x \geq 0$ and $y \geq 0$

3. *Loudspeaker Systems*[3]. In order for warehouses A, B and C to meet their orders, they need to be shipped 200, 300 and 400 loudspeaker units per month respectively. Also, number output of plants I and II are 400 and 600 units per month. Let x_1 represent number of units shipped from plant I to warehouse A and x_2 represent number of units shipped from plant I to warehouse B, and so on.

	Warehouse			
Plant	A	B	C	plant output
I	x_1	x_2	x_3	400
II	x_4	x_5	x_6	600
warehouse require	200	300	400	

What is shipping schedule? Set up system but do not solve.

(a) Identify system of linear equations which describes this situation.

i. System A

$$\begin{array}{rcrcrcrcrcrcr} x_1 & + & x_2 & + & x_3 & & & & & & & = & 400 \\ & & & & & & x_4 & + & x_5 & + & x_6 & = & 600 \\ x_1 & & & + & & & x_4 & & & & & = & 200 \\ & & x_2 & & & & & + & x_5 & & & = & 300 \\ & & & & x_3 & & & & & + & x_6 & = & 400 \end{array}$$

ii. System B

$$\begin{array}{rcrcrcrcrcrcr} x_1 & + & x_2 & + & x_3 & & & & & & & = & 600 \\ & & & & & & x_4 & + & x_5 & + & x_6 & = & 400 \\ x_1 & & & + & & & x_4 & & & & & = & 200 \\ & & x_2 & & & & & + & x_5 & & & = & 300 \\ & & & & x_3 & & & & & + & x_6 & = & 400 \end{array}$$

(b) Identify associated augmented matrix for system.

i. Augmented matrix A

$$\left[\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 400 \\ 0 & 0 & 0 & 1 & 1 & 1 & 600 \\ 1 & 0 & 0 & 1 & 0 & 0 & 200 \\ 0 & 1 & 0 & 0 & 1 & 0 & 300 \\ 0 & 0 & 1 & 0 & 0 & 1 & 400 \end{array}\right]$$

ii. Augmented matrix B

$$\left[\begin{array}{cccccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 600 \\ 0 & 0 & 0 & 1 & 1 & 1 & 400 \\ 1 & 0 & 0 & 1 & 0 & 0 & 200 \\ 0 & 1 & 0 & 0 & 1 & 0 & 300 \\ 0 & 0 & 1 & 0 & 0 & 1 & 400 \end{array}\right]$$

[3] Tan, Example 4, p 187, 1997.

(c) Identify associated coefficient and constant matrices for system.

i. Coefficient and constant matrices A

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 600 \\ 400 \\ 200 \\ 300 \\ 400 \end{bmatrix}$$

ii. Coefficient and constant matrices B

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 400 \\ 600 \\ 200 \\ 300 \\ 400 \end{bmatrix}$$

(d) How does system change if twice as many units are shipped from plant I to warehouse A as are shipped from plant II to warehouse A, $x_1 = 2x_4$?

i. System A

$$\begin{array}{rcrcrcrcrcrcrcr} x_1 & + & x_2 & + & x_3 & & & & & & & & & = & 400 \\ & & & & & & x_4 & + & x_5 & + & x_6 & = & 600 \\ x_1 & & & + & & & x_4 & & & & & = & 200 \\ & & x_2 & & & & & + & x_5 & & & = & 300 \\ & & & & x_3 & & & & & + & x_6 & = & 400 \\ x_1 & & & & & + & 2x_4 & & & & & = & 0 \end{array}$$

ii. System B

$$\begin{array}{rcrcrcrcrcrcrcr} x_1 & + & x_2 & + & x_3 & & & & & & & & & = & 400 \\ & & & & & & x_4 & + & x_5 & + & x_6 & = & 600 \\ x_1 & & & + & & & x_4 & & & & & = & 200 \\ & & x_2 & & & & & + & x_5 & & & = & 300 \\ & & & & x_3 & & & & & + & x_6 & = & 400 \\ x_1 & & & & & - & 2x_4 & & & & & = & 0 \end{array}$$

2.2 Linear Systems Having One or No Solutions

Gauss–Jordan elimination method used to solve system of linear equations with either one or no solutions.

Exercise 2.2 (Linear Systems Having One or No Solutions)

1. *Solving Systems of Linear Equations, A First Look.*

(a) Linear system of equations,

$$\begin{aligned} 1x + 0y &= 2 \\ 0x + 1y &= 3 \end{aligned}$$

has solution (intersection point) at

i. $(x, y) = (2, 0)$
ii. $(x, y) = (3, 0)$
iii. $(x, y) = (2, 3)$

(b) Linear system of equations,

$$\begin{aligned} 1x + 0y + 0z &= 2 \\ 0x + 1y + 0z &= -3 \\ 0x + 0y + 1z &= 1 \end{aligned}$$

have solution (intersection point) at

i. $(x, y, z) = (2, -3, 1)$
ii. $(x, y, z) = (0, -3, 1)$
iii. $(x, y, z) = (2, -3, 0)$

Notice diagonal of 1s makes this solution easy to read.

(c) Consider following two linear system of equations.

i. linear system of equations A

$$\begin{aligned} 2x + 4y + 2z &= 6 \\ -2x + y + 3z &= 4 \\ 3x + 2y + z &= 5 \end{aligned}$$

ii. linear system of equations B

$$\begin{aligned} 1x + 0y + 0z &= 1 \\ 0x + 1y + 0z &= 0 \\ 0x + 0y + 1z &= 2 \end{aligned}$$

Easier to identify solution for **system A** / **system B**. Gauss–Jordan method transforms system A to system B, in *reduced row-echelon form*, to solve for intersection point (if there is an intersection point).

2. *Reduced row-echelon form: TI-84+.*

(a) *First example: unique solution.* System of equations and related augmented matrix are

$$\begin{array}{rcrcrcl} 2x & + & 4y & + & 2z & = & 6 \\ -2x & + & y & + & 3z & = & 4 \\ 3x & + & 2y & + & z & = & 5 \end{array} \qquad \left|\begin{array}{ccc|c} 2 & 4 & 2 & 6 \\ \text{-2} & 1 & 3 & 4 \\ 3 & 2 & 1 & 5 \end{array}\right|$$

2nd MATRIX EDIT ENTER 3 ENTER 4 ENTER, then enter data: 2 ENTER 4 ENTER ... 5 ENTER.

Reduced row-echelon form is

$$\begin{array}{ccccc} x & & & = & 1 \\ & y & & = & 0 \\ & & z & = & 2 \end{array} \qquad \left|\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right|$$

2nd MATRIX MATH rref(2nd MATRIX [A]) ENTER.

Three planes intersect at unique point

i. $(x, y, z) = (2, 1, 0)$
ii. $(x, y, z) = (0, 2, 1)$
iii. $(x, y, z) = (1, 0, 2)$

(b) *Another example: unique solution.* Consider following system,

$$\left[\begin{array}{ccc|c} 1 & \frac{10}{9} & \frac{10}{9} & 600 \\ 15 & 10 & 5 & 6000 \\ 15 & 20 & 15 & 9600 \end{array}\right]$$

2nd MATRIX EDIT ENTER 3 ENTER 4 ENTER, then: 1 ENTER 10/9 ENTER ... 9600 ENTER.

Reduced row-echelon form is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 200 \\ 0 & 1 & 0 & 240 \\ 0 & 0 & 1 & 120 \end{array}\right]$$

2nd MATRIX MATH rref(2nd MATRIX [A]) ENTER.

and so solution is

i. $(x, y, z) = (200, 240, 120)$
ii. $(x, y, z) = (120, 240, 200)$
iii. $(x, y, z) = (240, 120, 200)$

(c) *No solution.* Consider following system,

$$\left[\begin{array}{cc|c} -3 & 6 & 8 \\ 2 & -4 & 7 \end{array}\right]$$

2nd MATRIX EDIT ENTER 2 ENTER 3 ENTER, then: -3 ENTER 6 ENTER ... 7 ENTER.

Reduced row-echelon form is

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

2nd MATRIX MATH rref(2nd MATRIX [A]) ENTER.

and so, since $0x + 0y = 1$,
solution is $(x, y) =$ $\mathbf{(0, 1)}$ / $\mathbf{(2, 2)}$ / **inconsistent (no solution)**

(d) *No solution, again.* Consider following system,

$$\left[\begin{array}{ccc|c} 2 & -4 & 0 & 7 \\ 6 & 5 & 1 & 0 \\ -3 & 6 & 0 & 8 \end{array}\right]$$

2nd MATRIX EDIT ENTER 3 ENTER 4 ENTER, then: 1 ENTER -4 ENTER ... 8 ENTER.

Reduced row-echelon form is

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{2}{17} & 0 \\ 0 & 1 & \frac{1}{17} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

2nd MATRIX MATH rref(2nd MATRIX [A]) ENTER, then MATH Enter Enter to get fractions.

and so, since $0x + 0y + 0z = 1$,
solution is $(x, y, z) =$ $\mathbf{(0,0,1)}$ / $\mathbf{(0,0,0)}$ / **inconsistent (no solution)**

2.3 Linear Systems Having Many Solutions

In addition to one or no solution, it is possible to an infinity of solutions for a system of linear equations.

Exercise 2.3 (Linear Systems Having Many Solutions)

1. *A first example.* Consider following system of equations.

$$\begin{array}{rcrcrcr} -x & + & 3y & - & 2z & = & 1 \\ 2x & - & 5y & + & 2z & = & 0 \\ 2x & - & y & - & 6z & = & 8 \end{array}$$

with augmented matrix form

$$\left[\begin{array}{ccc|c} -1 & 3 & -2 & 1 \\ 2 & -5 & 2 & 0 \\ 2 & -1 & -6 & 8 \end{array}\right]$$

2nd MATRIX EDIT ENTER 3 ENTER 4 ENTER, then: -1 ENTER 3 ENTER ... -6 ENTER.

Reduced row-echelon form is

$$\left[\begin{array}{ccc|c} 1 & 0 & -4 & 5 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

2nd MATRIX MATH rref(2nd MATRIX [A]) ENTER.

two nonzero rows and *three* variables where

$x - 4z = 5$, $y - 2z = 2$ and $0x + 0y + 0z = 0$; in other words, *parametric* solution

$$\begin{aligned} x &= 5 + 4z \\ y &= 2 + 2z \\ z &= \text{any number} \end{aligned}$$

or $(x, y, z) = (5 + 4z, 2 + 2z, z)$.

If $z = 0$, $(x, y, z) = (5 + 4(0), 2 + 2(0), 0) = \mathbf{(9, 4, 1)}$ / $\mathbf{(5, 2, 0)}$
If $z = 1$, $(x, y, z) = (5 + 4(1), 2 + 2(1), 1) = \mathbf{(9, 4, 1)}$ / $\mathbf{(5, 2, 0)}$

2. *Another example.* Reduced row–echelon form

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

where $x + w = 3$, $y = -3$, $z - w = 2$ and $0x + 0y + 0z + 0w = 0$; in other words,

$$\begin{aligned} x &= 3 - w \\ y &= -3 \\ z &= 2 + w \\ w &= \text{any number} \end{aligned}$$

is system with infinity of solutions where
$(x, y, z, w) = \mathbf{(3, -3, 2, 0)}$ / $\mathbf{(3, -3, 2, \boldsymbol{w})}$ / $\mathbf{(3 - \boldsymbol{w}, -3, 2 + \boldsymbol{w}, \boldsymbol{w})}$

If $w = 0$, $(x, y, z, w) = (3 - 0, -3, 2 + 0, 0) = \mathbf{(3, -3, 2, 0)}$ / $\mathbf{(3, -3, 2, 1)}$
If $w = 1$, $(x, y, z, w) = (3 - 1, -3, 2 + 1, 1) = \mathbf{(3, -3, 2, 0)}$ / $\mathbf{(2, -3, 3, 1)}$
where $w = 0$ easier to solve than $w = 1$.

3. *Another example.* Reduced row–echelon form

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & -1 & 2 \end{array}\right]$$

where $x + w = 3$, $y = -3$ and $z - w = 2$; in other words,

$$\begin{aligned} x &= 3 - w \\ y &= -3 \\ z &= 2 + w \\ w &= \text{any number} \end{aligned}$$

is system with infinity of solutions where
$(x, y, z, w) = \mathbf{(3, -3, 2, 0)}$ / $\mathbf{(3, -3, 2, \boldsymbol{w})}$ / $\mathbf{(3 - \boldsymbol{w}, -3, 2 + \boldsymbol{w}, \boldsymbol{w})}$

4. *Another example.* Reduced row–echelon form

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

is system with infinity of solutions where
$(x_1, x_2, x_3, x_4) = \mathbf{(3, -3, 2, 0)}$ / $\mathbf{(3, -3, 2, x_4)}$ / $\mathbf{(3 + x_4, -3, 2 - x_4, x_4)}$

If $x_4 = 0$, $(x_1, x_2, x_3, x_4) = \mathbf{(3, -3, 2, 0)}$ / $\mathbf{(3, -3, 2, 1)}$
If $x_4 = 1$, $(x_1, x_2, x_3, x_4) = \mathbf{(3, -3, 2, 0)}$ / $\mathbf{(3, -3, 2, 1)}$

5. *And another example.* Consider following system of equations.

$$\begin{array}{rcrcrcr} 2x & - & y & + & z & = & 4 \\ 4x & - & 2y & + & 2z & = & 8 \end{array}$$

with augmented matrix form

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 4 \\ 4 & -2 & 2 & 8 \end{array}\right]$$

2nd MATRIX EDIT ENTER 2 ENTER 4 ENTER, then: 2 ENTER -1 ENTER ... 8 ENTER.
Reduced row-echelon form is

$$\left[\begin{array}{ccc|c} 1 & -0.5 & 0.5 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

2nd MATRIX MATH rref(2nd MATRIX [A]) ENTER.
one nonzero row and *three* variables, where
$x - 0.5y + 0.5z = 2$ and $0x + 0y + 0z = 0$; in other words,

$$\begin{array}{rcl} x & = & 2 + 0.5y - 0.5z \\ y & = & \text{any number} \\ z & = & \text{any number} \end{array}$$

or $(x, y, z) = (2+0.5y-0.5z, y, z)$, a dependent system with infinity of solutions.

If $y = z = 0$, $(x, y, z) = (2 + 0.5(0) - 0.5(0), 0, 0) = \mathbf{(2, 0, 0)}$ / $\mathbf{(1.5, 0, 1)}$
If $y = 0$, $z = 1$, $(x, y, z) = (2 + 0.5(0) - 0.5(1), 0, 1) = \mathbf{(2, 0, 0)}$ / $\mathbf{(1.5, 0, 1)}$
where $y = z = 0$ easier to solve than $y = 0$, $z = 1$.

6. *And another example.* Consider following system of equations.

$$\begin{array}{rcrcrcrcr} 2x & - & y & + & u & & & = & 4 \\ 4x & - & 2y & & & + & v & = & 8 \end{array}$$

could be rewritten, by switching u, v with x, y,

$$\begin{array}{rcrcrcrr} u & & & + & 2x & - & y = & 4 \\ & & v & + & 4x & - & 2y = & 8 \end{array}$$

which is reduced row-echelon form

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 4 \\ 0 & 1 & 4 & -2 & 8 \end{array}\right]$$

with *two* nonzero rows and *four* variables,
where $u + 2x - y = 4$ and $v + 4x - 2y = 8$; in other words,

$$\begin{array}{rcl} u & = & 4 - 2x + y \\ v & = & 8 - 4x + 2y \\ x & = & \text{any number} \\ y & = & \text{any number} \end{array}$$

or $(u, v, x, y) = (4 - 2x + y, 8 - 4x + 2y, x, y)$.

If $x = y = 0$, $(4 - 2(0) + 0, 8 - 4(0) + 2(0), 0, 0) = \mathbf{(4, 8, 0, 0)}$ / $\mathbf{(3, 6, 1, 1)}$
If $x = y = 1$, $(4 - 2(1) + 1, 8 - 4(1) + 2(1), 1, 1) = \mathbf{(4, 8, 0, 0)}$ / $\mathbf{(3, 6, 1, 1)}$
where $x = y = 0$ easier to solve than $x = y = 1$.

7. *And another example.* Reduced row–echelon form

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

is system with infinity of solutions where
$(x_1, x_2, x_3, x_4) = \boldsymbol{(3, -3, 2, x_4)}$ / $\boldsymbol{(3, -3, x_3, x_4)}$ / $\boldsymbol{(3 - x_4, -3, x_3, x_4)}$
where x_3 and x_4 are any real numbers.

8. *Row of zeroes, but unique solution.* Reduced row–echelon form

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{array}\right]$$

has *unique* solution

(a) $(x, y) = (4, 2)$
(b) $(x, y, z) = (2, 4, 0)$
(c) $(x, y) = (2, 4)$
(d) $(x, y) = (2, y)$

although third equation is dependent on other two equations.

9. *Row of zeroes, but inconsistent (no) solution.* Augmented matrix form

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{array}\right]$$

has reduced row–echelon form

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

which indicates

(a) one point of intersection (unique solution)
(b) inconsistent (no) solution, but first equation depends on third
(c) inconsistent (no) solution, but equations depend on one another

10. *Special case: homogeneous system.* Consider homogeneous system,

$$\begin{array}{ccccccc} & & y & + & 2z & = & 0 \\ x & & & + & z & = & 0 \\ 3x & & & + & 3z & = & 0 \end{array}$$

with augmented matrix form with *zero* constant matrix,

$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 3 & 0 & 3 & 0 \end{array}\right]$$

2nd MATRIX EDIT ENTER 3 ENTER 4 ENTER, then: 0 ENTER 1 ENTER ... 0 ENTER.
Reduced row-echelon form is

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

2nd MATRIX MATH rref(2nd MATRIX [A]) ENTER.
two nonzero rows and *three* variables where
$x + z = 0$, $y + 2z = 0$ and $0x + 0y + 0z = 0$; in other words,

$$\begin{aligned} x &= -z \\ y &= -2z \\ z &= \text{any number} \end{aligned}$$

or $(x, y, z) = (-z, -2z, z)$.

If $z = 0$, $(x, y, z) = (-(0), -2(0), 0) = \mathbf{(0, 0, 0)}$ / $\mathbf{(5, 2, 0)}$
which always true for a homogeneous system of equations; many other solutions also possible.

11. *Properties of reduced row-echelon form (rref).*

   (a) **True** / **False**. Matrix of rref has properties[4]:

   - First element 1 in nonzero row.
   - First element 1 in nonzero row is only nonzero entry in *column*.
   - Row with fewer leading zeroes above row with more leading zeroes.

   (b) Identify as reduced row-echelon form or not[5] and number of solutions:

      i. **is** / **is not** rref and **no** / **one** / **many** solutions

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 200 \\ 0 & 1 & 0 & 240 \\ 0 & 0 & 1 & 120 \end{array}\right]$$

      ii. **is** / **is not** rref and **no** / **one** / **many** solutions

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 200 \\ 0 & 1 & 0 & 240 \\ 0 & 0 & 0 & 120 \end{array}\right]$$

      iii. **is** / **is not** rref and **no** / **one** / **many** solutions

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 200 \\ 0 & 1 & 0 & 240 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

[4]Roughly, whatever else occurs, upper left corner of matrix *must* be identity.
[5]If not sure whether rref or not, type in calculator and use rref.

iv. **is** / **is not** rref and **no** / **one** / **many** solutions

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 200 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 240 \end{array}\right]$$

v. **is** / **is not** rref and **no** / **one** / **many** solutions

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 200 \\ 0 & 1 & 2 & 240 \end{array}\right]$$

vi. **is** / **is not** rref and **no** / **one** / **many** solutions

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 200 \\ 0 & 2 & 2 & 240 \end{array}\right]$$

vii. **is** / **is not** rref and **no** / **one** / **many** solutions

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

viii. **is** / **is not** rref and **no** / **one** / **many** solutions

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 1 & 1 \end{array}\right]$$

12. *Gauss–Jordan Method, all steps.* Goal of Gauss–Jordan method is to force initial augmented matrix into reduced row-echelon form.

    (a) *Unique solution.* Consider system

$$\begin{aligned} 3x + 7y &= 27 \\ 5x + 4y &= 22 \end{aligned}$$

2nd MATRIX EDIT ENTER 2 ENTER 3 ENTER, then: 2 ENTER 7 ENTER ... 22 ENTER.

i. Multiply first row by $\frac{1}{3}$.

$$\begin{array}{lccccc} \frac{1}{3}R_1 \to R_1 & \_\_\_\_ x & + & \_\_\_\_ y & = & \_\_\_\_ \\ & 5x & + & 4y & = & 22 \end{array}$$

2nd MATRIX MATH *row(1/3, [A], 1) STO→ [B], then MATH ENTER

ii. Multiply first row by five, subtract result from second row.

$$\begin{array}{lccccc} & x & + & \frac{7}{3}y & = & 9 \\ R_2 - 5R_1 \to R_2 & \_\_\_\_ x & + & \_\_\_\_ y & = & \_\_\_\_ \end{array}$$

2nd MATRIX MATH *row+((-)5, [B], 1, 2) STO→ [C], then MATH ENTER

iii. Multiply second row by $-\frac{3}{23}$.

$$\begin{array}{rcccccc} & x & + & \frac{7}{3}y & = & 9 \\ -\frac{3}{23}R_2 \to R_2 & \underline{\qquad}x & + & \underline{\qquad}y & = & \underline{\qquad} \end{array}$$

2nd MATRIX MATH *row((-)3/23, [C], 2) STO→ [D]

iv. Multiply second row by $\frac{7}{3}$, subtract result from first row.

$$\begin{array}{rcccccc} R_1 - \frac{7}{3}R_2 \to R_1 & \underline{\qquad}x & + & \underline{\qquad}y & = & \underline{\qquad} \\ & 0x & + & y & = & 3 \end{array}$$

2nd MATRIX MATH *row+(−7/3, [D], 2, 1) STO→ [E]

v. Unique (one) solution (intersection point) is

A. $(x, y) = (2, 1)$

B. $(x, y) = (2, 3)$

C. $(x, y) = (3, 2)$

(b) *Using TI-84+: inconsistent (no) Solution.* Consider system,

$$\begin{aligned} 2x - 4y &= 7 \\ -3x + 6y &= 8 \end{aligned}$$

2nd MATRIX EDIT ENTER 2 ENTER 3 ENTER, then data: 2 ENTER (-)4 ENTER ... 8 ENTER.

i. Multiply first row by $\frac{1}{2}$.

$$\begin{array}{rcccccc} \frac{1}{2}R_1 \to R_1 & \underline{\qquad}x & + & \underline{\qquad}y & = & \underline{\qquad} \\ & \underline{\qquad}x & + & \underline{\qquad}y & = & \underline{\qquad} \end{array}$$

2nd MATRIX MATH *row(1/2, [A], 1) STO→ [B]

ii. Multiply first row by three, add result to second row.

$$\begin{array}{rcccccc} & \underline{\qquad}x & + & \underline{\qquad}y & = & \underline{\qquad} \\ R_2 + 3R_1 \to R_2 & \underline{\qquad}x & + & \underline{\qquad}y & = & \underline{\qquad} \end{array}$$

or,

$$\begin{aligned} x - 2y &= \frac{7}{2} \\ 0x + 0y &= \frac{37}{2} \end{aligned}$$

2nd MATRIX MATH (*row+ 3, [B], 1, 2) STO→ [C]

iii. Since second row impossible (nonsense) equality, "$0x + 0y = \frac{37}{2}$",

A. no points of intersection (inconsistent solution)

B. one point of intersection (unique solution)

(c) *Pivot* on 2: change to 1, zero all other elements in column.

$$\left[\begin{array}{cc|c} \boxed{2} & -4 & 7 \\ -3 & 6 & 8 \end{array}\right]$$

becomes

$$\left[\begin{array}{cc|c} 1 & -2 & \frac{7}{2} \\ 0 & 0 & \frac{37}{2} \end{array}\right]$$

by following row operations

i. $\frac{1}{2}R_1 \rightarrow R_1,\ R_2 + 3R_1 \rightarrow R_2$
ii. $\frac{1}{3}R_1 \rightarrow R_1,\ R_2 + 4R_1 \rightarrow R_2$

13. *Linear equations geometrically: planes in three–dimensional space.*

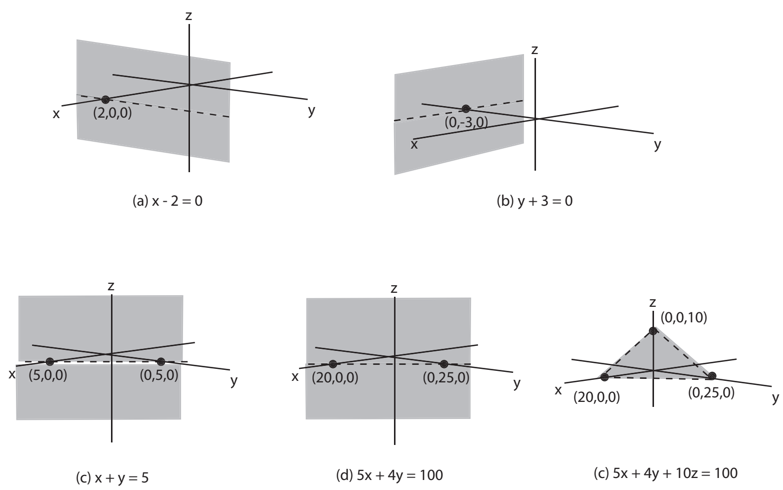


Figure 2.1 (Planes in three–dimensional space)

(a) *Figure (a).*
Equation $x = 2$ describes a **point** / **line** / **plane**, parallel z–y plane.
Equation $x = 2$ is equivalent to equation $x - 2 = 0$.
Plane $x - 2 = 0$ has x–intercept $x = 2$ but no y–intercept or z–intercept.

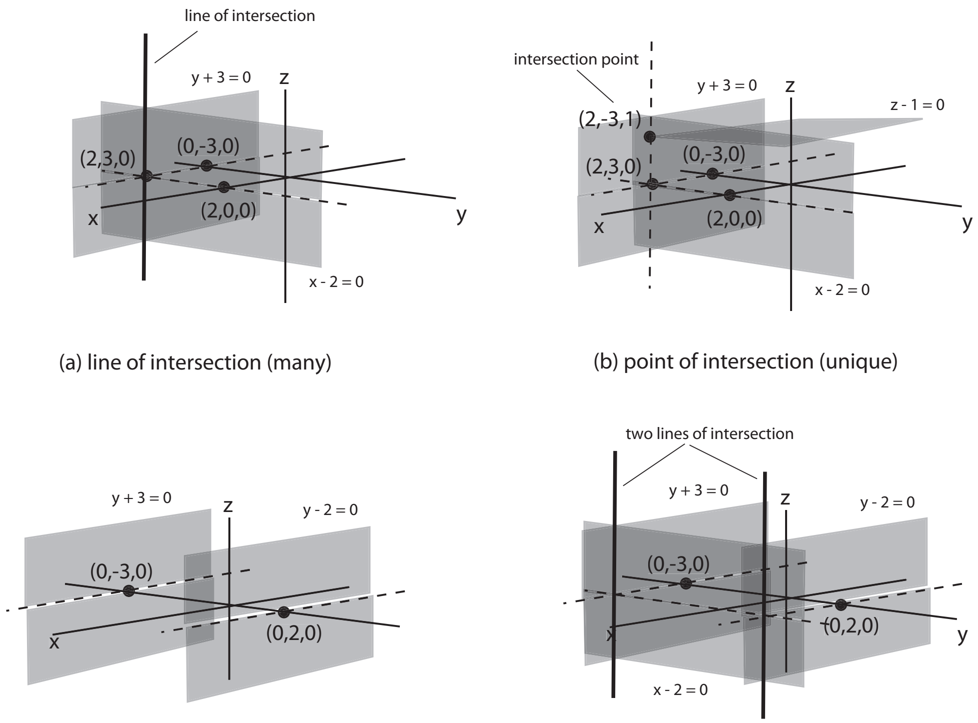
(b) *Figure (b).*
Equation $y + 3 = 0$ describes a **point** / **line** / **plane** parallel to z–x plane.
Plane $y + 3 = 0$ has y–intercept $y = -3$ but no x–intercept or z–intercept.

(c) *Figure (c).*
Equation $x + y = 5$ describes a **point** / **line** / **plane** parallel to z–axis
which has x–intercept $\boldsymbol{x = 1}$ / $\boldsymbol{x = 3}$ / $\boldsymbol{x = 5}$ (Hint: What is x when $y = 0$?)
and y–intercept $\boldsymbol{y = 1}$ / $\boldsymbol{y = 3}$ / $\boldsymbol{y = 5}$ (Hint: What is y when $x = 0$?)

(d) *Figure (d).*
Equation $5x + 4y = 100$ describes **point** / **line** / **plane** parallel to z–axis
with x–intercept $\boldsymbol{x = 20}$ / $\boldsymbol{x = 25}$ / $\boldsymbol{x = 30}$
and y–intercept $\boldsymbol{y = 20}$ / $\boldsymbol{y = 25}$ / $\boldsymbol{y = 30}$

(e) *Figure (e)*
Equation $5x + 4y + 10z = 100$ describes a **point** / **line** / **plane**
with x–intercept $\boldsymbol{x = 20}$ / $\boldsymbol{x = 25}$ / $\boldsymbol{x = 30}$ (Hint: Set $y = 0$ and $z = 0$.)
and y–intercept $\boldsymbol{y = 20}$ / $\boldsymbol{y = 25}$ / $\boldsymbol{y = 30}$ (Hint: Set $x = 0$ and $z = 0$.)
and z–intercept $\boldsymbol{z = 5}$ / $\boldsymbol{z = 10}$ / $\boldsymbol{z = 30}$ (Hint: Set $x = 0$ and $y = 0$.)

14. *Intersection of planes.*



(a) line of intersection (many)

(b) point of intersection (unique)

(c) no intersection (inconsistent)

(d) no (common) intersection (inconsistent)

Figure 2.2 (Intersection of planes)

(a) *Figure (a), line of intersection.*
Two planes (system of equations), $x - 2 = 0$ and $y + 3 = 0$, or

$$\begin{aligned} 1x + 0y + 0z &= 2 \\ 0x + 1y + 0z &= -3 \end{aligned}$$

have

i. no points in common (inconsistent solution)
ii. one point in common (unique solution)
iii. many points (a line) in common (dependent solution)

in particular, $(x, y, z) = (2, -3, z)$.

(b) *Figure (b), point of intersection.*
Three planes (system of equations), $x - 2 = 0$, $y + 3 = 0$ and $z - 1 = 0$ or

$$\begin{aligned} 1x + 0y + 0z &= 2 \\ 0x + 1y + 0z &= -3 \\ 0x + 0y + 1z &= 1 \end{aligned}$$

have

   i. no points in common (inconsistent solution)
   ii. one point in common (unique solution)
   iii. many points in common (dependent solution)

   in other words, $(x, y, z) = (2, -3, 1)$.

(c) *Figure (c), no common intersection points.*
   Two planes (system of equations), $y - 2 = 0$ and $y + 3 = 0$ or

$$\begin{aligned} 0x + 1y + 0z &= 2 \\ 0x + 1y + 0z &= -3 \end{aligned}$$

   have

   i. no points in common (inconsistent solution)
   ii. one point in common (unique solution)
   iii. many points in common (dependent solution)

   since they are parallel to one another.
   2nd MATRIX EDIT ENTER 2 ENTER 4 ENTER, then: 0 ENTER 1 ENTER ... -3 ENTER, then 2nd MATRIX MATH rref(2nd MATRIX [A]) ENTER.

(d) *Figure (d), no common intersection points.*
   Three planes (system of equations), $y - 2 = 0$, $y + 3 = 0$ and $x - 2 = 0$ or

$$\begin{aligned} 0x + 1y + 0z &= 2 \\ 0x + 1y + 0z &= -3 \\ x + 0y + 0z &= 2 \end{aligned}$$

   have

   i. no points in *common* (inconsistent solution)
   ii. one point in common (unique solution)
   iii. many points in common (dependent solution)

   although they do intersect in two *different* lines.
   2nd MATRIX EDIT ENTER 3 ENTER 4 ENTER, then: 0 ENTER 1 ENTER ... 2 ENTER, then 2nd MATRIX MATH rref(2nd MATRIX [A]) ENTER.