Lecture Notes

Statistics 345

Probability and Statistics

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by

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Preface

The point of this course is to introduce the mathematical theory of probability. Some familiarity with both differentiation and integration is a necessary prerequisite to this course.

These lecture notes are a necessary component for a student to successfully complete this course. Without the lecture notes, a student will not be able to participate in the course. Without the text, a student will not have complete information about the course material. More than this,

- The lecture notes are *based* on the text.
- Although the material covered in each is very similar, the *presentation* of the material in the workbook is quite different from the presentation given in the text. The text consists essentially of definitions, formulas, worked out examples and exercises; these lecture notes, on the other hand, consists *solely* of exercises to be worked out by the student.
- The overheads presented during each lecture are based *exclusively* on the lecture notes. A student is to use these lecture notes to follow along with during a lecture.
- The lecture notes have a number of fill–in–the–blank, multiple choice, true/false and other kinds of interactive exercises which a student completes during lecture time. Rather than spend most time writing down what is given on the overhead during the lecture, a student can simply fill in the lecture notes.
- Many of the exercises given in the lecture notes are based on the text. A student should try the exercises out of both the lecture notes and text.

On the one hand, these lecture notes are, as you will see, quite a bit more elaborate than typical lecture notes, which are usually a summary of what the instructor finds important in a recommended course text. On the other hand, these lecture notes are not quite a text, because although it has many exercises, it does not have quite enough exercises to qualify it as a complete text. In short, these lecture notes aspire to be text and, in the next few years, when enough exercises have been collected, and when most of the typographical errors have been weeded out, it will become a text.

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Chapter 1

Basics of Probability

1.1 Introduction

In this chapter, we look at basic properties of probability.

1.2 Basic Concepts and Rules

We discuss random experiment, sample space and events; relative frequency approximation and theoretical approach to probability, law of large numbers and subjective probabilities.

Exercise 1.2 (Basic Concepts and Rules)

1. *Definitions.*

   - *Random experiment.*
     Process which results in a sample space where each outcome (element) is assigned a chance of occurrence.
   - *Sample space, S.*
     List of all possible outcomes (or simple events) of experiment.
   - *Event, A; simple event, a_i.*
     Event is subset of sample space of experiment; simple event if subset consists of one outcome.
   - *Relative frequency approximation to probability*:
     Probability of event A is approximated by the relative frequency an event A occurs in the total number of repetitions (trials) of a random experiment, the fraction (proportion):

$$P(A) \approx \frac{\text{number of times } A \text{ occurs}}{\text{total number of trials of experiment}}$$

- *Theoretical (classical) definition of probability*:
  Probability of event A in experiment, assuming equally likely outcomes in sample space S, where $n(A)$ is the number of outcomes in A, and where $n(S)$ is the number of outcomes in S, is:

$$P(A) = \frac{n(A)}{n(S)}.$$

- *Law of large numbers*:
  Under repeated trials of a random experiment, the relative frequency approaches and stays close to the theoretical probability. Since proportion is a special kind of average (of 0s and 1s), we sometimes say a probability is an average in the long run.

2. *Coin tossing.*

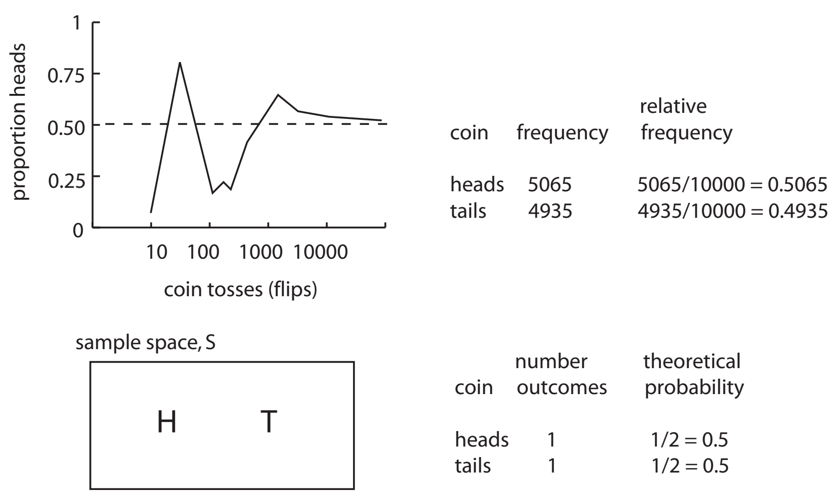


Figure 1.1: Terminology: coin tossing

(a) *Random experiment.*
Flipping a coin is a *probability experiment*. It is generally *un*known, when flipping a coin, whether coin comes up heads (Hs) or tails (Ts). However, it is known there are only two possible outcomes
(i) **{H, T}** (ii) **{H, H}** (iii) **{T, T}** .

(b) *Sample space.*
Sample space for flipping a coin is $S = \{\text{H, T}\}$. Sample space is
(i) **set** (ii) **subset** (iii) **element** of all possible outcomes.

(c) *Event, simple event.*
Flipping a head {H} is an example of an *event*, E. An event is a

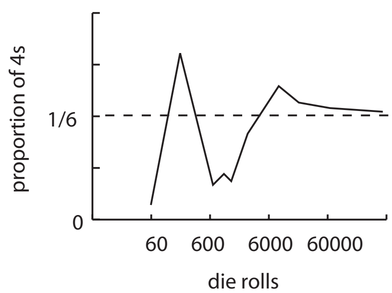
(i) **set** (ii) **subset** (iii) **element** of all possible outcomes. Since only one outcome, {H}, this event is a *simple* event, $E = e_1 = \{H\}$

(d) *Approximating probability with relative frequency.*
Since 5065 tosses of 10000 tosses are heads we approximate probability of tossing a head by $P(H) \approx \frac{5065}{10000} =$ (i) **0.4935** (ii) **0.5** (iii) **0.5065**.

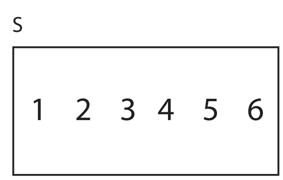
(e) *Theoretical probability.*
Since a coin can be tossed only as a head (H) or tail (T) and *assuming equally* likely outcomes, $P(H) = \frac{1}{2} =$ (i) **0.4935** (ii) **0.5** (iii) **0.5065**.

(f) (i) **True** (ii) **False**. As number of coin tosses increase, proportion of total tosses which are heads will approach and stay close to expected *probability* of tossing a head. This is an example of *law of large numbers*.

3. *Die rolling.*



die	frequency	relative frequency
1	9965	9965/60000 = 0.166
2	10035	10035/60000 = 0.167
3	10100	10100/60000 = 0.168
4	9900	9900/60000 = 0.165
5	10200	10200/60000 = 0.170
6	9800	9800/60000 = 0.163



die	number outcomes	theoretical probability
1	1	1/6
2	1	1/6
3	1	1/6
4	1	1/6
5	1	1/6
6	1	1/6

Figure 1.2: Terminology: die rolling

(a) *Random experiment.*
Die rolling is *probability experiment*. Value to be rolled unknown, but six possible outcomes (i) **known** (ii) **unknown**.

(b) *Sample space.*
Sample space is (i) **{1, 2, 3, 4, 5}** (ii) **{1, 2, 3, 4, 5, 6}** .

(c) *Event, simple event.*
Examples of events are
(choose one or *more*!) (i) **{1}** (ii) **{1, 2}** (iii) **{1, 2, 3, 6}** .

(d) *Approximating probability with relative frequency.*
Since 9900 tosses of 60000 rolls are 4s we approximate probability of rolling a 4 by $P(4) \approx \frac{9900}{60000} =$ (i) **0.163** (ii) **0.164** (iiI) **0.165**.

(e) *Theoretical probability.*
Since a die can be rolled either 1, 2, 3, 4, 5 or 6 and assuming equally likely outcomes, $P(4) = \frac{1}{6} \approx$ (i) **0.163** (ii) **0.165** (iii) **0.167**.

(f) (i) **True** (ii) **False**. As number of die rolls increase, proportion of total rolls which are 4s will approach and stay close to expected probability of rolling a 4. This is an example of *law of large numbers.*

4. *Chance error, % chance error and law of large numbers: die rolling*
Results of die rolling experiment are given in the table below.

no of rolls	no of 4s	expected 4s	difference	% difference
60	9	10	-1	$-\frac{1}{60} \times 100 \approx -1.7\%$
120	21	20	1	$\frac{1}{120} \times 100 \approx 0.8\%$
180	34	30	4	$\frac{4}{180} \times 100 \approx 2.2\%$
240	42	40	2	$\frac{2}{240} \times 100 \approx 0.8\%$
600	110	100	10	$\frac{10}{600} \times 100 \approx 1.7\%$
1200	234	200	34	$\frac{34}{1200} \times 100 \approx 2.8\%$
2400	390	400	-10	$-\frac{10}{2400} \times 100 \approx -0.4\%$

(a) After 60 rolls, (i) **6** (ii) **7** (iii) **9** are 4s;
after 2400 rolls, (i) **390** (ii) **400** (iii) **409** are 4s.

(b) If fair, after 60 rolls, we *expect* $\frac{60}{6} =$ (i) **4** (ii) **5** (iii) **10** are 4s;
after 2400 rolls, we expect $\frac{2400}{6} =$ (i) **390** (ii) **400** (iii) **409** are 4s.

(c) *Chance error* after 60 rolls $9 - 10 =$ (i) $\mathbf{-1}$ (ii) **0** (iii) **1** are 4s;
after 2400 rolls, chance error $390 - 400 =$ (i) $\mathbf{-10}$ (ii) $\mathbf{-5}$ (iii) **10**.

(d) Chance error (i) **small** (ii) **large** (iii) **same** for small number of rolls but
(i) **smaller** (ii) **same** (iii) **larger** for a large number of rolls.

(e) *Percentage chance error*, 60 rolls $\frac{9-10}{60} \times 100 =$
(i) $\mathbf{-1.7\%}$ (ii) **0%** (iii) **1.7%**;
after 2400 rolls, $\frac{390-400}{2400} \times 100 \approx$ (i) $\mathbf{-0.4\%}$ (ii) $\mathbf{-0.0\%}$ (iii) **2.0%**.

(f) Percentage chance error (i) **increases** (ii) **decreases** (iii) **remains same** as number of rolls increases: *observed variability* around $\frac{1}{6}$ decreases.

(g) So *observed* proportion of 4s (i) **converges towards** (ii) **diverges from** the *expected* proportion of 4s, $\frac{1}{6}$, by the law of large numbers.

(h) Law of large numbers (i) **does** (ii) **does not** describe the *expected variability* of chance error around $\frac{1}{6}$, but is known to be a bell-shaped histogram.

5. *Dice.* In two rolls of fair die, let event A be the "sum of dice is five". Let event B be the event "no fours are rolled".

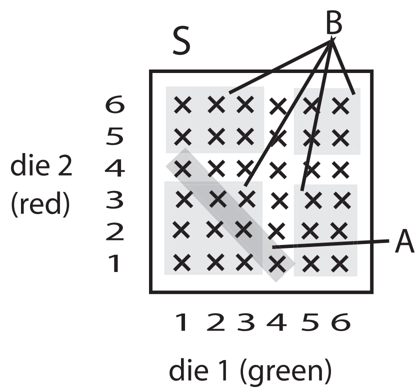


Figure 1.3: Venn diagram for tossing two dice

   (a) $P(A) =$ (i) $\mathbf{\frac{1}{36}}$ (ii) $\mathbf{\frac{2}{36}}$ (iii) $\mathbf{\frac{3}{36}}$ (iv) $\mathbf{\frac{4}{36}}$.

   (b) $P(B) =$ (i) $\mathbf{\frac{24}{36}}$ (ii) $\mathbf{\frac{25}{36}}$ (iii) $\mathbf{\frac{26}{36}}$ (iv) $\mathbf{\frac{27}{36}}$.

6. *Subjective probability.* If random experiment cannot be repeated it is not possible to calculate a relative frequency approximation to probability $P(A)$, and, if not sensible to assume equally likely outcomes, it is not possible to calculate a theoretical probability $P(A)$. In these cases, it is sometimes possible to subjectively estimate $P(A)$ using relevant information. Which situation(s) might involve subjective probability? Choose none, one or more.

   (i) probability of getting at least a 7 on a roll of a pair of dice

   (ii) probability of 2 heads in three tosses of a coin

   (iii) probability of rain tomorrow

   (iv) probability car needs a tune-up in 2 months

7. *Discrete versus Continuous (Interval): Potatoes.* An automated process fills one bag after another with Idaho potatoes. Although each filled bag should weigh 50 pounds, in fact, because of the differing shapes and weights of each potato, each bag weighs ...

   (a) ...either 49 pounds or 50 pounds or 51 pounds, $S = \{49, 50, 51\}$, each with equal probability. The chance a bag chosen at random is 50 or more pounds, $A = \{50, 51\}$ is

      i. $P(A) = \frac{n(A)}{n(S)} = \frac{2}{3}$

ii. $P(A) = \frac{51-50}{51-49} = \frac{1}{2}$

(b) ... anywhere from 49 pounds to 51 pounds, $S = [49, 51]$. The chance a bag chosen at random is 50 or more pounds, $A = [50, 51]$ is

i. $P(A) = \frac{n(A)}{n(S)} = \frac{2}{3}$

ii. $P(A) = \frac{51-50}{51-49} = \frac{1}{2}$

1.3 Counting Problems

Counting can be an important part of determining various probabilities. To help in understanding the various counting rules, we find out about an important method of visualizing counting techniques, "marbles in buckets", and discuss the notions of whether order matters or not, sampling with or without replacement and whether objects (marbles or buckets) are distinguishable or not. We look at the fundamental counting principle and a number of counting rules, summarized in the *twelve-fold way* table below and which includes, most importantly, permutations and combinations. Finally, all of the counting rules are used to calculate various probabilities.

Exercise 1.3 (Counting Problems)

1. *Fundamental Counting principle.* If (chance) experiment 1 results in n_1 possible outcomes and for each outcome from experiment 1 there results n_2 possible outcomes from experiment 2, and for each outcome from the first two experiments there results n_3 possible outcomes from experiment 3, ..., then, together, there are $n_1 \cdot n_2 \cdots n_r$ possible r–tuple outcomes from the r experiments.

   (a) If there are three eligible treasurer candidates (T_1, T_2 and T_3) and two eligible secretary candidates (S_1 and S_2), the possible (treasurer, secretary) pairings include $\{(T_1, S_1), (T_1, S_2), (T_2, S_1), (T_2, S_2), (T_3, S_1), (T_3, S_2)\}$ or $3 \times 2 =$ (i) **6** (ii) **8** (iii) **32** pairs.

   (b) The number of ways of choosing a treasurer and secretary pair, when there are four eligible treasurer candidates and eight eligible secretary candidates, is $4 \times 8 =$ (i) **6** (ii) **8** (iii) **32** pairs.

   (c) The number of ways of rolling a pair of dice, when there are six possible values for red die 1 and six possible values for green die 2, is
   $6 \times 6 =$ (i) $\mathbf{6^3}$ (ii) **18** (iii) **36** ways.

   (d) The number of ways of rolling a six-sided die three times is
   $6 \times 6 \times 6 =$ (i) $\mathbf{6^3}$ (ii) **18** (iii) **36** ways.

2. *Marbles in buckets analogy.* It is often useful to think of the analogy of counting the number of ways of placing marbles in a number of side–by–side buckets when counting the possible number of sample outcomes. Assuming only *at most one*

marble is allowed in each bucket, four possible situations arise, according to whether the marbles and buckets are labelled (distinguishable) or not, as shown in Figure 1.4. Determine the number of arrangements of marbles in buckets in each case.

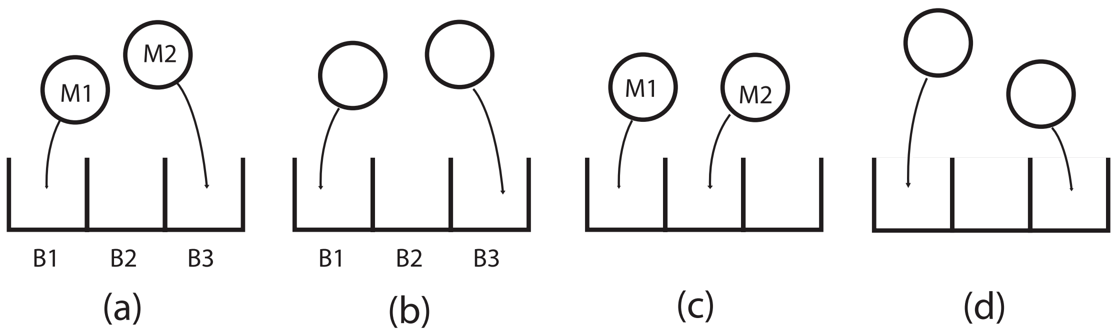


Figure 1.4: Marbles In Buckets

(a) *Figure 1.4(a).* Arrangements: (i) **0** (ii) **1** (iii) **3** (ii) **6** ways. Hint: There are six possible ways of placing two labelled marbles in three labelled buckets. Once M1 is placed in one of three buckets, M2 can be placed one of the other two buckets, a total of $3 \times 2 = 6$ arrangements. Placing marbles (M1, M2) in buckets (B1, B2) is different from placing marbles (M2, M1) in the buckets (B1, B2), for example, and so would count as two arrangements instead of one arrangement.

(b) *Figure 1.4(b).* Arrangements: (i) **0** (ii) **1** (iii) **3** (ii) **6** ways. Hint: There are three possible ways of placing two indistinguishable marbles in three labelled buckets. The two marbles can be placed in buckets (B1, B2), (B1, B3) or (B2, B3) only.

(c) *Figure 1.4(c).* Arrangements: (i) **0** (ii) **1** (iii) **3** (ii) **6** ways. Hint: There is only one way of placing two labelled marbles in three indistinguishable buckets: two of the indistinguishable buckets have one marble and one of the indistinguishable buckets does not have a marble. It is assumed even the order of the buckets is unknown, cannot be used tell the buckets apart and the identity of each marble is disguised once placed in an indistinguishable bucket.

(d) *Figure 1.4(d).* Arrangements: (i) **0** (ii) **1** (ii) **3** (iii) **6** ways. Hint: There is only one way of placing two indistinguishable marbles in three indistinguishable buckets: two of the indistinguishable buckets have one marble and one of the indistinguishable buckets does not have a marble.

(e) *Order matters or not, order is important or not.* Order matters the most, the most arrangements are counted for (i) Figure 1.4(a) (ii) Figure 1.4(b) (iii) Figure 1.4(c) (iv) Figure 1.4(d). Hint: when both the marbles and buckets are labelled, this allows one arrangement to be distinguishable from another arrangement.

3. *Marbles in buckets analogy continued.* In addition to assuming only at most one marble is allowed in each bucket, two other situations arise, including requiring at least one marble in each bucket and the unrestricted case where none to as many marbles as are available are allowed in each bucket. Reconsider only Figure 1.4(a). Determine the number of arrangements of marbles in buckets in each case below.

   (a) *At most one marble in each bucket.*
   Arrangements: (i) **0** (ii) **1** (iii) **6** (ii) **9** ways. Hint: As before, there are six possible ways of placing two labelled marbles in three labelled buckets. Once M1 is placed in one of three buckets, M2 can be placed one of the other two buckets, a total of $3 \times 2 = 6$ outcomes. Notice, in all six cases, two buckets have one marble and one bucket does not have a marble.

   (b) *At least one marble in each bucket.*
   Arrangements: (i) **0** (ii) **1** (iii) **6** (ii) **9** ways. Hint: This is not possible, there are zero ways of doing this, because there are only two marbles and so one bucket must have no marbles.

   (c) *Unrestricted: none, one or two marbles in each bucket.*
   Arrangements: (i) **0** (ii) **1** (iii) **6** (iv) **9** ways. Hint: Since marbles M1 and M2 can appear together in one bucket, there are a total of 6 "at most one" marble arrangements plus 3 more arrangements of having 2 marbles in one of the three buckets, $3^2 = 9$ ways.

   (d) *Sampling with or without replacement.*
   (i) **True** (ii) **False** Sampling is assumed without replacement only in the situation when *at most one* marble is allowed in each bucket, otherwise, sampling with replacement is assumed. Sampling with replacement allows more than one marble in each bucket.

4. *Twelve-Fold Way Table.* Some counting methods, of placing r marbles into n buckets, can be categorized into the following twelve-fold way table. In this table, the two most important formulas covered in this course are highlighted in red, formulas 1B and 2B; furthermore, D: distinguishable (labelled), I: indistinguishable (identical, unlabelled), also, a combination is

$$_nC_r = \frac{n!}{r!(n-r)!} = \binom{n}{r}$$

and a permutation is

$$ {}_nP_r = \frac{n!}{(n-r)!} = n(n-1)(n-2)\cdots(n-r+1) $$

less important, Stirling numbers of second kind are $S(r,n) = \frac{1}{n!}\sum_{k=0}^{n}(-1)^k \begin{pmatrix} n \\ k \end{pmatrix}(n-k)^r$, the integer partitioning number $p_n(r)$ represents number of ways integer r can be expressed as sum of n positive numbers, where order does not matter.

	marbles, r	buckets, n	0, 1 or more marbles per bucket, unrestricted A	at most 1 marble per bucket, ≤ 1 B	at least 1 marble per bucket, ≥ 1 C
1	D	D	n^r	${}_nP_r$	$n!S(r,n)$
2	I	D	${}_{r+n-1}C_r$	${}_nC_r$	${}_{r-1}C_{n-1}$
3	D	I	$\sum_{k=1}^{n} S(r,k)$	1 if $r \leq n$ 0 if $r > n$	$S(r,n)$
4	I	I	$\sum_{k=1}^{n} p_k(r)$	1 if $r \leq n$ 0 if $r > n$	$p_n(r)$

Counting rule 1A: 3–digit street numbers. Let marbles represent the three possible place-values, 100s, 10s and 1s and buckets represent the 10 possible digits, 0, 1, . . . , 9, and so street number 144 consists of a 100s place-value with digit 1 and also a 10s place-value and a 1s place-value which are both digit 4, as indicated in Figure 1.5.

(a) Since the r marbles are each labelled, they are distinguishable from one another; the n buckets are also labelled, and since anywhere from none to three of the marbles can be placed in each bucket, counting rule 1A applies: n^r, as shown in the 12-fold way table. Since the $r = 3$ possible place-values, the 100s, 10s and 1s, can be any of $n = 10$ possible digits 0, 1, . . . , 9, the total number of three–digit street numbers is
$n^r = 10^3 = 10 \times 10 \times 10 =$ (i) **90** (ii) **100** (iii) **900** (iv) **1000** .

```
10^3 # fundamental counting principle
```

```
[1] 1000
```

(b) Number of 3–digit numbers if 100s place-value cannot be zero is
$9 \times 10 \times 10 =$ (i) **90** (ii) **100** (iii) **900** (iv) **1000**.

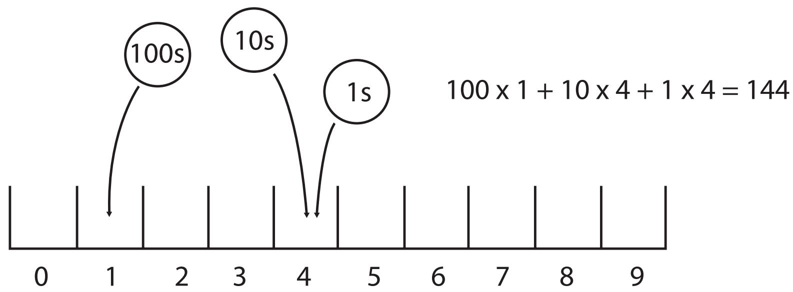


Figure 1.5: Marbles In Buckets: 3–Digit Street Numbers

(c) Number of 3–digit numbers if 100s place-value must be 3, is
$1 \times 10 \times 10 =$ (i) **90** (ii) **100** (iii) **900** (iv) **1000**.

(d) 3–digit numbers if 100s place-value is 3 and 10s place-value cannot be 9,
$1 \times 9 \times 10 =$ (i) **90** (ii) **100** (iii) **900** (iv) **1000**.

(e) Number of 3–digit numbers with exactly two 3s, is, since there are three ways of having exactly two 3s in a 3–digit street number, is
$1 \times 1 \times 9 + 1 \times 9 \times 1 + 9 \times 1 \times 1 =$ (i) **9** (ii) **18** (iii) **27**.

(f) Since digits can be repeated in 3-digit street numbers, such as repeating the digit 4 in street number 144, the digits have been (i) **sampled with replacement** (ii) **sampled without replacement** in this case. Furthermore, since the order of digits in a street number changes the street number; for example, street number 144 is a different from street number 414, order matters, (i) **is** (ii) **is not** important.

5. *Counting rule 1B, permutations: parking cars.* As shown in Figure 1.6, to count number of ways 5 distinguishable cars can park in 3 distinguishable parking spots, let the marbles represent the parking spots P1, P2 and P3 and the buckets represent the cars C1, C2, C3, C4 and C5 where only at most 1 marble (parking spot) can be assigned one car at one time. Consequently, five different cars could occupy the first parking spot, only four could occupy the second parking spot (since one car is in the first parking spot) and three could occupy the final parking spot in Figure . Consequently, there are

$$_5P_3 = 5 \times 4 \times 3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = 60$$

ways of doing this.

```
factorial(5)/factorial(5-3) # permutation
```

```
[1] 60
```

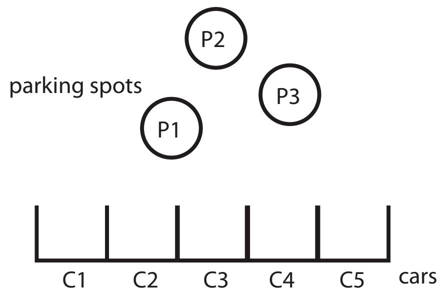


Figure 1.6: Marbles in buckets: parking cars

(a) Ways of parking $n = 7$ cars in $r = 4$ spots is
$_7P_4 = \frac{7!}{(7-4)!}$ = (i) **840** (ii) **479, 001, 600**.

```
factorial(7)/factorial(7-4) # permutation
```

```
[1] 840
```

(b) Ways of parking $n = 12$ cars in $r = 12$ spots is
$_{12}P_{12} = \frac{12!}{(12-12)!} = 12!$ = (i) **840** (ii) **479, 001, 600**.

```
factorial(12)/factorial(12-12) # permutation
```

```
[1] 479001600
```

(c) The assumptions here are (choose one or more):

i. cars are distinguishable from one another,
ii. parking spots are distinguishable from one another,
iii. sample without replacement, at most one car appears in each spot, (a car can*not* appear in different parking spots simultaneously),
iv. order matters, is important. (different arrangements in each group of cars counted differently).

(d) *Factorial notation.*
(i) **True** (ii) **False** If $n > 0$ is an integer, then n-factorial is

$$n! = n \times (n-1) \times (n-2) \times \cdots 2 \times 1.$$

Also, $0! = 1$.

6. *Permutations and fundamental counting rule: parking cars.* Consider Figure 1.7. Nine different cars are to be driven into nine side–by–side parking spots.

(a) Three of the cars are driven by statistics students and there are three parking spots reserved for statistics students only. Six of the cars are driven by

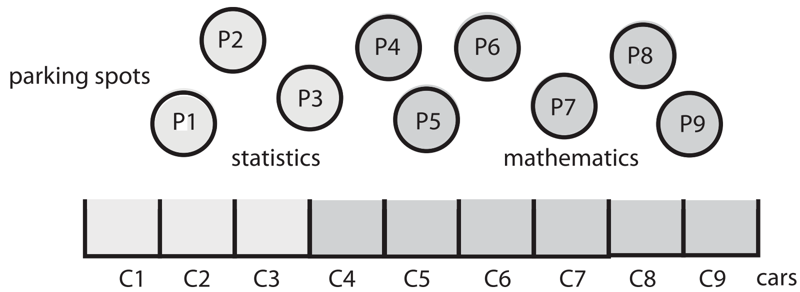


Figure 1.7: Marbles in buckets: statistics and mathematics parking spots

mathematics students and there are 6 parking spots reserved for these students only. Combining the permutation and fundamental counting rules, the number of ways cars can be parked is

$$_3P_3 \times {}_6P_6 = 3!6! = (6)(720) =$$

(i) **4,320** (ii) **362,880**.

```
factorial(3)*factorial(6) # permutation and fundamental counting rule
```

```
[1] 4320
```

(b) Allowing all cars to park in any parking spot, the number of ways is $9! =$ (i) **4,320** (ii) **362,880**.

```
factorial(9) # factorial
```

```
[1] 362880
```

7. *Counting rule 2B, combinations: 3-card hands.* As shown in Figure 1.8, to count number of ways 3 cards can be dealt from 5 distinguishable cards, let the marbles represent one of the three cards and the buckets represent the card ranks 10, J, Q, K and A. The 3-card hands are indistinguishable in the sense they are played in exactly the same way no matter what order ($3! = 6$ ways) they appear in the hand. Using counting rule 2B,

$$_5C_3 = \binom{5}{3} = \binom{5}{3 \quad 5-3} = \binom{5}{5-3} = \frac{5!}{3!(5-3)!} = \frac{_5P_3}{3!} =$$

(i) **10** (ii) **60** (iii) **120**,
where, notice, combinations are permutations with the "order divided out".

```
choose(5,3) # combination
```

```
[1] 10
```

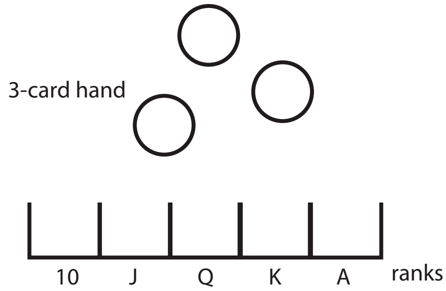


Figure 1.8: Marbles in buckets: 3-card hands

8. *Combination and multiplication counting rules.* Choose committees from a group of 6 women and 9 men.

   (a) The number of different committees consisting of 4 women and 3 men which can be formed is

   $$\binom{6}{4}\binom{9}{3} = \frac{6!}{4!(6-4)!} \cdot \frac{9!}{3!(9-3)!} =$$

   (i) $\frac{\mathbf{4!}}{\mathbf{6!(6-4)!}} \cdot \frac{\mathbf{9!}}{\mathbf{3!(9-3)!}}$ (ii) **1260** (iii) **3780** (iv) **5005**

   ```
   choose(6,4)*choose(9,3) # combination and fundamental counting rule
   [1] 1260
   ```

   (b) The number of different committees consisting of (4 women and 3 men) or (3 women and 4 men) is

   $$\binom{6}{4}\binom{9}{3} + \binom{6}{3}\binom{9}{4} =$$

   (i) **1260** (ii) **3780** (iii) **5005**.

   ```
   choose(6,4)*choose(9,3) + choose(6,3)*choose(9,4) # combinations and fundamental counting rule
   [1] 3780
   ```

   (c) The number of different committees consisting of six (6) people is

   $$\binom{6+9}{6} = \binom{6}{0}\binom{9}{6} + \binom{6}{1}\binom{9}{5} + \cdots + \binom{6}{6}\binom{9}{0} =$$

   (i) **1260** (ii) **3780** (iii) **5005**.

   ```
   choose(6+9,6) # combination
   [1] 5005
   ```

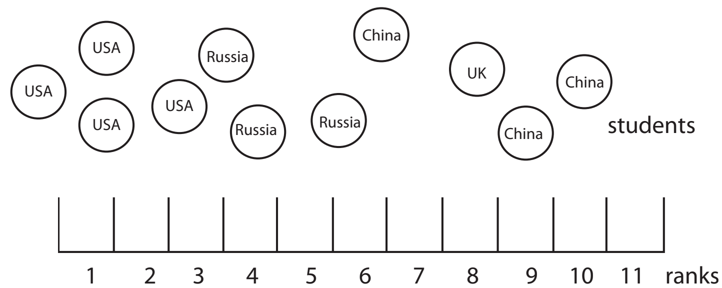


Figure 1.9: Marbles in buckets: ranks in statistics contest

9. *Multinomial: combination and multiplication counting rules.* Eleven students are ranked 1 (first) to 11 (last) in a statistics contest. As shown in Figure 1.9, four of the students are from the United States, three are from Russia, three are from China and one is from the United Kingdom.

   (a) The (student) ranks can be divided among the four different countries in the following *multinomial* number of ways:

$$\binom{11}{4}\binom{7}{3}\binom{4}{3}\binom{1}{1} = \binom{11}{4\ 3\ 3\ 1} = \frac{11!}{4!3!3!1!} =$$

   (i) $\mathbf{2,520}$ (ii) $\mathbf{46,200}$ (iii) $\frac{\mathbf{11!}}{\mathbf{4!7!}} \cdot \frac{\mathbf{7!}}{\mathbf{3!4!}}$.

```
choose(11,4)*choose(7,3)*choose(4,3)*choose(1,1) # combinations and fundamental counting rule
factorial(11)/(factorial(4)*factorial(3)*factorial(3)*factorial(1)) # factorials
```

```
[1] 46200
```

   (b) If the United States has two students in the top four, one in the middle four and one in the bottom three, the (student) ranks can be divided among the four different countries in the following number of ways:

$$\binom{4}{2}\binom{2}{1}\binom{1}{1}\binom{7}{2\ 3\ 2} =$$

   (choose two) (i) $\frac{\mathbf{11!}}{\mathbf{4!3!3!1!}}$ (ii) $\mathbf{2,520}$ (iii) $\mathbf{46,200}$ (iv) $\frac{\mathbf{4!}}{\mathbf{2!2!}} \cdot \frac{\mathbf{2!}}{\mathbf{1!1!}} \cdot \frac{\mathbf{1!}}{\mathbf{1!0!}} \cdot \frac{\mathbf{7!}}{\mathbf{2!3!2!}}$.

   Hint: $\boxed{\boxed{\text{US}}\ \boxed{\text{US}}\ \boxed{\text{O}}\ \boxed{\text{O}}\ |\ \boxed{\text{US}}\ \boxed{\text{O}}\ \boxed{\text{O}}\ \boxed{\text{O}}\ |\ \boxed{\text{US}}\ \boxed{\text{O}}\ \boxed{\text{O}}}$

   The US can appear in 2 of the top four, 1 of middle four, 1 of bottom three and any other student, "O", of the seven remaining, are then placed accordingly.

```
choose(4,2)*choose(2,1)*choose(1,1)*(factorial(7)/(factorial(2)*factorial(3)*factorial(2)))
```

```
[1] 2520
```

(c) *Number of unique arrangements of n objects.* (i) **True** (ii) **False**. If arranging n objects where n_1 are identical, n_2 are identical, ..., n_r are identical and where $n = n_1 + n_2 + \cdots + n_r$, then number of unique arrangements of n objects is

$$\frac{n!}{n_1!n_2!\cdots n_r!}.$$

10. *Combinations in probability: cards.* Four cards are dealt at random from a deck of 52 playing cards, what is the probability of dealing 3 jacks?

(a) Number of ways of dealing four cards is

$$\binom{52}{4} =$$

(i) **2,520** (ii) **270,725** (iii) $\frac{\mathbf{4!}}{\mathbf{3!1!}} \cdot \frac{\mathbf{48!}}{\mathbf{1!47!}}$.

```
choose(52,4) # cards combination
```

```
[1] 270725
```

(b) Number of ways of dealing four cards with three jacks is

$$\binom{4}{3}\binom{48}{1} =$$

(choose two) (i) **192** (ii) **270,725** (iii) $\frac{\mathbf{4!}}{\mathbf{3!1!}} \cdot \frac{\mathbf{48!}}{\mathbf{1!47!}}$.

```
choose(4,3)*choose(48,1) # combinations and fundamental counting rule
```

```
[1] 192
```

(c) So probability of dealing four cards with three jacks is

$$\frac{\binom{4}{3}\binom{48}{1}}{\binom{52}{4}} \approx$$

(i) **0.0071** (ii) **0.00071** (iii) **0.000071**.

```
choose(4,3)*choose(48,1)/choose(52,4) # probability
```

```
[1] 0.0007092068
```

11. *Binomial theorem.* Using

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

where $\left(\begin{array}{c} n \\ k \end{array} \right)$ are called *binomial coefficients*, so for example

$$\begin{aligned}
(x+y)^3 &= \sum_{k=0}^{3} \left(\begin{array}{c} 3 \\ k \end{array} \right) x^k y^{3-k} \\
&= \left(\begin{array}{c} 3 \\ 0 \end{array} \right) x^0 y^3 + \left(\begin{array}{c} 3 \\ 1 \end{array} \right) x^1 y^2 + \left(\begin{array}{c} 3 \\ 2 \end{array} \right) x^2 y^1 + \left(\begin{array}{c} 3 \\ 3 \end{array} \right) x^3 y^0 \\
&= \frac{3!}{0!3!} x^0 y^3 + \frac{3!}{1!2!} x^1 y^2 + \frac{3!}{2!1!} x^2 y^1 + \frac{3!}{3!0!} x^3 y^0 =
\end{aligned}$$

(i) $\boldsymbol{y^3}$ (ii) $\boldsymbol{y^3 + 3xy^2 + 3x^2y + x^3}$ (iii) $\boldsymbol{x^3}$.