## 7.5 The Multiplication Principle

*Multiplication principle* of counting is if $n_1$ possible first choices, $n_2$ second choices and so on, then total of $n_1 \cdot n_2 \cdot ...$ choices. Counting can be an important part of determining various probabilities. To help in understanding various counting techniques, we first discuss notions of whether order matters or not and whether to sample with or without replacement. We look at two ways of visualizing counting techniques: tree diagrams and "marbles in a box".

**Exercise 7.5 (The Multiplication Principle)**

1. *Order matters or not when counting.*

   (a) *Committees (Generic).* Three possible committees of two people from Jim, Sue and Ali are (Jim, Sue), (Jim, Ali) and (Sue, Ali). We **would** / **would not** include both (Jim, Sue) and (Sue, Jim) in the count because *order* of people chosen for this committee *does not matter*.

   (b) *Committees (Different Roles).* We **would** / **would not** include (Jim, Sue) and (Sue, Jim) in a count of (chair, secretary) committees because of two roles in committee; *order* of people chosen *does matter*.

   (c) *Street numbers.* When counting 3–digit street numbers, **order matters** / **order does not matter**. Are (9,3,4) and (3,4,9) two street numbers (order matters) or one street number (order does not matter)?

   (d) *Cards.* When dealing five cards for a poker hand, **order matters** / **order does not matter**. Are (10,J,Q,2,3) and (2,3,J,Q,10) two hands (order matters) or one hand (order does not matter)?

   (e) *Cars.* When parking cars in parking spots (P1, P2, P3), **order matters** / **order does not matter**. Are (Ford,GM,Toyota) and (GM,Toyota,Ford) two different parking arrangements (order matters) or one parking arrangement (order does not matter)?

   (f) *(Indistinguishable) Cars.* When parking *indistinguishable* cars in parking spots (P1, P2, P3), **order matters** / **order does not matter**. Are (car,car,car) and (car,car,car) two different parking arrangements (order matters) or one parking arrangement (order does not matter)?

   (g) We count **more** / **less** if we assume order matters.

2. *Sampling with or without replacement (repetition or not) when counting.*

   (a) *Committees.* Three possible committees of two people from Jim, Sue and Ali are (Jim, Sue), (Jim, Ali) and (Sue, Ali). We **would** / **would not** include (Jim, Jim) in count because Jim can only be chosen once; we *sample without replacement*.

(b) *Street numbers.* When counting 3–digit street numbers, **sample with replacement** / **sample without replacement**. Is it possible to have street numbers with two of three digits the same (sample with replacement) or not (sample without replacement)?

(c) *Cards.* When dealing five cards for a poker hand, **sample with replacement** / **sample without replacement**. Is it possible to have two identical cards in a hand (sample with replacement) or not (sample without replacement), playing with one deck of cards?

(d) *Cars.* When parking cars in parking spots (P1, P2, P3), **sample with replacement** / **sample without replacement**. Is it possible to park same car in two different spots (sample with replacement) or not (sample without replacement)?

(e) We count **more** / **less** if we sample with replacement.

3. *Marbles in a box used for counting outcomes.*

(a) Number of 3–digit street numbers, as indicated in "marbles in boxes" counting method below: (choose *two*!) $\mathbf{10 \times 10 \times 10}$ / **100** / **1000**.

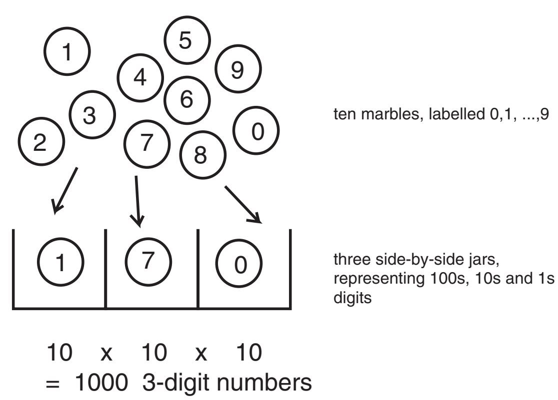


Figure 7.9 (Marbles in a box)

(b) Number of three–digit numbers if first number cannot be zero: (circle *one or more*!) $\mathbf{9 \times 10 \times 10}$ / **900** / **999**.

(c) Number of three–digit numbers if first number must be 3: (circle *one or more*!) $\mathbf{1 \times 10 \times 10}$ / **100** / **900**.

(d) Number three–digit numbers if first number is 3, second cannot be 9: $\mathbf{9 \times 9 \times 10}$ / $\mathbf{9 \times 10 \times 10}$ / $\mathbf{1 \times 9 \times 10}$.

(e) Number of *four*–digit numbers if first number cannot be zero: $\mathbf{9 \times 10 \times 10}$ / $\mathbf{9 \times 10 \times 10 \times 10}$ / $\mathbf{1 \times 9 \times 10 \times 10}$.

(f) Number of three–digit numbers with exactly two 3s: $\mathbf{1 \times 1 \times 9 = 9}$ / $\mathbf{1 \times 9 \times 1 = 9}$ / $\mathbf{9 \times 1 \times 1 = 9}$ / $\mathbf{9 + 9 + 9 = 27}$.

Hint: Correct answer is fourth one, 27, and is obtained by adding first three possible answers together. Each of first three answers represent a different way to have exactly two 3s in three digits.

(g) Number of three-digit numbers with at least one zero?
$\mathbf{10 \times 10 \times 10}$ / $\mathbf{9 \times 9 \times 9}$ / $\mathbf{10^3 - 9^3 = 271}$

Hint: All three-digit numbers minus three-digit numbers that do *not* contain zero.

4. *Tree diagrams used for multiplication rule of counting outcomes.*

(a) As shown in tree diagram, there are $\mathbf{3 + 2 = 5}$ / $\mathbf{3 \times 2 = 6}$ possible (treasurer, secretary) pairs from three eligible treasurer candidates ($T_1$, $T_2$ and $T_3$) and two eligible secretary candidates ($S_1$ and $S_2$), including $\{(T_1, S_1), (T_1, S_2), (T_2, S_1), (T_2, S_2), (T_3, S_1), (T_3, S_2)\}$.

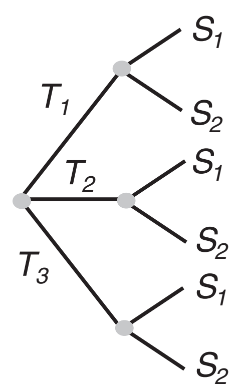


Figure 7.10 (Tree diagram)

(b) Number of treasurer and secretary pairs, when *four* eligible treasurer candidates, *eight* eligible secretary candidates, is
$\mathbf{4 + 8 = 12}$ / $\mathbf{4 \times 8 = 24}$ / $\mathbf{32}$.

(c) Number of treasurer, secretary and president triplets, when *three* eligible treasurer candidates, *five* eligible secretary candidates and *two* eligible president candidates

$\mathbf{3 + 5 + 2 = 10}$ / $\mathbf{20}$ / $\mathbf{3 \times 5 \times 2 = 30}$.

5. *More multiplication principle.*

(a) How many different ways can a line of 5 females and 4 males waiting to buy books at bookstore be arranged where females are grouped together at head of line and males are grouped together at back of line?

$$\underbrace{(5 \times 4 \times 3 \times 2 \times 1)}_{females} \times \underbrace{(4 \times 3 \times 2 \times 1)}_{males} =$$

$\mathbf{20}$ / $\mathbf{2880}$ / $\mathbf{362,880}$

(b) How many different ways can a line of 5 females and 4 males waiting to buy books at bookstore be arranged where females are grouped together and males are grouped together?
**2880 / 5760 / 362, 880**
Hint: Either line of females, males or males, females.

## 7.6 Permutations

- *Permutation*: count of ordered arrangement of $r$ from $n$ distinct objects, sampled without replacement

$$P(n, r) = n(n-1)\cdots(n-r+2)(n-r+1) = \frac{n!}{(n-r)!}, \quad r \leq n$$

- *Permutation of nondistinct items*: count of ordered arrangement of $n$ objects into $k_1$, $k_2$, ..., $k_m$ groups, each group consisting of indistinguishable objects, sampled without replacement

$$\frac{n!}{k_1!k_2!\cdots k_m!}$$

- *Permutation of distinct items with replacement*: count of ordered arrangement of $r$ of $n$ distinct objects, sampled *with* replacement

$$n^r$$

**Exercise 7.6 (Permutations)**

1. *Factorial notation used in formulas for counting outcomes.*

   (a) Special mathematical notation, called *factorial notation*, denoted by an exclamation mark, "!", is often used in formulas used to count outcomes in a sample space. For example,

   $$5! = 5 \times 4 \times 3 \times 2 \times 1 =$$

   **100 / 110 / 120**.
   (Use your calculator: type five (5), then MATH PRB 4:! ENTER.)

   (b) 7! is equal to (*circle none, one or more*)

      i. $7 \times 6!$
      ii. 5040
      iii. $7 \times 6 \times 5!$

iv. $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

(Use your calculator: type seven (7), then MATH PRB 4:! ENTER.)

(c) $\frac{7!}{5!}$ is equal to (*circle none, one or more*)

i. $\frac{7\times6\times5\times4\times3\times2\times1}{5\times4\times3\times2\times1}$

ii. $7 \times 6$

iii. 42

(d) $\frac{7!}{5!3!}$ is equal to (circle none, one or more)

i. $\frac{7\times6\times5\times4\times3\times2\times1}{(5\times4\times3\times2\times1)(3\times2\times1)}$

ii. $\frac{7\times6\times5\times4}{5\times4\times3\times2\times1}$

iii. $\frac{7\times6}{3\times2\times1}$

iv. $\frac{42}{6}$

(e) $(7-3)!$ is equal to
(circle none, one or more) $\mathbf{7! - 3!}$ / $\mathbf{4!}$ / $\mathbf{4 \times 3 \times 2 \times 1}$ / **24**.

(f) $\frac{7!}{(7-3)!}$ is equal to (circle none, one or more)

i. $\frac{7\times6\times5\times4\times3\times2\times1}{4\times3\times2\times1}$

ii. $7 \times 6 \times 5$

iii. 210

(g) *By definition* (in other words, accept as true that), $0! = 1$, and so $0!$ is equal to **1!** / **2!** / **3!**.

2. *Permutations (order matters): parking cars.*

(a) How many different ways can three of five cars be parked in three different side–by–side parking spots? As shown in marbles in boxes diagram below, since five different cars occupy first parking spot, only four occupy second parking spot (since one car is in first parking spot) and three could occupy final parking spot, number of permutations is
$\mathbf{5 + 4 + 3 = 12}$ / $\mathbf{5 \times 4 = 20}$ / $\mathbf{5 \times 4 \times 3 = 60}$.

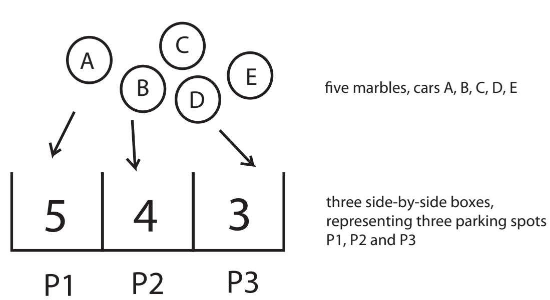


Figure 7.11 (Permutations: arrangements of parked cars)

(b) *Assumptions.* In marbles in boxes analogy, side–by–side boxes represent parking spots and marbles represent cars, A, B, C, D and E. Since (A,B,C) and (B,A,C) count as two different parking arrangements, even though same three cars are used for both, "marbles in boxes" counting method assumes **order matters** / **order does not matter**.
Since same car cannot appear in two different parking spots,
**sample with replacement** / **sample without replacement**
Cars **distinguishable** / **indistinguishable** from one another.

(c) Count parking four of $n = 5$ cars in $r = 4$ parking spots (one or more)

i. $5 \times 4 \times 3 \times 2 = 120$
ii. $\frac{5\times4\times3\times2\times1}{1}$
iii. $\frac{5!}{1!}$
iv. $\frac{n!}{(n-r)!}$, where $n = 5$ and $r = 4$
v. $P(5, 4) = 120$

Calculator: type 5, then MATH PRB 2:nPr ENTER 4 ENTER.

(d) Count parking four of $n = 7$ cars in $r = 4$ parking spots (one or more)

i. $P(7, 4) = 840$
ii. $\frac{n!}{(n-r)!}$, where $n = 7$ and $r = 4$
iii. $\frac{7!}{(7-4)!}$
iv. $\frac{7!}{3!}$
v. $7 \times 6 \times 5 \times 4$

Calculator: type seven (7), then MATH PRB 2:nPr ENTER 4 ENTER.

(e) Count parking eight of $n = 12$ cars in $r = 8$ parking spots (one or more)

i. $P(12, 8) = 19,958,400$
ii. $\frac{n!}{(n-r)!}$, where $n = 12$ and $r = 8$
iii. $\frac{12!}{(12-8)!}$
iv. $\frac{12!}{4!}$
v. $12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5$

(f) Count parking twelve of $n = 12$ cars in $r = 12$ parking spots (one or more)

i. $\frac{n!}{(n-r)!}$, where $n = 12$ and $r = 12$
ii. $P(12, 12) = \frac{12!}{(12-12)!}$
iii. $\frac{12!}{0!}$
iv. $12!$

(g) Number of ways of parking eight of $n = 8$ cars in $r = 8$ parking spots
$P(8, 8) =$ **7!** / **8!** / **9!**

3. *More permutations (order matters).*

   (a) *Permutations and multiplication principle.* Nine different cars are driven into nine parking spots. Three cars are driven by (distinguishable) statistics students and three parking spots are reserved for these statistics students only. Six cars are driven by (distinguishable) mathematics students and six parking spots reserved for these students only. Number of ways cars parked is $\mathbf{3!6! = (3)(6) = 18}$ / $\mathbf{3!6! = (6)(720) = 4320}$ / $\mathbf{9!}$ ways.

   (b) *Permutations.* Nine different cars are driven into nine parking spots. Three cars are driven by (distinguishable) statistics students and six cars are driven by (distinguishable) mathematics students. Number of ways cars parked is $\mathbf{3!6! = (3)(6) = 18}$ / $\mathbf{3!6! = (6)(720) = 4320}$ / $\mathbf{9!}$ ways.

   (c) *Permutations.* Number of ways a line of 5 females and 4 males waiting to buy books at bookstore can be arranged:
   $\mathbf{5!4! = (5)(4) = 20}$ / $\mathbf{5!4! = (120)(24) = 2880}$ / $\mathbf{9!}$

   (d) *Permutations and multiplication principle.* How many different ways can a line of 5 statistics students, 4 business students and 3 technology students, waiting to buy books at bookstore, be arranged where statistics students are grouped together at head of line, business students are grouped together in middle of line and technology students are grouped together at end of line? (circle *two*!) $\mathbf{5!4!3!}$ / $\mathbf{17280}$ / $\mathbf{12!}$

   (e) *Permutations and multiplication principle.* How many different ways can a line of 5 statistics students, 4 business students and 3 technology students, waiting to buy books at bookstore, be arranged where the students in each subject are grouped together in line? $\mathbf{5!4!3!}$ / $\mathbf{3!5!4!3!}$ / $\mathbf{12!}$

   Hint: Three subjects can be arranged in $3! = 6$ ways.

4. *And more permutations (order matters).*

   (a) *Permutations and indistinguishable objects.* How many different letter permutations can be formed from A, A, R, D, V, A, R, K? Since eight letters in total, with three indistinguishable A's and two indistinguishable R's, number of letter permutations is $\mathbf{8!}$ / $\mathbf{\frac{8!}{3!2!}}$ / $\mathbf{\frac{8!}{5!}}$

   Hint: *If* letters were all distinguishable from one another, 8! permutation.

   "Divide" out 3! arrangements of indistinguishable A's and 2! arrangements of indistinguishable R's.

   (b) *Permutations, indistinguishable objects and multiplication principle.* How many different ways can a line of 5 statistics students, 4 business students and 3 technology students be arranged, where students in each subject are grouped together in line and where each arrangement lists only subject of student? (circle *two*!) $\mathbf{3!5!4!3!}$ / $\mathbf{\frac{3!5!4!3!}{5!4!3!}}$ / $\mathbf{3!}$

   Hint: *If* students were all distinguishable from one another, 3!5!4!3! permutation.

"Divide" out 5! arrangements of statistic students, 4! arrangements of business students and 3! arrangements of technology students.

(c) *Permutations and complement.* Nine different cars are driven into nine parking spots, three of which are covered by an overhang. Three cars are driven by (distinguishable) statistics students and six cars are driven by (distinguishable) mathematics students. Number of ways at least one statistics student car in covered spot is
$\boldsymbol{P(9,3) - P(6,3) = 384}$ / $\boldsymbol{P(9,3) + P(6,3) = 624}$ ways.

Hint: *Total* ways to park 9 cars in 3 covered spots is $P(9,3)$. Ways to park *no* (0) statistics students cars (equivalently, *all/only* 6 mathematics students cars) in 3 covered spots is $P(6,3)$. Subtracting, gives ways to park *at least one* statistics students car in covered spot.

## 7.7 Combinations

Count of *un*ordered arrangement of $r$ from $n$ distinct objects, sampled without replacement

$$C(n,r) = \frac{n!}{r!(n-r)!}, \quad r \leq n.$$

**Exercise 7.7 (Combinations)**

1. *A first look: combinations (order does not matter).*

   (a) Number of ways of dealing three cards from five cards (10,J,Q,K,A) is calculated by assuming order matters (5 marbles, three side–by–side boxes),

   $$5 \times 4 \times 3 = 60$$

   and then "dividing out the order" (3! = 6 permutations of three cards),

   $$\frac{5 \times 4 \times 3}{3!} = \frac{60}{6} =$$

   **9** / **10** / **11** combinations. See figure below.

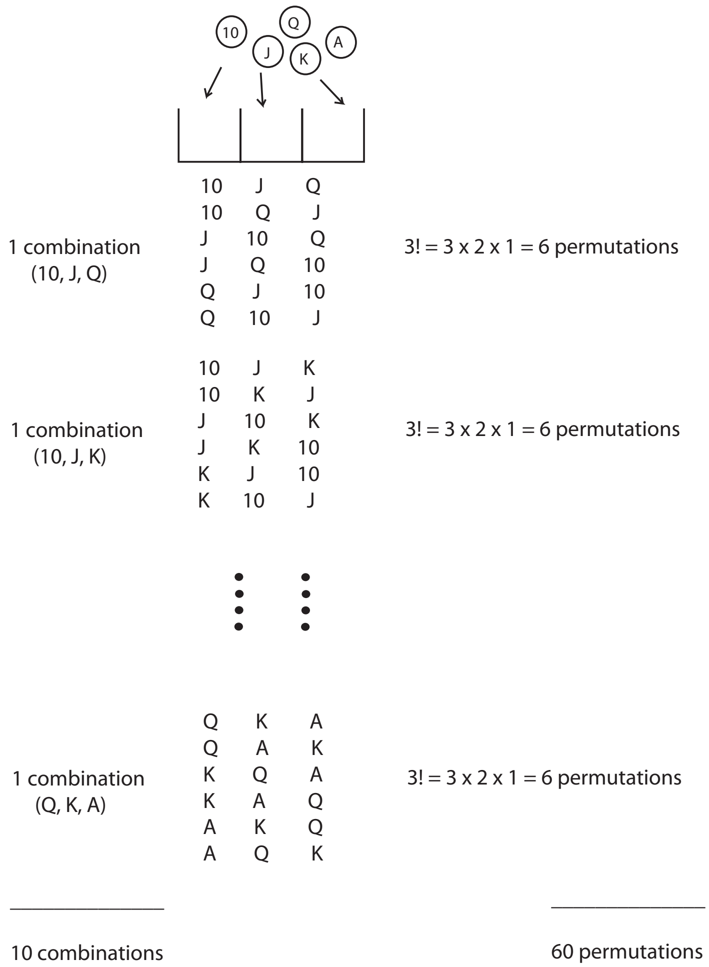


Figure 7.12 (Combinations: number of hands in 5 cards)

(b) Number of ways of dealing $r = 3$ of $n = 5$ cards (choose one or more)

i. $\frac{5\times4\times3}{3!}$

ii. $\frac{5\times4\times3\times2\times1}{(2\times1)3!}$

iii. $\frac{5\times4\times3\times2\times1}{(2\times1)(3\times2\times1)}$

iv. $C(5,3) = \frac{5!}{2!3!} = 10$

v. $C(n,r) = \frac{n!}{(n-r)!r!}$, where $n = 5$ and $r = 3$

(Use your calculator: type 5, then MATH PRB nCr ENTER 3 ENTER.)

(c) Number of ways of dealing $r = 3$ of $n = 11$ cards (choose one or more)

i. $C(n,r) = \frac{n!}{(n-r)!r!}$, where $n = 11$ and $r = 3$

ii. $\frac{11!}{(11-3)!3!}$

iii. $\frac{11!}{8!3!}$

iv. $\frac{11\times10\times9}{3\times2\times1}$

v. $C(11, 3) = 165$

(Use your calculator: type 11, then MATH PRB nCr ENTER 3 ENTER.)

2. *More combinations (order does not matter).*

(a) *Combinations and multiplication principle.* From a group of 6 women and 9 men, how many different committees consisting of 4 women and 3 men can be formed? (*one or more*!) $\boldsymbol{C(6,4)C(9,3)}$ / $\left(\frac{\mathbf{6 \cdot 5}}{\mathbf{4 \cdot 3 \cdot 2 \cdot 1}}\right)\left(\frac{\mathbf{9 \cdot 8 \cdot 7}}{\mathbf{3 \cdot 2 \cdot 1}}\right)$ / **1260**

Hint: *Four* (4) of 6 women ($C(6, 4)$) and 3 of 9 men are chosen ($C(9, 3)$).

(b) *Combinations, multiplication principle and inclusion-exclusion rule.* From a group of 6 women and 9 men, how many different committees consisting of 4 women and 3 men *or* 3 women and 4 men can be formed?
$\boldsymbol{C(6,4)C(9,3) + C(6,3)C(9,4)}$ / $\left(\frac{\mathbf{6 \cdot 5}}{\mathbf{4 \cdot 3 \cdot 2 \cdot 1}}\right)\left(\frac{\mathbf{9 \cdot 8 \cdot 7}}{\mathbf{3 \cdot 2 \cdot 1}}\right)$ / **1260**

(c) *Combinations and multiplication principle.* From a deck of 10 clubs, 9 spades, 11 diamonds and 12 hearts, how many different hands consisting of 3 clubs, 3 spades, 4 diamonds and 10 hearts can be formed? (circle *two*!)
$\boldsymbol{C(10,3)C(9,3)C(11,4)C(12,10)}$ / $\left(\frac{\mathbf{6 \cdot 5}}{\mathbf{4 \cdot 3 \cdot 2 \cdot 1}}\right)\left(\frac{\mathbf{9 \cdot 8 \cdot 7}}{\mathbf{3 \cdot 2 \cdot 1}}\right)$ / **219, 542, 400**

(d) *Combinations, multiplication principle and poker.*
Number of two–pair hands in 5–card poker hand?

i. Ways of dealing pair of *twos*, then pair of *threes*
$\boldsymbol{C(4,2)C(4,2)C(44,1)}$
$\boldsymbol{C(4,3)C(4,3)C(44,1)}$
$\boldsymbol{C(4,4)C(4,4)C(44,1)}$
where, remember, one other card from remaining 44 chosen.

ii. Since $C(13, 2)$ choices for *any* two pairs (without regard to their order, since both sets of cards are pairs),
$\boldsymbol{C(13,0) \times C(4,2)C(4,2)C(44,1)}$
$\boldsymbol{C(13,1) \times C(4,2)C(4,2)C(44,1)}$
$\boldsymbol{C(13,2) \times C(4,2)C(4,2)C(44,1)}$

(e) *Combinations, multiplication principle and poker.* How many different full house (three–of–a–kind and a pair, where order now does matter since one set of *three* cards is different from the other set of *two* cards) hands are possible in a 5–card poker hand?
$\boldsymbol{P(13,0) \times C(4,3)C(4,2)}$
$\boldsymbol{P(13,1) \times C(4,3)C(4,2)}$
$\boldsymbol{P(13,2) \times C(4,3)C(4,2)}$

(f) *Combinations and line of trees.* In a straight line of 8 trees, 3 have lost their leaves (and so 5 have not), how many orderings are there in which no two leafless trees are consecutive?
(Choose *one or more*!) $\boldsymbol{C(6,3)}$ / $\left(\frac{\mathbf{6 \cdot 5 \cdot 4}}{\mathbf{3 \cdot 2 \cdot 1}}\right)$ / **20** arrangements.

Since no two (assume indistinguishable) leafless trees side-by-side, they appear in six *possible positions*,

| L | T | L | T | L | T | L | T | L | T | L |

Six "L"'s represent where the three leafless trees may *possibly* appear, $C(6, 3)$.

## 3. *Using the TI–83 Calculator: Factorials, Permutations and Combinations.*

(a) A friend of mine went to a wine tasting event featuring Chardonnay wine. There were 20 wines available for tasting and she decided to try 8. Assuming the order of tasting is relevant, determine all the possible ways she can taste the 8 types of wine.

The answer to this question (assuming sampling without replacement where order matters) is: $(20)(19)\cdots(13) = \frac{20!}{12!}$.

The are (at least) two possible ways to use the TI–83 to perform this calculation. One involves the factorial key and the other involves the permutation key.

Following the factorial approach, enter:

- 20 MATH ◁ ∇ ∇ ∇ ENTER
- 12 MATH ◁ ∇ ∇ ∇ ENTER

to arrive at 5,079,110,400.

Following the permutation approach, enter:

- 20 MATH ◁ ENTER ∇ ENTER 8 ENTER

to, once again, arrive at 5,079,110,400.

(b) Shortly after being put into service, some buses manufactured by a certain company have developed cracks on the underside of the main frame. Suppose a particular city has 20 of these buses and cracks have actually appeared in 8 of them. How many different ways are there to select a sample of 5 buses from the 20 for a thorough inspection?

The answer, here, is: $\frac{20!}{5!15!}$. Using the calculator, the number is found by entering

20 MATH ◁ ∇ ∇ 5 ENTER

which gives 15,504.