Chapter 5

Linear Programming: The Simplex Method

We look at how to solve linear programming problems using an algebraic approach, called *simplex method (algorithm)*. This algorithm, unlike geometric approach described in previous chapter, is able to solve linear programming problems with more than two variables. We first look at solving a special kind of linear programming problem called *standard maximization problem* which involves *slack variables* and *pivoting*. In later sections, we look at solving *nonstandard linear programming problems* using both *Crown's Rules* and *duality*.

5.1 Slack Variables and Pivoting

Simplex method is an iterative procedure which corresponds, geometrically, to moving from one feasible corner point to another until optimal feasible point is located. Slack variables are introduced to ensure corner points are *feasible*, not outside solution region. Algebraically, hopping from one feasible corner point to another corresponds to repeatedly identifying *pivot column*, *pivot row* and, consequently, *pivot element*, in a succession of matrix tableaus. Having identified pivot element, a new tableau is created by *pivoting* (by using Gauss–Jordan method) on this element. We consider slack variables and pivoting in *standard maximization problem* in this section,

- objective function linear and is maximized,
- variables all nonnegative,
- structural constraints all of form $ax + by + \cdots \leq c$, where $c \geq 0$.

Exercise 5.1 (Slack Variables and Pivoting)

1. *Standard maximization problem?*

(a) Linear programming problem,

$$
\begin{array}{lccccccc}
\text{Maximize} & 4x & + & 2y & + & 3z & & \\
\text{subject to} & 0.1x & + & 0.25y & & & \leq & 40 \\
 & 0.22x & + & 0.3y & + & 0.4z & \leq & 100 \\
 & x & & & & & \geq & 0 \\
 & & & y & & & \geq & 0 \\
 & & & & & z & \geq & 0
\end{array}
$$

is / **is not** a standard maximization problem because

i. it obeys three conditions.
ii. objective function minimized (rather than maximized).
iii. constraints *not* all written with less than or equal to signs,"≤".
iv. right–hand–sides *not* all nonnegative.

(b) Linear programming problem,

$$
\begin{array}{lccccc}
\text{Minimize} & -2x & - & 3y & & \\
\text{subject to} & 5x & + & 4y & \leq & 32 \\
 & x & + & 2y & \leq & 10 \\
 & x & & & \geq & 0 \\
 & & & y & \geq & 0
\end{array}
$$

is / **is not** a standard maximization problem because

i. it obeys three conditions.
ii. objective function minimized (rather than maximized).
iii. constraints *not* all written with less than or equal to signs,"≤".
iv. right–hand–sides *not* all nonnegative.

(c) Linear programming problem,

$$
\begin{array}{lccccc}
\text{Maximize} & 2x & - & 3y & & \\
\text{subject to} & 3x & + & 5y & \geq & 20 \\
 & 3x & + & y & \leq & 16 \\
 & -2x & + & y & = & 1 \\
 & x & & & \geq & 0 \\
 & & & y & \geq & 0
\end{array}
$$

is / **is not** a standard maximization problem because

i. it obeys three conditions.
ii. objective function minimized (rather than maximized).
iii. constraints *not* all written with less than or equal to signs,"≤".
iv. right–hand–sides *not* all nonnegative.

(d) Linear programming problem,

Maximize	2x	-	3y		
subject to	3x	+	5y	$\leq$	20
	-3x	-	y	$\geq$	-16
	-2x	+	y	$\leq$	1
	x			$\geq$	0
			y	$\geq$	0

is / **is not** a standard maximization problem because

i. it obeys three conditions.
ii. objective function minimized (rather than maximized).
iii. constraints *not* all written with less than or equal to signs, "$\leq$".
iv. right–hand–sides *not* all nonnegative.

If both sides of constraint $-3x - y \geq -16$ multiplied by -1, LP problem **becomes** / **does not become** standard maximization problem.

2. *Slack variables.* Consider following standard maximization problem.

Maximize	6x	+	4y		
subject to	x	+	2y	$\leq$	8
	2x	+	y	$\leq$	10
	x			$\geq$	0
			y	$\geq$	0

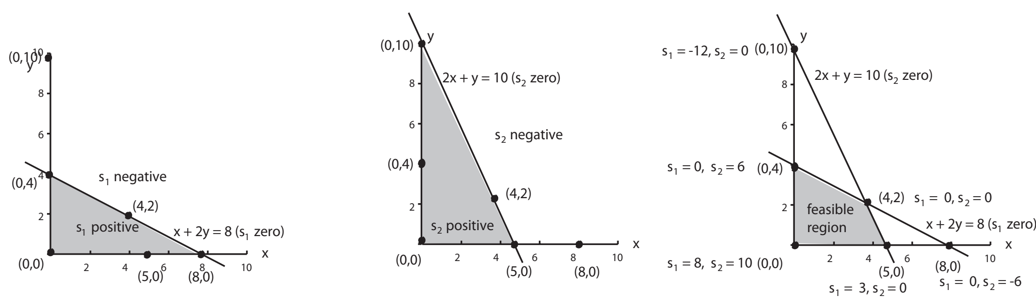


Figure 5.1 (Slack variables and feasible region)

(a) *Slack variable s_1 for $x + 2y \leq 8$.*
Introduce slack s_1 so $x + 2y + s_1 = 8$, or $s_1 = 8 - (x + 2y)$.

i. If $x + 2y = 8$, then $s_1 =$ **negative** / **zero** / **positive**
ii. If $x + 2y < 8$, then $s_1 =$ **negative** / **zero** / **positive**
iii. If $x + 2y > 8$, then $s_1 =$ **negative** / **zero** / **positive**

(b) *More on slack variable s_1 for $x + 2y \leq 8$.*

i. If $(x, y) = (0, 0)$, then $s_1 = 8 - (x + 2y) = 8 - (0 + 2(0)) =$ **0** / **3** / **8**
ii. If $(x, y) = (0, 4)$, then $s_1 = 8 - (0 + 2(4)) =$ **0** / **3** / **8**

iii. If $(x, y) = (0, 10)$, then $s_1 = 8 - (0 + 2(10)) =$ **0** / **−12** / **−20**
iv. If $(x, y) = (4, 2)$, then $s_1 = 8 - (4 + 2(2)) =$ **0** / **3** / **8**
v. If $(x, y) = (5, 0)$, then $s_1 = 8 - (5 + 2(0)) =$ **0** / **3** / **8**
vi. If $(x, y) = (8, 0)$, then $s_1 = 8 - (8 + 2(0)) =$ **0** / **3** / **8**

(c) *Slack variable s_2 for $2x + y \leq 10$.*
Introduce slack s_2 so $2x + y + s_2 = 10$, or $s_2 = 10 - (2x + y)$.

i. If $2x + y = 10$, then $s_2 =$ **negative** / **zero** / **positive**
ii. If $2x + y < 10$, then $s_2 =$ **negative** / **zero** / **positive**
iii. If $2x + y > 10$, then $s_2 =$ **negative** / **zero** / **positive**

(d) *More on slack variable s_2 for $2x + y \leq 10$.*

i. If $(x, y) = (0, 0)$, $s_2 = 10 - (2x + y) = 10 - (2(0) + 0) =$ **0** / **8** / **10**
ii. If $(x, y) = (0, 4)$, then $s_2 = 10 - (2(0) + 4) =$ **0** / **6** / **8**
iii. If $(x, y) = (0, 10)$, then $s_2 = 10 - (2(0) + 10) =$ **0** / **6** / **10**
iv. If $(x, y) = (4, 2)$, then $s_2 = 10 - (2(4) + 2) =$ **0** / **6** / **8**
v. If $(x, y) = (5, 0)$, then $s_2 = 10 - (2(5) + 0) =$ **0** / **6** / **8**
vi. If $(x, y) = (8, 0)$, then $s_2 = 10 - (2(8) + 0) =$ **0** / **−3** / **−6**

(e) *Summary for slack variables.*

corner	$s_1 = 8 - (x + 2y)$	$s_2 = 10 - (2x + y)$	feasible?
(0,0)	8	10	yes
(0,4)	0	6	yes
(0,10)	-12	0	no
(4,2)	0	0	yes
(5,0)	3	0	yes
(8,0)	0	-6	no

Slack variables (s_1, s_2) for all corner (feasible) points are
nonnegative / **zero** / **nonpositive**
Non-feasible (s_1, s_2) are **nonnegative** / **zero** / **nonpositive**

3. *Pivoting.* Consider following feasible region.

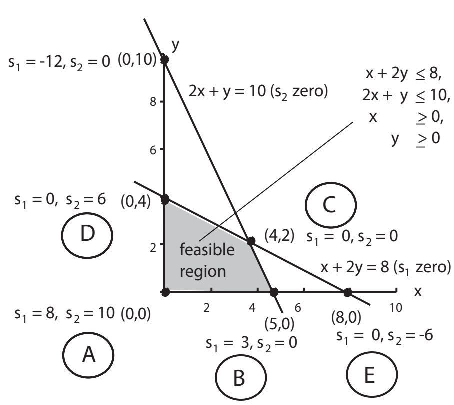


Figure 5.2 (Corner and other points and feasible region)

(a) Algebraically, corner point A where $(x, y) = (0, 0)$ and $(s_1, s_2) = (8, 10)$ is

x	y	s_1	s_2	
1	2	1	0	8
2	1	0	1	10

Type this 2 by 5 table into MATRIX [A]; use 2nd MATRIX EDIT.

where first row of this (augmented) matrix is
$\boldsymbol{x + 2y + s_1 = 8}$ / $\boldsymbol{2x + y + s_2 = 10}$
and second row of this (augmented) matrix is
$\boldsymbol{x + 2y + s_1 = 8}$ / $\boldsymbol{2x + y + s_2 = 10}$
(x, y) are *nonbasic* variables (where columns *not* 0s and 1s)
and (s_1, s_2) are *basic* (where columns are 0s and 1s) variables

(b) *Corner point A to B.* Pivot on element[1] 2,

x	y	s_1	s_2	
1	2	1	0	8
[2]	1	0	1	10

with resulting matrix,

$\xrightarrow{\frac{1}{2}R_2 \to R_2, \ R_1 - R_2 \to R_1}$

x	y	s_1	s_2	
0	1.5	1	-0.5	3
1	0.5	0	0.5	5

2nd MATRIX MATH *row(1/2,[A],2) STO 2nd MATRIX [B];
2nd MATRIX MATH *row+(−1,[B], $\underbrace{2}_{\text{pivot row}}$, $\underbrace{1}_{\text{zeroed}}$) STO 2nd MATRIX [C]

which corresponds to

i. corner point A where $(x, y) = (0, 0)$ and $(s_1, s_2) = (8, 10)$
ii. corner point B where $(x, y) = (5, 0)$ and $(s_1, s_2) = (3, 0)$
iii. corner point C where $(x, y) = (4, 2)$ and $(s_1, s_2) = (0, 0)$
iv. corner point D where $(x, y) = (0, 4)$ and $(s_1, s_2) = (0, 6)$

where nonbasic variables $(y, s_2) = \boldsymbol{(0, 0)}$ / $\boldsymbol{(5, 3)}$/ $\boldsymbol{(3, 5)}$
and basic $(x, s_1) = \boldsymbol{(0, 0)}$ / $\boldsymbol{(5, 3)}$ / $\boldsymbol{(3, 5)}$

(c) *Corner point B to C.* Pivot on element[2] 1.5,

x	y	s_1	s_2	
0	[1.5]	1	-0.5	3
1	0.5	0	0.5	5

[1]Force element 2 equal to 1 by dividing row 2 by 2, then zero element above 2 to zero by subtracting row 2 from row 1. We will find out later why element 2 is pivot.

[2]Force element 1.5 equal to 1 by dividing row 1 by 1.5, then zero element below 1.5 to zero by subtracting row 1 from row 2. We will find out later why element 1.5 is pivot.

with resulting matrix,

$$\xrightarrow{\frac{1}{1.5}R_1\to R_1,\ \ R_2-0.5R_1\to R_2} \begin{array}{cccc|c} x & y & s_1 & s_2 & \\ \hline 0 & 1 & \frac{2}{3} & -\frac{1}{3} & 2 \\ 1 & 0 & -\frac{1}{3} & \frac{2}{3} & 4 \end{array}$$

2nd MATRIX MATH *row(1/1.5,[C],1) STO 2nd MATRIX [D];

2nd MATRIX MATH *row+(−0.5,[D],1,2) STO 2nd MATRIX [E]

which corresponds to

i. corner point A where $(x, y) = (0, 0)$ and $(s_1, s_2) = (8, 10)$
ii. corner point B where $(x, y) = (5, 0)$ and $(s_1, s_2) = (3, 0)$
iii. corner point C where $(x, y) = (4, 2)$ and $(s_1, s_2) = (0, 0)$
iv. corner point D where $(x, y) = (0, 4)$ and $(s_1, s_2) = (0, 6)$

where nonbasic variables $(s_1, s_2) =$ **(0, 0)** / **(2, 4)**/ **(4, 2)**
and basic $(x, y) =$ **(0, 0)** / **(2, 4)** / **(4, 2)**

(d) *Corner point C to D.* Pivot on element $\frac{2}{3}$,

$$\begin{array}{cccc|c} x & y & s_1 & s_2 & \\ \hline 0 & 1 & \frac{2}{3} & -\frac{1}{3} & 2 \\ 1 & 0 & -\frac{1}{3} & \boxed{\frac{2}{3}} & 4 \end{array}$$

with resulting matrix,

$$\xrightarrow{\frac{3}{2}R_2\to R_2,\ \ R_1+\frac{1}{3}R_2\to R_1} \begin{array}{cccc|c} x & y & s_1 & s_2 & \\ \hline \frac{1}{2} & 1 & \frac{1}{2} & 0 & 4 \\ \frac{3}{2} & 0 & -\frac{1}{2} & 1 & 6 \end{array}$$

2nd MATRIX MATH *row(3/2,[E],2) STO 2nd MATRIX [F];

2nd MATRIX MATH *row+(1/3,[F],2,1) STO 2nd MATRIX [G]

which corresponds to

i. corner point A where $(x, y) = (0, 0)$ and $(s_1, s_2) = (8, 10)$
ii. corner point B where $(x, y) = (5, 0)$ and $(s_1, s_2) = (3, 0)$
iii. corner point C where $(x, y) = (4, 2)$ and $(s_1, s_2) = (0, 0)$
iv. corner point D where $(x, y) = (0, 4)$ and $(s_1, s_2) = (0, 6)$

where nonbasic variables $(x, s_1) =$ **(0, 0)** / **(2, 4)**/ **(4, 2)**
and basic $(y, s_2) =$ **(0, 0)** / **(4, 6)** / **(6, 4)**

(e) *Corner A to unfeasible E.*

$$\begin{array}{cccc|c} x & y & s_1 & s_2 & \\ \hline \boxed{1} & 2 & 1 & 0 & 8 \\ 2 & 1 & 0 & 1 & 10 \end{array}$$

with resulting matrix,

$\xrightarrow{R_2 - 2R_1 \to R_2}$

x	y	s_1	s_2	
1	2	1	0	8
0	-3	-2	1	-6

2nd MATRIX MATH *row+(−2,[A],1,2) STO 2nd MATRIX [H]

which corresponds to

i. corner point A where $(x, y) = (0, 0)$ and $(s_1, s_2) = (8, 10)$
ii. corner point B where $(x, y) = (5, 0)$ and $(s_1, s_2) = (3, 0)$
iii. corner point C where $(x, y) = (4, 2)$ and $(s_1, s_2) = (0, 0)$
iv. corner point D where $(x, y) = (0, 4)$ and $(s_1, s_2) = (0, 6)$
v. infeasible point $(x, y) = (8, 0)$ and $(s_1, s_2) = (0, -6)$

where nonbasic variables $(y, s_1) =$ $\mathbf{(0,0)}$ / $\mathbf{(8,-6)}$/ $\mathbf{(-6,8)}$
and basic $(x, s_2) =$ $\mathbf{(0,0)}$ / $\mathbf{(8,-6)}$ / $\mathbf{(-6,8)}$

4. *More pivoting: party problem.*

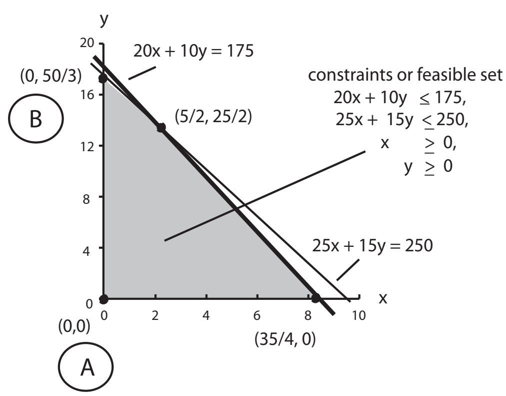


Figure 5.3 (Party problem: pivoting)

(a) *Initial corner point A.* Start at $(x, y) = (0, 0)$ and $(s_1, s_2) = (8, 10)$ where

x	y	s_1	s_2	
20	10	1	0	175
25	15	0	1	250

Type this 2 by 5 table into MATRIX [A]; use 2nd MATRIX EDIT.

where first row of this (augmented) matrix is
$\mathbf{20x + 10y + s_1 = 175}$ / $\mathbf{25x + 15y + s_2 = 250}$
and second row of this (augmented) matrix is
$\mathbf{20x + 10y + s_1 = 175}$ / $\mathbf{25x + 15y + s_2 = 250}$

(b) *Corner point A to B*
Pivot in column[3] y (not x!),

[3]We shortly learn why column y, not x.

x	y	s_1	s_2	
20	10	1	0	175
25	15	0	1	250

and pivot on element **10 / 15** in y,

$\frac{1}{15}R_2 \to R_2, \ R_1 - 10R_2 \to R_1 \longrightarrow$

x	y	s_1	s_2	
$\frac{10}{3}$	0	1	$-\frac{2}{3}$	$\frac{25}{3}$
$\frac{5}{3}$	1	0	$\frac{1}{15}$	$\frac{50}{3}$

2nd MATRIX MATH *row(1/15,[A],2) STO 2nd MATRIX [B];
2nd MATRIX MATH *row+(−10,[B],2,1) STO 2nd MATRIX [C],
then MATH ENTER for fractional form

which corresponds to corner point B where

i. $(x, y) = \left(\frac{50}{3}, 0\right)$ and $(s_1, s_2) = \left(0, \frac{50}{3}\right)$
ii. $(x, y) = \left(0, \frac{50}{3}\right)$ and $(s_1, s_2) = \left(\frac{25}{3}, 0\right)$

5.2 The Simplex Algorithm

Simplex algorithm is iterative procedure which involves repeatedly identifying *pivot column*, *pivot row* and, consequently, *pivot element*, in succession of matrix tableaus. Having identified pivot element, new tableau is created by pivoting on this element. This iterative procedure corresponds, geometrically, to moving from one feasible corner point to another until *optimal* feasible point is located. We consider simplex algorithm for standard maximization problem, which, recall, is

- objective function linear and is maximized,
- variables all nonnegative,
- structural constraints all of form $ax + by + \cdots \le c$, where $c \ge 0$.

Exercise 5.2 (The Simplex Algorithm)

1. *A first look.*

$$\begin{array}{lcccccc} \text{Maximize} & 6x & + & 4y & & \\ \text{subject to} & x & + & 2y & \le & 8 \\ & 2x & + & y & \le & 10 \\ & x & & & \ge & 0 \\ & & & y & \ge & 0 \end{array}$$

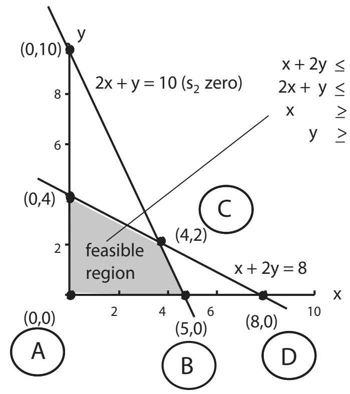


Figure 5.4 (Simplex algorithm)

(a) *Review.* **True / False.** Geometric solution to this linear programming problem involves calculating value of objective function at four corners,

corner	$f = 6x + 4y$ or $-6x - 4y + f = 0$
(0,0)	$f = 6(0) + 4(0) = 0$
(5,0)	$f = 6(5) + 4(0) = 30$
(4,2)	$f = 6(4) + 4(2) = \boxed{32}$
(0,4)	$f = 6(0) + 4(4) = 16$

choosing $(x, y) = (4, 2)$, where objective function f maximum at value 32.

(b) *Initial corner point A.* Start at $(x, y) = (0, 0)$, $(s_1, s_2) = (8, 10)$ and $f = 0$

x	y	s_1	s_2	f	
1	2	1	0	0	8
2	1	0	1	0	10
-6	-4	0	0	1	0

Type this 3 by 6 table into MATRIX [A]; use 2nd MATRIX EDIT.

where first row of this (augmented) matrix is constraint
$\boldsymbol{x + 2y + s_1 = 8}$ / $\boldsymbol{2x + y + s_2 = 10}$ / $\boldsymbol{-6x - 4y + f = 0}$
and second row of matrix is constraint
$\boldsymbol{x + 2y + s_1 = 8}$ / $\boldsymbol{2x + y + s_2 = 10}$ / $\boldsymbol{-6x - 4y + f = 0}$
and third row of matrix is objective function (*indicators*)
$\boldsymbol{x + 2y + s_1 = 8}$ / $\boldsymbol{2x + y + s_2 = 10}$ / $\boldsymbol{-6x - 4y + f = 0}$

(c) *Corner point A to B.*
Pivot with smallest quotient $\frac{10}{2} = 5$ *and* most negative indicator -6,

x	y	s_1	s_2	f		
1	2	1	0	0	8	Quotient: $\frac{8}{1} = 8$
$\boxed{2}$	1	0	1	0	10	Quotient: $\frac{10}{2} = 5$
-6	-4	0	0	1	0	

$\frac{1}{2}R_2 \to R_2,\quad R_1 - R_2 \to R_1,\quad R_3 + 6R_2 \to R_3 \longrightarrow$

x	y	s_1	s_2	f	
0	1.5	1	-0.5	0	3
1	0.5	0	0.5	0	5
0	-1	0	3	1	30

2nd MATRIX MATH *row(1/2,[A],2) STO 2nd MATRIX [B]

2nd MATRIX MATH *row+(−1,[B],2,1) STO 2nd MATRIX [C]

2nd MATRIX MATH *row+(6,[C],2,3) STO 2nd MATRIX [D].

which corresponds to corner point B where

i. $(x, y) = (0, 0)$, $(s_1, s_2) = (3, 0)$ and $f = 30$
ii. $(x, y) = (5, 0)$, $(s_1, s_2) = (3, 0)$ and $f = 30$

objective function f has **decreased** / **increased** from $f = 0$ to $f = 30$

(d) *Corner point B to C.*
Pivot with smallest quotient $\frac{3}{1.5} = 2$ *and* most negative indicator -1,

x	y	s_1	s_2	f		
0	[1.5]	1	-0.5	0	3	Quotient: $\frac{3}{1.5} \approx 2$
1	0.5	0	0.5	0	5	Quotient: $\frac{5}{0.5} = 10$
0	-1	0	3	1	30	

$\frac{1}{1.5}R_1 \to R_1,\quad R_2 - 0.5R_1 \to R_2,\quad R_3 + R_1 \to R_3 \longrightarrow$

x	y	s_1	s_2	f	
0	1	$\frac{2}{3}$	$-\frac{1}{3}$	0	2
1	0	$-\frac{1}{3}$	$\frac{2}{3}$	0	4
0	0	$\frac{2}{3}$	$\frac{8}{3}$	1	32

2nd MATRIX MATH *row(1/1.5,[D],1) STO 2nd MATRIX [E];

2nd MATRIX MATH *row+(−0.5,[E],1,2) STO 2nd MATRIX [F];

2nd MATRIX MATH *row+(1,[F],1,3) STO 2nd MATRIX [G].

which corresponds to corner point C where

i. $(x, y) = (4, 0)$, $(s_1, s_2) = (0, 2)$ and $f = 32$
ii. $(x, y) = (4, 2)$, $(s_1, s_2) = (0, 0)$ and $f = 32$

objective function f has **decreased** / **increased** from $f = 30$ to $f = 32$.
This is *optimal* solutions because all indicators are nonnegative.

(e) *Smallest quotient rule: Corner A to corner B or infeasible D.*
True / **False.** Pivoting on element 2, with *minimum* quotient $\frac{10}{2} = 5$,

x	y	s_1	s_2	f		
1	2	1	0	0	8	Quotient: $\frac{8}{1} = 8$
[2]	1	0	1	0	10	Quotient: $\frac{10}{2} = 5$, minimum
-6	-4	0	0	1	0	

results in matrix with corresponding feasible corner point B,

$\overset{\frac{1}{2}R_2 \to R_2,\quad R_1 - R_2 \to R_1,\quad R_3 + 6R_2 \to R_3}{\longrightarrow}$

x	y	s_1	s_2	f	
0	1.5	1	-0.5	0	3
1	0.5	0	0.5	0	5
0	-1	0	3	1	30

whereas pivoting on element 1,

x	y	s_1	s_2	f		
[1]	2	1	0	0	8	Quotient: $\frac{8}{1} = 8$
2	1	0	1	0	10	Quotient: $\frac{10}{2} = 5$, minimum
-6	-4	0	0	1	0	

results in matrix with corresponding infeasible point D,

$\overset{R_2 - 2R_1 \to R_2,\quad R_3 + 6R_1 \to R_3}{\longrightarrow}$

x	y	s_1	s_2	f	
1	2	1	0	0	8
0	-3	-2	1	0	-6
0	8	6	0	1	48

In other words, pivot using smallest quotient rule to ensure feasibility.

2. *Another problem.*

$$
\begin{array}{lcccccccc}
\text{Maximize} & 4x & + & 2y & + & 3z & & \\
\text{subject to} & 0.1x & + & 0.25y & & & \leq & 40 \\
 & 0.22x & + & 0.3y & + & 0.4z & \leq & 100 \\
 & x & & & & & \geq & 0 \\
 & & & y & & & \geq & 0 \\
 & & & & & z & \geq & 0
\end{array}
$$

(a) *Initial corner point.*
Start at $(x, y, z) = (0, 0, 0)$, $(s_1, s_2) = (40, 100)$ and $f = 0$.

x	y	z	s_1	s_2	f		
0.1	0.25	0	1	0	0	40	Quotient: ______
0.22	0.3	0.4	0	1	0	100	Quotient: ______
-4	-2	-3	0	0	1	0	

Type this 3 by 7 table into MATRIX [A]; use 2nd MATRIX EDIT.
Pivot column, most negative indicator, is column $\boldsymbol{x}$ / $\boldsymbol{y}$ / $\boldsymbol{z}$ / $\boldsymbol{s_1}$ / $\boldsymbol{s_2}$
Pivot row, smallest quotient, is row $\boldsymbol{R_1}$ / $\boldsymbol{R_2}$
So pivot element: **0.1** / **0.22** / **0.25**

(b) *Next corner point.*

$$\xrightarrow{\frac{1}{0.1}R_1 \to R_1,\quad R_2 - 0.22R_1 \to R_2,\ R_3 + 4R_1 \to R_3}$$

x	y	z	s_1	s_2	f		
1	2.5	0	10	0	0	400	Quotient: ______
0	-0.25	0.4	-2.2	1	0	12	Quotient: ______
0	8	-3	40	0	1	1600	

2nd MATRIX MATH *row(1/0.1,[A],1) STO 2nd MATRIX [B]

2nd MATRIX MATH *row+(−0.22,[B],1,2) STO 2nd MATRIX [C]

2nd MATRIX MATH *row+(4,[C],1,3) STO 2nd MATRIX [D].

Pivot column, most negative indicator, is column $\boldsymbol{x}$ / $\boldsymbol{y}$ / $\boldsymbol{z}$ / $\boldsymbol{s_1}$ / $\boldsymbol{s_2}$
Pivot row, smallest quotient[4], is row $\boldsymbol{R_1}$ / $\boldsymbol{R_2}$
So pivot element: **−0.25** / **0** / **0.4**

(c) *Final corner point.*

$$\xrightarrow{\frac{1}{0.4}R_2 \to R_2,\ R_3 + 3R_2 \to R_3}$$

x	y	z	s_1	s_2	f	
1	2.5	0	10	0	0	400
0	-0.6	1	-5.5	2.5	0	30
0	6.1	0	23.5	7.5	1	1690

2nd MATRIX MATH *row(1/0.4,[D],2) STO 2nd MATRIX [E]

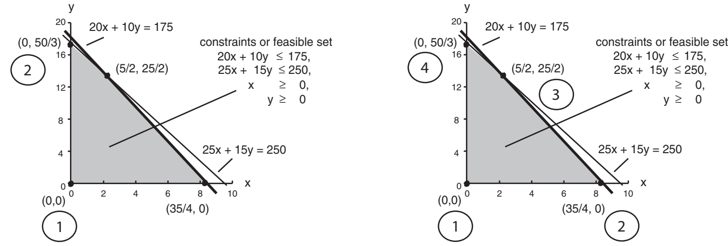
2nd MATRIX MATH *row+(3,[E],2,3) STO 2nd MATRIX [F].

Point $(x, y, z) = \mathbf{(400, 30, 0)}$ / $\mathbf{(0, 30, 400)}$ / $\mathbf{(400, 0, 30)}$,
with optimal (maximum) $f =$ **400** / **1600** / **1690**
because all indicators **nonpositive** / **zero** / **nonnegative**

(d) Since this problem involves *three* variables, (x, y, z), rather than two (x, y), **easy** / **difficult** to solve using geometric method.

3. *Acme party problem.*

$$\begin{array}{lcccccc} \text{Maximize} & \text{x} & + & \text{y} & & \\ \text{subject to} & 25\text{x} & + & 15\text{y} & \leq & 250 \\ & 20\text{x} & + & 10\text{y} & \leq & 175 \\ & \text{x} & & & \geq & 0 \\ & & & \text{y} & \geq & 0 \end{array}$$



(a) one way around (b) a longer way around

[4]Ignore undefined quotient, so R_1 has smallest quotient by default.

Figure 5.5 (Two paths to optimal solution)

(a) *Review.* **True** / **False.** Geometric solution to this linear programming problem involves calculating value of objective function at four corners,

corner	$f = x + y$ or $-x - y + f = 0$
(0,0)	$f = 0 + 0 = 0$
$\left(0, \frac{50}{3}\right)$	$f = 0 + \frac{50}{3} = \boxed{\frac{50}{3}}$
$\left(\frac{5}{2}, \frac{25}{2}\right)$	$f = \frac{5}{2} + \frac{25}{2} = 15$
$\left(\frac{35}{4}, 0\right)$	$f = \frac{35}{4}$

$(x, y) = \left(0, \frac{50}{3}\right)$, where objective function f maximum at value $\frac{50}{3}$.

(b) *Initial corner point.*
Start at $(x, y) = (0, 0)$, $(s_1, s_2) = (250, 175)$ and $f = 0$.

x	y	s_1	s_2	f		
25	15	1	0	0	250	Quotient: ______
20	10	0	1	0	175	Quotient: ______
-1	-1	0	0	1	0	

Type this 3 by 6 table into MATRIX [A]; use 2nd MATRIX EDIT.

Pivot column, most negative indicator, is column $\boldsymbol{x}$ / $\boldsymbol{y}$ / **either $\boldsymbol{x}$ or $\boldsymbol{y}$**
Using pivot column y, pivot row, smallest quotient, is row $\boldsymbol{R_1}$ / $\boldsymbol{R_2}$
So pivot element: **10** / **15** / **20**

(c) *Final corner point.*

$\frac{1}{15}R_1 \to R_1,\quad R_2 - 10R_1 \to R_2,\ R_3 + R_1 \to R_3 \longrightarrow$

x	y	s_1	s_2	P	
5/3	1	1/15	0	0	50/3
10/3	0	$-2/3$	1	0	25/3
2/3	0	1/15	0	1	50/3

2nd MATRIX MATH *row(1/15,[A],1) STO 2nd MATRIX [B]

2nd MATRIX MATH *row+(−10,[B],1,2) STO 2nd MATRIX [C].

2nd MATRIX MATH *row+(1,[C],1,3) STO 2nd MATRIX [D]

Point $(x, y) = \left(\boldsymbol{\frac{50}{3}, 0}\right)$ / $\left(\boldsymbol{0, \frac{50}{3}}\right)$
with optimal (maximum) $f = \mathbf{0}$ / $\boldsymbol{\frac{50}{3}}$
because all indicators **nonpositive** / **zero** / **nonnegative**
If pivot column x chosen, optimal in three steps, rather than one step.

4. *LP problem with no solution.*

Maximize	x	+	2y		
subject to	-3x	+	2y	$\leq$	5
	4x	-	3y	$\leq$	9
	x			$\geq$	0
			y	$\geq$	0

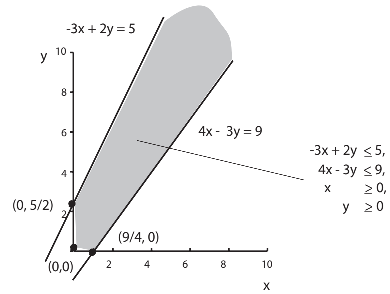


Figure 5.6 (Open bound, no optimal solution)

(a) *Review.* **True** / **False.** Geometric solution to this linear programming problem involves calculating value of objective function at corners,

corner	$f = x + 2y$ or $-x - 2y + f = 0$
$(0, 0)$	$f = 0 + 2(0) = 0$
$\left(0, \frac{5}{2}\right)$	$f = 0 + 2\left(\frac{5}{2}\right) = 5$
$\left(\frac{9}{4}, 0\right)$	$f = \frac{9}{4} + 2(0) = \frac{9}{4}$

However, for example, $(x, y) = (5, 10)$ is feasible because
$-3x + 2y = -3(5) + 10(10) = 85 \leq 5$
and $4x - 3y = 4(5) - 3(10) = -10 \leq 9$
and has objective function $f = 5 + 2(10) = 25$
which is larger than maximum $f = \frac{9}{4}$ at corner point $\left(\frac{9}{4}, 0\right)$.
In fact, objective function f unbounded (has no maximum);

(b) *Initial corner point.*
Start at $(x, y) = (0, 0)$, $(s_1, s_2) = (5, 9)$ and $f = 0$.

x	y	s_1	s_2	f		
-3	2	1	0	0	5	Quotient ______
4	-3	0	1	0	9	Quotient ______
-1	-2	0	0	1	0	

Type this 3 by 6 table into MATRIX [A]; use 2nd MATRIX EDIT.
Pivot column, most negative indicator, is column $\boldsymbol{x}$ / $\boldsymbol{y}$
Pivot row, smallest *nonnegative* quotient[5], is row $\boldsymbol{R_1}$ / $\boldsymbol{R_2}$
So pivot element: $\mathbf{-3}$ / $\mathbf{2}$ / $\mathbf{4}$

(c) *Simplex algorithm stops.*

$$\xrightarrow{\frac{1}{2}R_1 \to R_1, \quad R_2 + 3R_1 \to R_2, \;\; R_3 + 2R_1 \to R_3}$$

x	y	s_1	s_2	f		
-1.5	1	0.5	0	0	2.5	Quotient ______
-0.5	0	1.5	1	0	16.5	Quotient ______
-4	0	1	0	1	5	

[5]Negative quotient not used, so smallest quotient automatically $\frac{5}{2}$.

2nd MATRIX MATH *row(1/2,[A],1) STO 2nd MATRIX [B]

2nd MATRIX MATH *row+(3,[B],1,2) STO 2nd MATRIX [C].

2nd MATRIX MATH *row+(2,[C],1,3) STO 2nd MATRIX [D]

Pivot column, most negative indicator, is column $\boldsymbol{x}$ / $\boldsymbol{y}$
Pivot row, smallest *nonnegative* quotient: $\boldsymbol{R_1}$ / $\boldsymbol{R_2}$ / **neither** $\boldsymbol{R_1}$ **nor** $\boldsymbol{R_2}$
So pivot element: $\mathbf{-1.5}$ / $\mathbf{-0.5}$ / **does not exist**
so no optimal solution.

5. *LP problem with edge solution.*

$$\begin{array}{lcccccc} \text{Maximize} & 6x & + & 3y & & \\ \text{subject to} & x & + & 2y & \leq & 8 \\ & 2x & + & y & \leq & 10 \\ & x & & & \geq & 0 \\ & & & y & \geq & 0 \end{array}$$

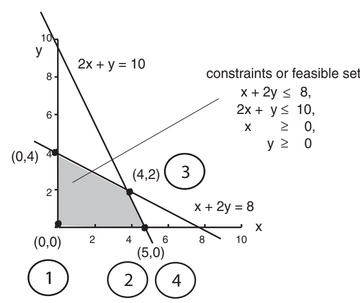


Figure 5.7 (LP problem with edge solution)

(a) Geometric solution to this linear programming problem involves calculating value of objective function at four corners,

vertex	$f = 6x + 3y$
(0, 0)	$f = 6(0) + 3(0) = 0$
(0, 4)	$f = 6(0) + 3(4) = 12$
(4, 2)	$f = 6(4) + 3(2) = \boxed{30}$
(5, 0)	$f = 6(5) + 3(0) = \boxed{30}$

where objective function f maximum at $(x, y) =$
$\mathbf{(4,2)}$ / $\mathbf{(5,0)}$ / **all points along edge between** $\mathbf{(5,0)}$ **and** $\mathbf{(4,2)}$

(b) *Initial corner point.*
Start at $(x, y) = (0, 0)$, $(s_1, s_2) = (8, 10)$ and $f = 0$.

x	y	s_1	s_2	f		
1	2	1	0	0	8	Quotient ______
2	1	0	1	0	10	Quotient ______
-6	-3	0	0	1	0	

Type this 3 by 6 table into MATRIX [A]; use 2nd MATRIX EDIT.
Pivot column, most negative indicator, is column $\boldsymbol{x}$ / $\boldsymbol{y}$
Pivot row, smallest nonnegative quotient, is row $\boldsymbol{R_1}$ / $\boldsymbol{R_2}$
So pivot element: **1** / **2** / **4**

(c) *Next corner.*

$$\frac{1}{2}R_2 \to R_2, \quad R_1 - R_2 \to R_1, \; R_3 + 6R_2 \to R_3 \; \longrightarrow$$

x	y	s_1	s_2	f		
0	1.5	1	-0.5	0	3	Quotient ______
1	0.5	0	0.5	0	5	Quotient ______
0	[0]	0	3	1	30	

2nd MATRIX MATH *row(1/2,[A],2) STO 2nd MATRIX [B]
2nd MATRIX MATH *row+(−1,[B],2,1) STO 2nd MATRIX [C].
2nd MATRIX MATH *row+(6,[C],2,3) STO 2nd MATRIX [D]
Point $(x, y) = \mathbf{(5, 0)}$ / $\mathbf{(0, 5)}$
with optimal (maximum) $f =$ **0** / **30**
because all indicators **nonpositive** / **zero** / **nonnegative**
However nonbasic y column has *zero* indicator,
indicating another corner just "as optimal".
Pivot column, zero indicator, is column $\boldsymbol{x}$ / $\boldsymbol{y}$
Pivot row, smallest nonnegative quotient, is row $\boldsymbol{R_1}$ / $\boldsymbol{R_2}$
So pivot element: **0.5** / **1.5** / **4**

(d) *Next corner along edge.*

$$\frac{1}{1.5}R_1 \to R_1, \quad R_2 - 0.5R_1 \to R_2, \; R_3 + 0R_1 \to R_3 \; \longrightarrow$$

x	y	s_1	s_2	f		
0	1	2/3	-1/3	0	2	Quotient ______
1	0	-1/3	2/3	0	4	Quotient ______
0	0	[0]	3	1	30	

2nd MATRIX MATH *row(1/1.5,[D],2) STO 2nd MATRIX [E]
2nd MATRIX MATH *row+(−0.5,[E],2,1) STO 2nd MATRIX [F].
2nd MATRIX MATH *row+(0,[F],2,3) STO 2nd MATRIX [G]
Point $(x, y) = \mathbf{(2, 4)}$ / $\mathbf{(4, 2)}$
with optimal (maximum) $f =$ **0** / **30**
because all indicators **nonpositive** / **zero** / **nonnegative**
Pivot column, zero indicator, is column $\boldsymbol{y}$ / $\boldsymbol{s_1}$
Pivot row, smallest nonnegative quotient, is row $\boldsymbol{R_1}$ / $\boldsymbol{R_2}$
So pivot element: $\mathbf{\frac{1}{3}}$ / $\mathbf{\frac{2}{3}}$ / **1**
Pivoting returns to corner (5,0), so edge between (5,0) and (4,2) optimal.

6. *Using EXCEL.*

$$\begin{array}{lccccc} \text{Maximize} & 6x & + & 4y & & \\ \text{subject to} & x & + & 2y & \leq & 8 \\ & 2x & + & y & \leq & 10 \\ & x & & & \geq & 0 \\ & & & y & \geq & 0 \end{array}$$

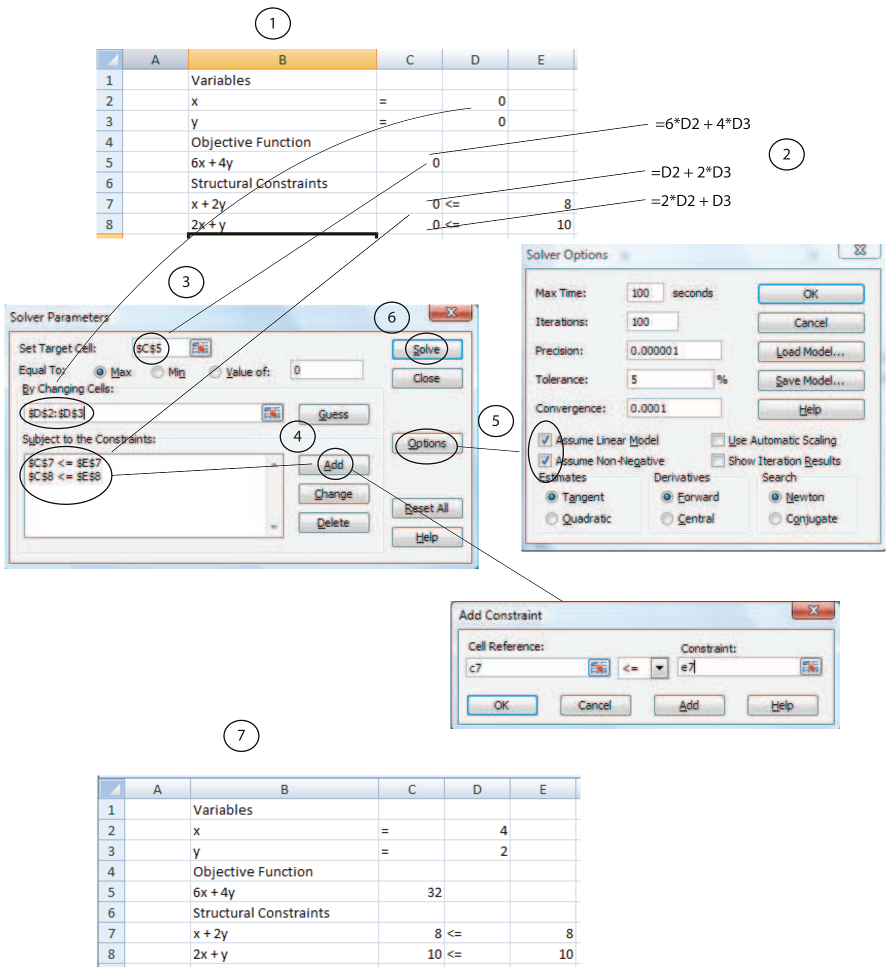


Figure 5.8 (Simplex algorithm using EXCEL)

(a) Step 1. Type variables, objective function and structural constraints as given in column B of figure.

(b) Step 2. Define cells as given in figure. They should agree with objective function and constraints.

(c) Step 3. Open Solver. It is under Data tab in Excel[6]. Type information into Target Cells and By Changing Cells as given in figure.

[6] If Solver is *not* available in Data tab, add it by clicking on large office button in left top corner of Excel, then on Excel Options a bottom of popup, then click on Add-Ins on left side, then add in Solver.

(d) Step 4. Click on Add, then type in information in Add Constraints pop up box as given in figure. Click Add after each constraint, then Cancel when done.

(e) Step 5. Click on Options and check both Assume Linear Model and Assume Non-Negative.

(f) Step 6. Click on Solve.

(g) Step 7. Observe solution, which should be $(x, y) = (4, 2)$, where objective function f maximum at value 32.