Chapter 13

Comparing Three or More Means

13.1 Comparing Three or More Means (One-Way Analysis of Variance)

A *one–way ANOVA* procedure tests multiple averages using test statistic

$$F_0 = \frac{\text{MST}}{\text{MSE}}$$

where

$$\text{MST} = \frac{\text{SST}}{k-1}, \qquad \text{MSE} = \frac{\text{SSE}}{n-k}$$

where k is number of groups, n is total number of observations, and where

$$\begin{aligned} \text{SST} &= n_1(\bar{x}_1 - \bar{x})^2 + n_2(\bar{x}_2 - \bar{x})^2 + \cdots + n_k(\bar{x}_k - \bar{x})^2 \\ \text{SSE} &= (n_1 - 1)s_1^2 + (n_2 - 1)s_2^2 + \cdots + (n_k - 1)s_k^2. \end{aligned}$$

which has a F distribution, with $(n-k,\, k)$ df, where n data points sampled randomly from k treatments (populations), and where k samples are independent of one another, each normally distributed and all with same variance, σ^2, and where MST is mean square due to treatment and MSE is mean square due to error.

Normality is checked using normal probability plots and equal variance is (roughly) checked by assuring largest SD is no more than twice the smallest SD or if sample size in each treatment group is the same.

Exercise 13.1 (Comparing Three or More Means (One-Way ANOVA))

1. *Test comparing multiple means, ANOVA: average drug responses A.*
   Fifteen different patients, chosen at random, subjected to three drugs. Test if at least one of the three mean patient responses to drug is different at $\alpha = 0.05$.

drug 1	drug 2	drug 3
5.90	5.51	5.01
5.92	5.50	5.00
5.91	5.50	4.99
5.89	5.49	4.98
5.88	5.50	5.02
$\bar{x}_1 \approx 5.90$	$\bar{x}_2 \approx 5.50$	$\bar{x}_3 \approx 5.00$

(Blank data table. Relabel var1 drug 1, var2 drug 2, var3 drug 3. Type data into these three columns. Data, Save data, 13.1.1 drug same data.)

(a) *Statement.* Choose one.

i. $H_0 : \mu_1 = \mu_2 = \mu_3$ vs $H_1 : \mu_1 \neq \mu_2, \mu_1 = \mu_3$.
ii. $H_0 : \mu_1 = \mu_2 = \mu_3$ vs $H_1 : \mu_1 \neq \mu_3, \mu_1 \neq \mu_2$.
iii. $H_0 : \mu_1 = \mu_2 = \mu_3$ vs $H_1 : \mu_i \neq \mu_j, i \neq j;\ i, j = 1, 2, 3$.
iv. H_0 : means same vs H_1 : at least one of the means different

No matter how the question is worded, null is *always* "means same" and alternative is *always* "at least one of the means different".

(b) *Test.*
p–value = (circle one) **0.00** / **0.035** / **0.043**.
(Stat, ANOVA, One Way, choose Compare selected columns drug 1, drug 2, drug 3, Calculate.)
Level of significance α = (choose one) **0.01** / **0.05** / **0.10**.

(c) *Conclusion.*
Since p–value $= 0.00 < \alpha = 0.05$,
(circle one) **do not reject** / **reject** null H_0 : means same.
Data indicates (circle one)
average drug responses same
at least one of average drug responses different

(d) *Related question.*
"H_1 : at least one of the means different" means: (choose *one or more*)

i. $\mu_1 \neq \mu_2$, but $\mu_2 = \mu_3$
ii. $\mu_1 \neq \mu_3$, but $\mu_2 = \mu_3$
iii. $\mu_2 \neq \mu_3$, but $\mu_1 = \mu_3$
iv. $\mu_1 \neq \mu_2$, $\mu_1 \neq \mu_3$ and $\mu_2 \neq \mu_3$

2. *Test comparing multiple means, ANOVA: average drug responses B.*
Fifteen different patients, chosen at random, subjected to three drugs. Test if at least one of the of three mean patient responses to drug is different at $\alpha = 0.05$.

drug 1	drug 2	drug 3
5.90	6.31	4.52
4.42	3.54	6.93
7.51	4.73	4.48
7.89	7.20	5.55
3.78	5.72	3.52
$\bar{x}_1 \approx 5.90$	$\bar{x}_2 \approx 5.50$	$\bar{x}_3 \approx 5.00$

(Blank data table. Relabel var1 drug 1, var2 drug 2, var3 drug 3. Type data into these three columns. Data, Save data, 13.1.2 drugs different data.)

(a) *Statement.* Choose none, one or more.

i. $H_0 : \mu_1 = \mu_2 = \mu_3$ vs $H_1 : \mu_1 \neq \mu_2, \mu_1 = \mu_3$.
ii. $H_0 : \mu_1 = \mu_2 = \mu_3$ vs $H_1 : \mu_1 \neq \mu_3, \mu_1 \neq \mu_2$.
iii. $H_0 : \mu_1 = \mu_2 = \mu_3$ vs $H_1 : \mu_i \neq \mu_j, i \neq j;\ i, j = 1, 2, 3$.
iv. H_0 : means same vs H_1 : at least one of the means different

(b) *Test.*
p–value = (circle one) **0.00** / **0.35** / **0.66**.
(Stat, ANOVA, One Way, choose Compare selected columns drug 1, drug 2, drug 3, Calculate.)
Level of significance α = (choose one) **0.01** / **0.05** / **0.10**.

(c) *Conclusion.*
Since p–value = 0.66 > α = 0.05,
(circle one) **do not reject** / **reject** null H_0 : means same.
Data indicates (circle one)
average drug responses same
at least one of average drug responses different

(d) *Related question.* ANOVA table is

Source	Sum Of Squares	Degrees of Freedom	Mean Squares	F
Treatment	2.033	2	1.0167	0.436
Error	27.985	12	2.3321	
Total	30.018	14		

The "SS" is "sum of squares", "MS" is "mean squares" and "df" is "degrees of freedom".
so test statistic is

$$F_0 = \frac{\text{MST}}{\text{MSE}} = \frac{1.0167}{2.3321} =$$

(circle one) **0.345** / **0.436** / **0.767**,
so, with $k - 1 = 3 - 1 = 2$ and $n - k = 15 - 3 = 12$ df,

$$\text{p–value} = P(F \geq 0.436) =$$

(circle one) **0.00 / 0.35 / 0.66**.

(Stat, Calculators, F, Num. DF: 2, Den. DF: 12, Prob(X $\geq$) 0.436 = [?]). The F statistic and p–value calculated here by hand may not match values calculated using accurate StatCrunch because of round–off error.

3. *Comparing drug response data set A and data set B.*

   (a) *Comparing averages, $\bar{x}_1$, $\bar{x}_2$ and $\bar{x}_3$, in two drug response data sets.*
   Of following two data sets, data set A,

drug 1	drug 2	drug 3
5.90	5.51	5.01
5.92	5.50	5.00
5.91	5.50	4.99
5.89	5.49	4.98
5.88	5.50	5.02
$\bar{x}_1 \approx 5.90$	$\bar{x}_2 \approx 5.50$	$\bar{x}_3 \approx 5.00$
$s_1 \approx 0.12$	$s_2 \approx 0.004$	$s_3 \approx 0.01$

   and data set B,

drug 1	drug 2	drug 3
5.90	6.31	4.52
4.42	3.54	6.93
7.51	4.73	4.48
7.89	7.20	5.55
3.78	5.72	3.52
$\bar{x}_1 \approx 5.90$	$\bar{x}_2 \approx 5.50$	$\bar{x}_3 \approx 5.00$
$s_1 \approx 1.82$	$s_2 \approx 1.27$	$s_3 \approx 1.30$

   three *average* patient responses within drugs in data set A are
   (circle one) **smaller than / same as / larger than**
   three *average* patient responses within drugs in data set B.

   (b) *Comparing SDs, s_1, s_2 and s_3, in two drug response data sets.*
   The *standard deviations* in patient responses within drugs in data A are (circle one) **smaller than / the same as / larger than** the *standard deviations* in patient responses within drugs in data B.

   (c) *Comparing averages, taking into account SDs.*
   As shown in figure below, since standard deviations within drugs in data set A are smaller than they are for data set B, we are "more certain" about where averages are in data set A, than we are about where averages are in data set B. Consequently, it is "easiest" to tell if average patient responses are different from one another in data set (choose one) **A / B**. This is why p–value is smaller for data set A, than it is for data set B even though averages are same in both sets.

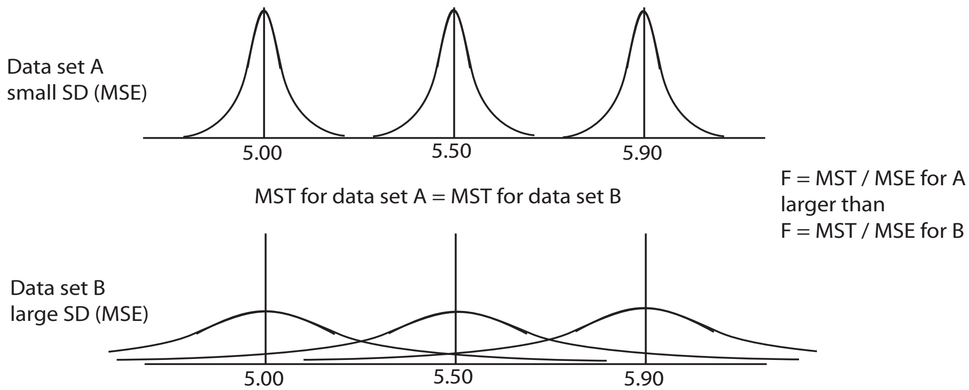

Figure 13.1 (Comparing average drug responses)

(d) *Comparing F test statistics.*
Since mean square treatment (MST) same for data sets A and B, but mean square error (MSE) smaller for data set A than for data set B,

$$F = \frac{\text{MST}}{\text{MSE}}$$

is (choose one) **smaller** / **same as** / **larger** for A than for B, so greater chance of rejecting null (means same) for A than for B.

4. *Test comparing multiple means, ANOVA: yet more drug responses.*
Twelve different patients, chosen at random, are subjected to three drugs. Test if at least one of the three average patient responses to drug is different at $\alpha = 0.05$.

drug 1	drug 2	drug 3
5.90	5.50	5.01
5.92	5.53	5.00
5.91		4.99
5.89		4.98
5.88		5.02
$\bar{x}_1 \approx 5.90$	$\bar{x}_2 \approx 5.52$	$\bar{x}_3 \approx 5.00$
$s_1 \approx 0.016$	$s_2 = 0.021$	$s_3 \approx 0.016$

(Blank data table. Relabel var1 drug 1, var2 drug 2, var3 drug 3. Type data into these three columns. Data, Save data, 13.1.3 drugs unequal sample size.)

(a) *Check assumptions.*

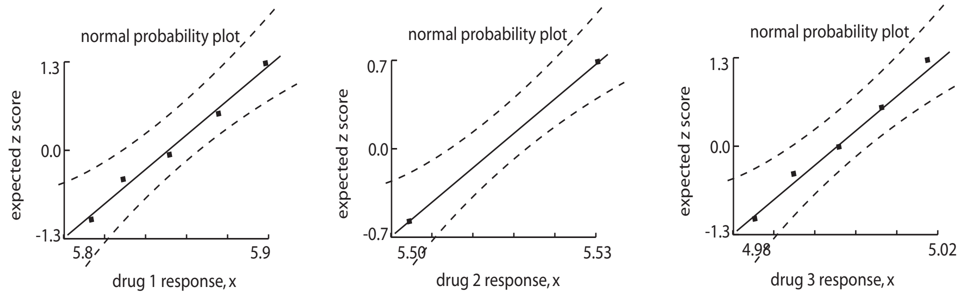

Figure 13.2 (Normal probability plots for drug responses.)

i. *Data normal?*
Normal probability plots for *all* three samples indicates
normal / not normal because data within dotted bounds.
(Graphics, QQ Plot, Select Columns: drug 1, drug 2, drug 3, Create Graph!. It is also possible to check normality of residuals from ANOVA model.)

ii. *Constant (equal) variance (SD)?*
Since maximum SD, $s_2 \approx 0.021$, is less than twice minimum SD, $s_1 = s_3 \approx 0.016$, $\frac{s_2}{s_1} \approx \frac{0.021}{0.016} \approx 1.3 < 2$, data indicates **equal / unequal** SD for different drug responses.
(Stat, Summary Stats, Columns, Select Columns: drug 1, drug 2, drug 3, Calculate.)

(b) *Statement.* Choose none, one or more.

i. $H_0 : \mu_1 = \mu_2 = \mu_3$ vs $H_1 :\ \mu_1 \neq \mu_2, \mu_1 = \mu_3$.
ii. $H_0 : \mu_1 = \mu_2 = \mu_3$ vs $H_1 :\ \mu_1 \neq \mu_3, \mu_1 \neq \mu_2$.
iii. $H_0 : \mu_1 = \mu_2 = \mu_3$ vs $H_1 :\ \mu_i \neq \mu_j, i \neq j;\ i, j = 1, 2, 3$.
iv. H_0 : means same vs H_1 : at least one of the means different

(c) *Test.*
p–value = (circle one) **0.00 / 0.035 / 0.043**.
(Stat, ANOVA, One Way, choose Compare selected columns drug 1, drug 2, drug 3, Calculate.)
Level of significance α = (choose one) **0.01 / 0.05 / 0.10**.

(d) *Conclusion.*
Since p–value $= 0.00 < \alpha = 0.05$,
(circle one) **do not reject / reject** null H_0 : means same.
The data indicates (circle one)
drug response means same
at least one of drug response means different

5. *Test comparing multiple means, ANOVA: rats city.*
Rats are counted at twelve different city locations, chosen at random. Test claim mean rat count per square meter in at least one of the three city areas is different at 1%.

sewers	parks	city hall
3	1	5
5	3	8
7	2	9
4		8
		10

(Blank data table. Relabel var1 sewers, var2 parks, var3 city hall. Type data into these three columns. Data, Save data, 13.1.4 rats data.)

(a) *Statement.* Choose none, one or more.

i. $H_0 : \mu_1 = \mu_2 = \mu_3$ vs $H_1 :\ \mu_1 \neq \mu_2, \mu_1 = \mu_3$.

ii. $H_0 : \mu_1 = \mu_2 = \mu_3$ vs $H_1 :\ \mu_1 \neq \mu_3, \mu_1 \neq \mu_2$.

iii. $H_0 : \mu_1 = \mu_2 = \mu_3$ vs $H_1 :\ \mu_i \neq \mu_j, i \neq j;\ i, j = 1, 2, 3$.

iv. H_0 : means same vs H_1 : at least one of the means different

(b) *Test.*

p–value = (circle one) **0.002** / **0.035** / **0.043**.

(Stat, ANOVA, One Way, choose Compare selected columns city hall, parks, sewers, Calculate.)

Level of significance α = (choose one) **0.01** / **0.05** / **0.10**.

(c) *Conclusion.*

Since p–value = $0.002 < \alpha = 0.01$,

(circle one) **do not reject** / **reject** null H_0 : means same.

The data indicates (circle one)

rat count means same

at least one of rat count means different

(d) *How are averages different?*

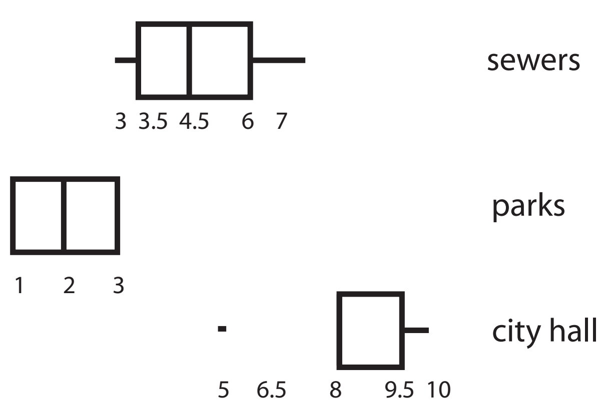

Figure 13.3 (Side-by-side boxplots for rat count.)

(Graphics, Boxplot, Select Columns: sewers, parks, city hall, Next, check use fences to identify outliers, Draw boxes horizontally, Create Graph!).

Rat count, is, from smallest to largest (choose one)

i. sewers, parks, city hall
ii. parks, sewers, city hall
iii. city hall, sewers, parks

6. *Using formulas for ANOVA: comparing average drug responses.*
Twelve different patients, chosen at random, are subjected to three drugs. Test if at least one of the three average patient responses to drug is different at $\alpha = 0.05$.

drug 1	drug 2	drug 3
$\bar{x}_1 = 5.90$	$\bar{x}_2 = 5.52$	$\bar{x}_3 = 5.00$
$s_1^2 = 0.000256$	$s_2^2 = 0.000441$	$s_3^2 = 0.000256$
$n_1 = 5$	$n_2 = 2$	$n_3 = 5$

Since StatCrunch does not deal with summarized data for ANOVA, this analysis must be done "by hand".

(a) *Statement.* Choose none, one or more.
i. $H_0 : \mu_1 = \mu_2 = \mu_3$ vs $H_1 : \mu_1 \neq \mu_2, \mu_1 = \mu_3$.
ii. $H_0 : \mu_1 = \mu_2 = \mu_3$ vs $H_1 : \mu_1 \neq \mu_3, \mu_1 \neq \mu_2$.
iii. $H_0 : \mu_1 = \mu_2 = \mu_3$ vs $H_1 : \mu_i \neq \mu_j, i \neq j;\ i, j = 1, 2, 3$.
iv. H_0 : means same vs H_1 : at least one of the means different

(b) *Test.*
Since grand average is

$$\bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2 + n_3\bar{x}_3}{n_1 + n_2 + n_3} \approx \frac{5 \cdot 5.90 + 2 \cdot 5.52 + 5 \cdot 5.00}{5 + 2 + 5} \approx$$

(Calculator: (5*5.9 + 2*5.52 + 5*5)/12.)
(circle one) **4.56** / **5.46** / **6.45** and

$$\begin{aligned} \text{SST} &= n_1(\bar{x}_1 - \bar{x})^2 + n_2(\bar{x}_2 - \bar{x})^2 + n_3(\bar{x}_3 - \bar{x})^2 \\ &= 5(5.90 - 5.46)^2 + 2(5.52 - 5.46)^2 + 5(5.00 - 5.46)^2 \approx \end{aligned}$$

(Calculator: (5*(5.9-5.46)∧2 + 2*(5.52 - 5.46)∧2 + 5*(5 - 5.46)∧2.)
(circle one) **0.56** / **1.06** / **2.03** and

$$\begin{aligned} \text{SSE} &= (n_1 - 1)s_1^2 + (n_2 - 1)s_2^2 + (n_3 - 1)s_3^2 \\ &= (5 - 1)0.000256 + (2 - 1)0.000441 + (5 - 1)0.000256 \approx \end{aligned}$$

(Calculator: (4*0.000256 + 1*0.000441 + 4*0.000256.)
(circle one) **0.0006** / **0.0016** / **0.0025**
and so MST $= \frac{\text{SST}}{k-1} = \frac{2.03}{3-1} =$
(Calculator: 2.03/2.)

(circle one) **0.53** / **0.81** / **1.02**
and $\text{MSE} = \frac{\text{SSE}}{n-k} = \frac{0.0025}{12-3} =$
(Calculator: 0.0025/9.)
(circle one) **0.00027** / **0.0027** / **0.027**
which gives ANOVA table

Source	Sum Of Squares	Degrees of Freedom	Mean Squares	F
Treatment	2.03	2	1.02	3778
Error	0.0025	9	0.00027	
Total	2.0325	11		

where, notice, test statistic is

$$F_0 = \frac{\text{MST}}{\text{MSE}} = \frac{1.02}{0.00027} \approx$$

(Calculator: 1.02/0.00027.)
(circle one) **1908** / **2310** / **3778**,
so, with $k - 1 = 3 - 1 = 2$ and $n - k = 12 - 3 = 9$ df,

$$\text{p–value} = P(F \geq 3778) =$$

(circle one) **0.00** / **0.35** / **0.66**.
(Stat, Calculators, F, Num. DF: 2, Den. DF: 9, Prob(X $\geq$) 3778 = ?)
Level of significance $\alpha =$ (choose one) **0.01** / **0.05** / **0.10**.

(c) *Conclusion.*
Since p–value $= 0.00 < \alpha = 0.05$,
(circle one) **do not reject** / **reject** null H_0 : means same.
Data indicates (circle one)
drug response means same
at least one of drug response means different

13.2 Post Hoc Tests on One-Way Analysis of Variance

Not covered.

13.3 The Randomized Complete Block Design

Not covered.

13.4 Two-Way Analysis of Variance)

Not covered.

Summary Section

HYPOTHESIS TESTS		mean μ	variance σ^2	proportion p
	one	chapter 10	chapter 10	chapter 10
sample	two	chapter 11	chapter 11	chapter 11
	multiple	$F_0 = \frac{\text{MST}}{\text{MSE}}$	not covered	chapter 12