1.4 Axioms of Probability and the Addition Rule

A capital letter A, for example, denotes a set of elements (or *outcomes*). The n elements in set (or *event*) A are denoted $A = \{a_1, a_2, ..., a_n\}$. The set that contains *no* elements is called a *null* or *empty set* and denoted by $\emptyset$. The set that contains *all* elements under consideration is the *universal set* (or *sample space*), S. Set A is a subset of B, $B \subset A$, if every element in A is also in B. Events A and B are *mutually exclusive* or *disjoint* if $A \cap B = \phi$. Mutually exclusive events cannot happen together. Some ways events can be manipulated:

- *intersection*, described using "and" or "cap" symbol, $\cap$,
- *union*, described using "or" or "cup" symbol, $\cup$,
- *complement*, using symbol, $\bar{A}$ or A'.

Some properties for sets A, B and C include

$$\begin{aligned}
A \cup B &= B \cup A \quad \text{(commutative law)} \\
A \cap B &= B \cap A \quad \text{(commutative law)} \\
A \cup (B \cup C) &= (A \cup B) \cup C = A \cup B \cup C \quad \text{(associative law)} \\
A \cap (B \cap C) &= (A \cap B) \cap C \quad \text{(associative law)} \\
A \cap (B \cup C) &= (A \cap B) \cup (A \cap C) \quad \text{(distributive law)} \\
A \cup (B \cap C) &= (A \cup B) \cap (A \cup C) \quad \text{(distributive law)} \\
(\overline{A \cup B}) &= \bar{A} \cap \bar{B} \quad \text{(DeMorgan's law)} \\
(\overline{A \cap B}) &= \bar{A} \cup \bar{B} \quad \text{(DeMorgan's law)} \\
(A \cup B) \cap C &\neq A \cup (B \cap C) \quad \text{(in general)} \\
A \subset B &\Rightarrow \bar{B} \subset \bar{A}
\end{aligned}$$

Axioms of Events. A collection of events of sample space S is called a *field* $\mathcal{E}$ if

1. $S \in \mathcal{E}$,
2. $\mathcal{E}$ is closed under complements: $A \in \mathcal{E} \Rightarrow \bar{A} \in \mathcal{E}$,
3. $\mathcal{E}$ is closed under finite unions: $A, B \in \mathcal{E} \Rightarrow A \cup B \in \mathcal{E}$.

In fact, this collection of events is a *σ-field* if also closed under *countable* unions,

$$A_1, A_2, \ldots \in \mathcal{E} \Rightarrow A_1 \cup A_2 \cup \cdots \in \mathcal{E}.$$

Axioms of Probability. Given sample space S with σ-field $\mathcal{E}$, a set function P defined on $\mathcal{E}$ is a *probability measure* if

1. $0 \leq P(A) \leq 1, \quad A \in \mathcal{E}$,
2. $P(S) = 1$,
3. and if $A_1, A_2, \ldots \in \mathcal{E}$ is a disjoint sequence, $A_i \cap A_j = \emptyset$, $i \neq j$, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P\left(A_i\right).$$

Also, $P(\bar{A}) = 1 - P(A)$; if $A \subset B$, then $P(A) \leq P(B)$.

Frequentist Probability. If all the outcomes are equally likely, and S is finite sample space, and A is an event, then

$$P(A) = \frac{n(A)}{n(S)}$$

where $n(A)$ and $n(S)$ are number of items in event A and sample space S.

The *additive law of probability* for two events is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B),$$

and, more generally, for n events, the *inclusion-exclusion* law is

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{j=1}^{n} \left((-1)^{j+1} \sum_{1 \leq i_1 < \cdots < i_j \leq n} P\left(A_{i_1} \cap \cdots \cap A_{i_j}\right) \right).$$

Exercise 1.4 (Axioms of Probability and the Addition Rule)

1. *Axioms of Events: Three Coins.* Three coins are flipped.

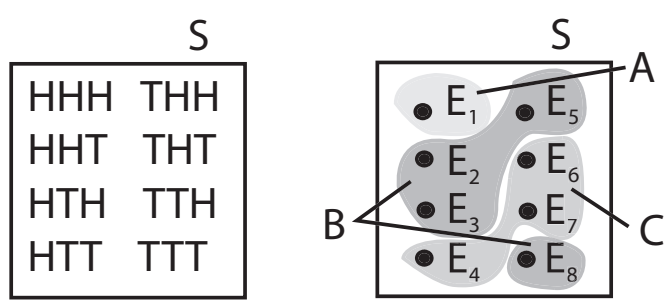

Figure 1.10: Two Venn diagrams for flipping three coins

(a) A (chance) *experiment* is a process which results in a set of observations where each observation has a chance of occurrence. Flipping a fair coin three times is an example of a chance experiment.
(i) **True** (ii) **False**

(b) The *sample space*, S, is a list of all the possible outcomes (or *sample points*) of a chance experiment. The sample space for flipping a fair coin three times is (choose one)
(i) **{HHH, HHT, HTH, HTT}**
(ii) **{THH, THT, TTH, TTT}**
(iii) **{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}**
Sample space really just means "set".

(c) A *discrete sample space* has a finite or countable number of sample points. The sample space for flipping a fair coin three times is an example of a discrete sample space.
(i) **True** (ii) **False**

(d) An *event* is a *subset* of a sample space. The event "flipping exactly three heads" is $A = \{E_1\} = \{HHH\}$. The event "flipping exactly one head" is $C =$ (choose one or more)
(i) $\{HTT, THT, TTH\}$
(ii) $\{E_4, E_6, E_7\}$
(iii) $E_4 \cup E_6 \cup E_7$
(iv) $E_4 \cap E_6 \cap E_7$

(e) A *simple event* consists of one sample point. Choose one correct statement.
(i) Events A, B and C are all examples of simple events.
(ii) Only events A and B are examples of simple events.
(iii) Only event A is an example of a simple event.

2. *Axioms of Probability: (More) Three Coins.* Assume the coin is fair; that is, the chance of any one of eight mutually exclusive simple events occurring is $\frac{1}{8}$: $P(E_1) = \frac{1}{8}$, $P(E_2) = \frac{1}{8}$, $\ldots$, $P(E_8) = \frac{1}{8}$.

(a) Axiom 1 implies any event has a probability between (and including) zero (0) and one (1); for example, $0 \leq P(A) = P(E_1) = \frac{1}{8} \leq 1$.
(i) **True** (ii) **False**

(b) Since all simple events are mutually exclusive, axiom 3 implies
$P(B) = P(E_2 \cup E_3 \cup E_5 \cup E_8) = P(E_2) + P(E_3) + P(E_5) + P(E_8) =$
(i) $\mathbf{\frac{1}{8}}$ (ii) $\mathbf{\frac{3}{8}}$ (iii) $\mathbf{\frac{4}{8}}$ (iv) $\mathbf{\frac{8}{8}}$.

(c) Axiom 3 implies the chance of flipping exactly one head,
$P(C) = P(E_4 \cup E_6 \cup E_7) = P(E_4) + P(E_6) + P(E_7) =$
(i) $\mathbf{\frac{1}{8}}$ (ii) $\mathbf{\frac{3}{8}}$ (iii) $\mathbf{\frac{4}{8}}$ (iv) $\mathbf{\frac{8}{8}}$.

(d) Axiom 3 implies, since events A, B and C are mutually exclusive,
$P(A \cup B \cup C) = P(A) + P(B) + P(C) = \frac{1}{8} + \frac{4}{8} + \frac{3}{8}$.
(i) **True** (ii) **False**

(e) Axiom 2 implies $P(S) = P(A \cup B \cup C) = 1$.
(i) **True** (ii) **False**

3. *Axioms of probability.* Consider a random experiment where positive integer n has probability

$$P(\{n\}) = \left(\frac{1}{4}\right)^n.$$

(a) $P(\{1\}) = \left(\frac{1}{4}\right)^1 =$ (i) $\mathbf{\frac{1}{4}}$ (ii) $\mathbf{\frac{1}{16}}$ (iii) $\mathbf{\frac{1}{64}}$ (iv) $\mathbf{\frac{21}{64}}$.

(b) $P(\{2\}) = \left(\frac{1}{4}\right)^2 =$ (i) $\mathbf{\frac{1}{4}}$ (ii) $\mathbf{\frac{1}{16}}$ (iii) $\mathbf{\frac{1}{64}}$ (iv) $\mathbf{\frac{21}{64}}$.

(c) $P(\{3\}) = \left(\frac{1}{4}\right)^3 =$ (i) $\mathbf{\frac{1}{4}}$ (ii) $\mathbf{\frac{1}{16}}$ (iii) $\mathbf{\frac{1}{64}}$ (iv) $\mathbf{\frac{21}{64}}$.

(d) For event $A = \{1, 2, 3\} = \{1\} \cup \{2\} \cup \{3\}$,
$P(\{1\} \cup \{2\} \cup \{3\}) = P(\{1\}) + (\{2\}) + \{3\}) =$
(i) $\mathbf{\frac{1}{4}}$ (ii) $\mathbf{\frac{1}{16}}$ (iii) $\mathbf{\frac{1}{64}}$ (iv) $\mathbf{\frac{21}{64}}$.

(e) For event $B = \{4, 5\} = \{4\} \cup \{5\}$,
$P(\{4\} \cup \{5\}) = P(\{4\}) + (\{5\}) =$ (i) $\mathbf{0}$ (ii) $\mathbf{\frac{5}{1024}}$ (iii) $\mathbf{\frac{341}{1024}}$ (iv) $\mathbf{\frac{683}{1024}}$.

(f) For event $A \cap B$, $P(A \cap B) =$ (i) $\mathbf{0}$ (ii) $\mathbf{\frac{5}{1024}}$ (iii) $\mathbf{\frac{341}{1024}}$ (iv) $\mathbf{\frac{683}{1024}}$.

(g) For event $A \cup B = \{1, 2, 3, 4, 5\}$,
$P(A \cup B) = P(A) + (B) =$ (i) $\mathbf{0}$ (ii) $\mathbf{\frac{5}{1024}}$ (iii) $\mathbf{\frac{341}{1024}}$ (iv) $\mathbf{\frac{683}{1024}}$.

(h) For event $C = \{n : n \geq 6\}$,
$P(C) = 1 - P(\bar{C}) = 1 - P(A \cup B) =$ (i) $\mathbf{0}$ (ii) $\mathbf{\frac{5}{1024}}$ (iii) $\mathbf{\frac{341}{1024}}$ (iv) $\mathbf{\frac{683}{1024}}$.

4. *Additive law for two events: dice.* In two rolls of fair die, let event A be the "sum of dice is five". Let event B be the event "no fours are rolled".

(a) $P(A) =$ (i) $\mathbf{\frac{1}{36}}$ (ii) $\mathbf{\frac{2}{36}}$ (iii) $\mathbf{\frac{3}{36}}$ (iv) $\mathbf{\frac{4}{36}}$.

(b) $P(B) =$ (i) $\mathbf{\frac{24}{36}}$ (ii) $\mathbf{\frac{25}{36}}$ (iii) $\mathbf{\frac{26}{36}}$ (iv) $\mathbf{\frac{27}{36}}$.

(c) $P(A \cap B) =$ (i) $\mathbf{\frac{1}{36}}$ (ii) $\mathbf{\frac{2}{36}}$ (iii) $\mathbf{\frac{3}{36}}$ (iv) $\mathbf{\frac{4}{36}}$.

(d) So $P(A \cup B) = P(A) + P(B) - P(A \cap B) =$
(i) $\mathbf{\frac{26}{36}}$ (ii) $\mathbf{\frac{27}{36}}$ (iii) $\mathbf{\frac{28}{36}}$ (iv) $\mathbf{\frac{29}{36}}$.

(e) Event A and event B
(i) **are disjoint** (ii) **are not disjoint**.

5. *Addition rule: fathers, sons and college.*
Data from a sample of 80 families in a midwestern city gives record of college attendance by fathers (F) and their oldest sons (S).

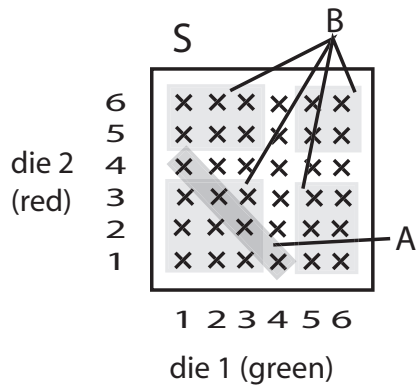

Figure 1.11: Venn diagram for tossing two dice

	son attended college	son did not attend college	
father attended college	18	7	25
father did not attend college	22	33	55
	40	40	80

(a) Probability son, in a randomly chosen family, attended college, is $P(S) = \mathbf{\frac{18}{40}}$ / $\mathbf{\frac{18}{25}}$ / $\mathbf{\frac{40}{80}}$ / $\mathbf{\frac{25}{80}}$.

(b) Probability father, in a randomly chosen family, attended college, is $P(F) = \mathbf{\frac{18}{40}}$ / $\mathbf{\frac{18}{25}}$ / $\mathbf{\frac{40}{80}}$ / $\mathbf{\frac{25}{80}}$.

(c) Probability son *and* father both attended college is $P(S \cap F) =$ (circle one) $\mathbf{\frac{18}{40}}$ / $\mathbf{\frac{18}{25}}$ / $\mathbf{\frac{40}{80}}$ / $\mathbf{\frac{18}{80}}$.

(d) Probability son *or* father both attended college is $P(S \cup F) = P(S) + P(F) - P(S \cap F) = \frac{40}{80} + \frac{25}{80} - \frac{18}{80} =$ (choose one) $\mathbf{\frac{45}{80}}$ / $\mathbf{\frac{46}{80}}$ / $\mathbf{\frac{47}{80}}$ / $\mathbf{\frac{48}{80}}$.

6. *More addition rule.*

(a) If $P(E) = \frac{3}{36}$, $P(F) = \frac{9}{36}$ and $P(E \cap F) = \frac{2}{36}$.
$P(E \textit{ or } F) = P(E) + P(F) - P(E \cap F) = \mathbf{\frac{10}{36}}$ / $\mathbf{\frac{11}{36}}$ / $\mathbf{\frac{12}{36}}$.

(b) $P(E) = 0.25$, $P(F) = 0.10$ and $P(E \cap F) = 0.03$,
$P(E \cup F) = P(E) + P(F) - P(E \cap F) =$ **0.30** / **0.32** / **0.33**.

(c) **True** / **False**. Addition rule determines chance of E "or" F.

(d) **True** / **False**. Events E and F are disjoint if $P(E \cap F) = 0$.

1.5 Conditional Probability and the Multiplication Rule

We will look at conditional probability

$$P(F|E) = \frac{P(E \text{ and } F)}{P(E)} = \frac{P(F \text{ and } E)}{P(E)},$$

and general multiplication rule,

$$P(E \text{ and } F) = P(E) \cdot P(F|E).$$

provided $P(E) \neq 0$.

Exercise 1.5 (Conditional Probability and the Multiplication Rule)

1. *Conditional chance: fathers, sons and college.*

	son attended college, S	son did not attend college, $\bar{S}$	
father attended college, F	18	7	25
father did not attend college, $\bar{F}$	22	33	55
	40	40	80

   (a) Probability a son attended college given a father attended college, in a randomly chosen family, is
   $P(S \mid F) =$ (i) $\mathbf{\frac{18}{40}}$ (ii) $\mathbf{\frac{18}{25}}$ (iii) $\mathbf{\frac{40}{80}}$ (iv) $\mathbf{\frac{25}{80}}$.

   (b) Probability a son did not attend college given a father attended college, in a randomly chosen family, is
   $P(\bar{S} \mid F) =$ (i) $\mathbf{\frac{7}{25}}$ (ii) $\mathbf{\frac{18}{25}}$ (iii) $\mathbf{\frac{7}{18}}$ (iv) $\mathbf{\frac{25}{7}}$.

   (c) $P(S \mid \bar{F}) =$ (i) $\mathbf{\frac{55}{22}}$ (ii) $\mathbf{\frac{33}{55}}$ (iii) $\mathbf{\frac{22}{55}}$ (iv) $\mathbf{\frac{22}{80}}$.

   (d) $P(\bar{S} \mid \bar{F}) =$ (i) $\mathbf{\frac{22}{55}}$ (ii) $\mathbf{\frac{33}{80}}$ (iii) $\mathbf{\frac{22}{33}}$ (iv) $\mathbf{\frac{33}{55}}$.

   (e) $P(F \mid S) =$ (i) $\mathbf{\frac{18}{40}}$ (ii) $\mathbf{\frac{18}{25}}$ (iii) $\mathbf{\frac{18}{22}}$ (iv) $\mathbf{\frac{25}{80}}$
   (i) **equals** (ii) **does not equal** $P(S \mid F) = \frac{18}{25}$.

   (f) *Using the formula.*

$$P(S \mid F) = \frac{P(S \cap F)}{P(F)} = \frac{18/80}{25/80} =$$

   (i) $\mathbf{\frac{18}{40}}$ (ii) $\mathbf{\frac{18}{25}}$ (iii) $\mathbf{\frac{40}{80}}$ (iv) $\mathbf{\frac{25}{80}}$.

2. *Using the formula: green grass mowed.* Let G be the event the grass is green and M the event the grass needs to be mowed and furthermore, $P(G) = 0.8$, $P(M) = 0.3$ and $P(G \cap M) = 0.2$.

(a) Probability grass is green, given it was mowed, is

$$P(G \mid M) = \frac{P(G \cap M)}{P(M)} = \frac{0.2}{0.3} \approx$$

(i) **0.25** (ii) **0.50** (iii) **0.67** (iv) **0.8**.

(b) Probability grass is mowed, given it was green, is

$$P(M \mid G) = \frac{P(G \cap M)}{P(G)} = \frac{0.2}{0.8} =$$

(i) **0.25** (ii) **0.50** (iii) **0.67** (iv) **0.8**.

3. *General multiplication rule.* A deck is shuffled and three cards are dealt.

(a) Chance first card dealt is an ace is
$P(ace) =$ (i) $\mathbf{\frac{1}{52}}$ (ii) $\mathbf{\frac{4}{52}}$ (iii) $\mathbf{\frac{3}{51}}$ (iv) $\mathbf{\frac{1}{51}}$.

(b) Chance *second* card dealt is a jack, given first card dealt is an ace, is
$P(jack \mid ace) =$ (i) $\mathbf{\frac{1}{52}}$ (ii) $\mathbf{\frac{4}{50}}$ (iii) $\mathbf{\frac{4}{51}}$ (iv) $\mathbf{\frac{1}{51}}$.
This is conditional probability since chance of one event depends on occurrence of another event.

(c) Probability first card is an ace and second card is a jack is
$P(ace \cap jack) = P(ace) \cdot P(jack \mid ace) =$
(i) $\mathbf{\frac{1}{52}} \times \mathbf{\frac{3}{51}}$ (ii) $\mathbf{\frac{4}{52}} \times \mathbf{\frac{4}{51}}$ (iii) $\mathbf{\frac{4}{51}}$ (iv) $\mathbf{\frac{1}{51}}$.
This is an example of general multiplication rule because it involves product of unconditional probability and conditional probability.

(d) Probability *third* card dealt is a jack, conditional on first two cards dealt are a jack and an ace, is
$P(jack \mid (ace \cap jack)) =$ (i) $\mathbf{\frac{1}{50}}$ (ii) $\mathbf{\frac{4}{52}}$ (iii) $\mathbf{\frac{3}{51}}$ (iv) $\mathbf{\frac{3}{50}}$.
This is another example of a conditional probability.

(e) Probability of an ace, jack and another jack is
$P(ace \cap jack \cap jack)$
$= P(ace) \cdot P(jack \mid ace) \cdot P(jack \mid (ace \cap jack)) =$
(i) $\mathbf{\frac{4}{52}} \times \mathbf{\frac{3}{51}} \times \mathbf{\frac{2}{50}}$
(ii) $\mathbf{\frac{4}{52}} \times \mathbf{\frac{4}{51}} \times \mathbf{\frac{3}{50}}$
(iii) $\mathbf{\frac{4}{50}} \times \mathbf{\frac{3}{49}} \times \mathbf{\frac{2}{48}}$
This is another example of the general multiplication rule.

4. *Marbles.* Two marbles are chosen at random without replacement from a bucket containing 5 red, 4 blue and 4 green marbles. Find following probabilities.

(a) *Choosing two red marbles.*
$P(RR) = P(R)P(R|R) = \frac{5}{13} \times \frac{4}{12} =$
(i) $\mathbf{\frac{12}{156}}$ (ii) $\mathbf{\frac{20}{44}}$ (iii) $\mathbf{\frac{20}{156}}$ (iv) $\mathbf{\frac{44}{156}}$.

(b) *Choosing two blue marbles.*
$P(BB) = P(B)P(B|B) = \frac{4}{13} \times \frac{3}{12} =$
(i) $\mathbf{\frac{12}{156}}$ (ii) $\mathbf{\frac{20}{44}}$ (iii) $\mathbf{\frac{20}{156}}$ (iv) $\mathbf{\frac{44}{156}}$.

(c) *Choosing two green marbles.*
$P(GG) = P(G)P(G|G) = \frac{4}{13} \times \frac{3}{12} =$
(i) $\mathbf{\frac{12}{156}}$ (ii) $\mathbf{\frac{20}{44}}$ (iii) $\mathbf{\frac{20}{156}}$ (iv) $\mathbf{\frac{44}{156}}$.

(d) *Choosing two marbles of same color.*
$P(S) = P(RR) + P(BB) + P(GG) = \frac{20}{156} + \frac{12}{156} + \frac{12}{156} =$
(i) $\mathbf{\frac{12}{156}}$ (ii) $\mathbf{\frac{20}{44}}$ (iii) $\mathbf{\frac{20}{156}}$ (iv) $\mathbf{\frac{44}{156}}$.

(e) *Choosing two red marbles, given they are of same color.*
$P(RR|S) = \frac{P(RR \cap S)}{P(S)} = \frac{P(RR)}{P(S)} = \frac{20/156}{44/156} =$
(i) $\mathbf{\frac{12}{156}}$ (ii) $\mathbf{\frac{20}{44}}$ (iii) $\mathbf{\frac{20}{156}}$ (iv) $\mathbf{\frac{44}{156}}$.

5. *Urns.* Urn A has 6 red and 10 blue marbles; urn B has 4 red and 14 blue marbles. A fair coin is tossed. If the coin comes up heads, a marble from urn A is chosen, otherwise a marble from urn B is chosen.

(a) *Choosing a red marble.*

i. The chance a head is tossed is $P(H) =$ (circle one)
(i) $\mathbf{\frac{1}{2}}$ (ii) $\mathbf{\frac{1}{3}}$ (iii) $\mathbf{\frac{1}{4}}$ (iv) $\mathbf{\frac{1}{5}}$.

ii. If a head is tossed, the chance a red marble is chosen is $P(R|H) =$
(i) $\mathbf{\frac{6}{10}}$ (ii) $\mathbf{\frac{6}{12}}$ (iii) $\mathbf{\frac{6}{14}}$ (iv) $\mathbf{\frac{6}{16}}$.

iii. If a tail is tossed, the chance a red marble is chosen is $P(R|T) =$
(i) $\mathbf{\frac{4}{10}}$ (ii) $\mathbf{\frac{4}{12}}$ (iii) $\mathbf{\frac{4}{14}}$ (iv) $\mathbf{\frac{4}{18}}$.

iv. The chance a red marble is chosen is
$P(R) = P(R|H)P(H) + P(R|T)P(T) = \frac{6}{16} \times \frac{1}{2} + \frac{4}{18} \times \frac{1}{2} =$
(i) $\mathbf{\frac{43}{144}}$ (ii) $\mathbf{\frac{45}{153}}$ (iii) $\mathbf{\frac{46}{165}}$ (iv) $\mathbf{\frac{53}{180}}$.

(b) *Choosing a blue marble.*
$P(B) = P(B|H)P(H) + P(B|T)P(T) = \frac{10}{16} \times \frac{1}{2} + \frac{14}{18} \times \frac{1}{2} =$
(i) $\mathbf{\frac{43}{144}}$ (ii) $\mathbf{\frac{76}{144}}$ (iii) $\mathbf{\frac{97}{144}}$ (iv) $\mathbf{\frac{101}{144}}$.
(Also, $P(B) = 1 - P(R) = 1 - \frac{43}{144} = \frac{101}{144}$.)

(c) *Choosing two red marbles, sample with replacement.*
Let R_1 and R_2 represent first and second red marbles.
$P(R_1 \cap R_2) = P(R_1)P(R_2) = \frac{43}{144} \times \frac{43}{144} =$
(i) **0.012** (ii) **0.034** (iii) **0.089** (iv) **0.123**.

(d) *Choosing two red marbles, sample without replacement.*
$P(R_1 \cap R_2) = P(R_1)P(R_2|R_1) =$
$\frac{43}{144} \times \left\{ \left(\frac{5}{15} \times \frac{1}{2} + \frac{4}{18} \times \frac{1}{2} \right) + \left(\frac{6}{16} \times \frac{1}{2} + \frac{3}{17} \times \frac{1}{2} \right) \right\} =$
(i) **0.042** (ii) **0.076** (iii) **0.099** (iv) **0.165**.

6. *Addition and Multiplication rules.* **True** (ii) **False**.
   When "and" is involved, "multiply": $P(E \cap F) = P(E) \cdot P(F|E)$.
   When "or" is involved, "add": $P(E \cup F) = P(E) + P(F) - P(E \cap F)$;

1.6 Bayes' Theorem

The two sets $\{A_1, A_2\}$ are a *partition* of sample space S if

$$A_1 \cup A_2 = S, \quad \text{and} \quad A_1 \cap A_2 = \emptyset,$$

and so the *law of total probability for two events* is

$$P(B) = P(B \mid A_1)P(A_1) + P(B \mid A_2)P(A_2),$$

and corresponding *Bayes Rule for event* A_i, $i = 1, 2$, is

$$P(A_i|B) = \frac{P(B \mid A_i)P(A_i)}{P(B \mid A_1)P(A_1) + P(B \mid A_2)P(A_2)}.$$

In general, let $A_1, \ldots, A_k$ be mutually exclusive and exhaustive events (a partition of sample space S) with $P(A_j) > 0$ for all j. Then for new evidence B, where $P(B) > 0$, the *law of total probability for* k *events* is

$$P(B \mid A_1)P(A_1) + \cdots + P(B \mid A_k)P(A_k),$$

and corresponding *Bayes Rule for event* A_i, $i = 1, 2, \ldots, k$, is

$$P(A_i \mid B) = \frac{P(B \mid A_i)P(A_i)}{P(B \mid A_1)P(A_1) + \cdots + P(B \mid A_k)P(A_k)}.$$

Exercise 1.6 (Bayes' Theorem)

1. *Sensitivity and specificity of disease.* Suppose a particular test has a 95% chance of detecting a disease if the person has it (this is called the sensitivity of the test), $P(T^+ \mid D^+) = 0.95$, and a 90% chance of correctly indicating the disease is absent if the person really does not have the disease (this is called the specificity of the test), $P(T^- \mid D^-) = 0.90$. Suppose 8% of the population has the disease, $P(D^+) = 0.08$. Consider figure below.

   (a) What is the chance a person tests negative given they have the disease?

$$\begin{aligned} P(T^- \mid D^+) &= 1 - P(T^+ \mid D^+) \\ &= 1 - 0.95 = \end{aligned}$$

   (i) **0.004** (ii) **0.050** (iii) **0.168** (iv) **0.452**.

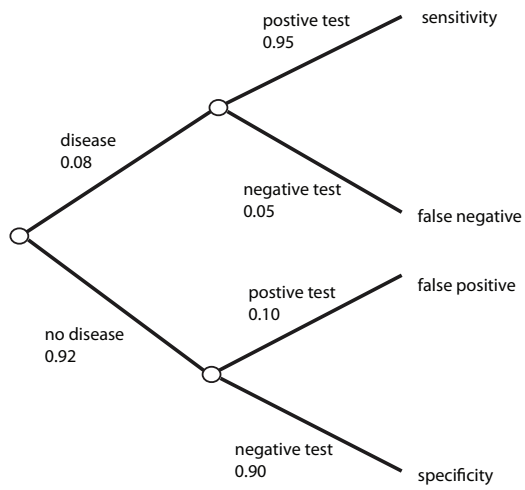

Figure 1.12: Testing diseases

(b) Chance a person has the disease but/and tests negative, a *false negative*?

$$\begin{aligned} P(T^- \cap D^+) &= P(T^- \mid D^+)P(D^+) \\ &= (0.05)(0.08) \; = \end{aligned}$$

(i) **0.004** (ii) **0.050** (iii) **0.100** (iv) **0.452**.

(c) Chance a person tests positive given they do not have the disease?

$$\begin{aligned} P(T^+ \mid D^-) &= 1 - P(T^- \mid D^-) \\ &= 1 - 0.90 \; = \end{aligned}$$

(i) **0.092** (ii) **0.100** (iii) **0.168** (iv) **0.452**.

(d) Person does not have the disease and tests positive, a *false positive*?

$$\begin{aligned} P(T^+ \cap D^-) &= P(T^+ \mid D^-)P(D^-) \\ &= (0.10)(0.92) \; = \end{aligned}$$

(i) **0.010** (ii) **0.092** (iii) **0.168** (iv) **0.452**.

(e) What is the probability a randomly chosen person will test positive?

$$\begin{aligned} P(T^+) &= P(T^+ \cap D^+) + P(T^+ \cap D^-) \\ &= P(T^+ \mid D^+)P(D^+) + P(T^+ \mid D^-)P(D^-) \\ &= (0.95)(0.08) + (0.10)(0.92) \; = \end{aligned}$$

(i) **0.010** (ii) **0.092** (iii) **0.168** (iv) **0.452**.

(f) Suppose a randomly chosen person does test positive. What is the probability this person really has the disease?

$$\begin{aligned} P(D^+ \mid T^+) &= \frac{P(T^+ \mid D^+)P(D^+)}{P(T^+ \mid D^+)P(D^+) + P(T^+ \mid D^-)P(D^-)} \\ &= \frac{(0.95)(0.08)}{(0.95)(0.08) + (0.10)(0.92)} \approx \end{aligned}$$

(i) **0.010** (ii) **0.092** (iii) **0.168** (iv) **0.452**.

2. *Rental agencies.*A firm rents cars from three rental agencies where $P(A_1) = \alpha$, $P(A_2) = \alpha$, $P(A_3) = \frac{1}{2}\alpha$. Also, $P(T \mid A_1) = 0.09$, $P(T \mid A_2) = 0.2$ and $P(T \mid A_3) = 0.13$. A car is chosen at random.

(a) The probability a car is rented from agency A_1, given this car needs a tune–up, is

$$\begin{aligned} P(A_1 \mid T) &= \frac{P(T \mid A_1)P(A_1)}{P(T \mid A_1)P(A_1) + P(T \mid A_2)P(A_2) + P(T \mid A_3)P(A_3)} \\ &= \frac{(0.09)(\alpha)}{(0.09)(\alpha) + (0.2)(\alpha) + (0.13)\left(\frac{1}{2}\alpha\right)} = \end{aligned}$$

(i) **0.183** (ii) **0.254** (iii) **0.424** (iv) **0.452**.

(b) Also,

$$\begin{aligned} P(A_3 \mid T) &= \frac{P(T \mid A_3)P(A_3)}{P(T \mid A_1)P(A_1) + P(T \mid A_2)P(A_2) + P(T \mid A_3)P(A_3)} \\ &= \frac{(0.13)\left(\frac{1}{2}\alpha\right)}{(0.09)(\alpha) + (0.2)(\alpha) + (0.13)\left(\frac{1}{2}\alpha\right)} \approx \end{aligned}$$

(i) **0.183** (ii) **0.254** (iii) **0.424** (iv) **0.452**.

(c) The probability a car comes from agency A_1, given this car does *not* need a tune–up, is

$$\begin{aligned} P(A_1|T^c) &= \frac{P(T^c \mid A_1)P(A_1)}{P(T^c \mid A_1)P(A_1) + P(T^c \mid A_2)P(A_2) + P(T^c \mid A_3)P(A_3)} \\ &= \frac{(0.91)(\alpha)}{(0.91)(\alpha) + (0.8)(\alpha) + (0.87)\left(\frac{1}{2}\alpha\right)} \approx \end{aligned}$$

(i) **0.183** (ii) **0.254** (iii) **0.424** (iv) **0.452**.

1.7 Independent Events

The *multiplicative law of probability* for two events is given by

$$P(A \cap B) = P(B)P(A|B),$$

and also more generally, for n events,

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_n|A_1 \cap \cdots \cap A_{n-1}),$$

and if events $A_1, \ldots, A_n$ are independent, then, for every subset $A_{i_1}, \ldots, A_{i_r}$ of them,

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_r}).$$

So, for example, events A, B and C are independent if *both* pairwise independent,

$$P(A \cap B) = P(A)P(B), \quad P(A \cap C) = P(A)P(C), \quad P(B \cap C) = P(B)P(C)$$

and also triwise independent,

$$P(A \cap B \cap C) = P(A)P(B)P(C).$$

Also, A and B are independent, then so are A and $\bar{B}$.

Exercise 1.7 (Independent Events)

1. *Independence (sampling with replacement)*
   versus dependence (sampling without replacement): Box of Tickets.
   Two things are *independent* if the chances for the second given the first are the same, no matter how the first turns out; otherwise, the two things are *dependent*. Consider the following box with six tickets.

   $$\boxed{\boxed{1_a}\ \boxed{2_a}\ \boxed{1_b}\ \boxed{3_b}\ \boxed{2_c}\ \boxed{3_c}}$$

   (a) Two tickets are drawn at random from the box. The first ticket drawn is *not* replaced in the box; that is, there are only five tickets remaining in the box when the second ticket is drawn. The chance the second ticket is a "2" *given that* the first ticket is a "2" is (i) $\frac{1}{5}$ (ii) $\frac{2}{5}$ (iii) $\frac{3}{5}$.

   (b) Two tickets are sampled with*out* replacement at random from the box. That is, there are only five tickets remaining in the box when the second ticket is drawn. The chance the second ticket is a "2" *given that* the first ticket is a "1" is (i) $\frac{1}{5}$ (ii) $\frac{2}{5}$ (iii) $\frac{3}{5}$.

   (c) When sampling at random with*out* replacement, the chance the second ticket of two drawn from the box is any given number will *depend* on what number was drawn on the first ticket.
   (i) **True** (ii) **False**

(d) Two tickets are sampled *with* replacement at random from the box. That is, all *six* tickets remain in the box when the second ticket is drawn. The chance the second ticket is a "2" *given that* the first ticket is a "1" is
(i) $\mathbf{\frac{1}{6}}$ (ii) $\mathbf{\frac{2}{6}}$ (iii) $\mathbf{\frac{3}{6}}$.

(e) When sampling at random with replacement, the chance the second ticket of two drawn from the box is "2", no matter what the first ticket is, will *always* be (i) $\mathbf{\frac{1}{6}}$ (ii) $\mathbf{\frac{2}{6}}$ (iii) $\mathbf{\frac{3}{6}}$.

(f) When sampling at random with replacement, the draws are independent of one another; without replacement, the draws are dependent.
(i) **True** (ii) **False**

2. *Independence and dependence: box of coins.*

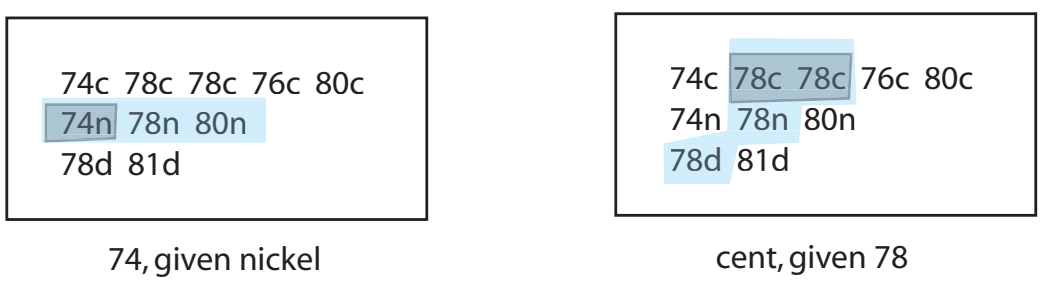

Figure 1.13: Conditional probability: box of coins

(a) *Choosing 1974.*
Chance a coin chosen at random from box is a 1974 coin is
$P(1974) =$ (i) $\mathbf{\frac{1}{10}}$ (ii) $\mathbf{\frac{2}{10}}$ (iii) $\mathbf{\frac{3}{10}}$ (iv) $\mathbf{\frac{4}{10}}$.

(b) *Choosing 1974, given nickel.*
Of three coins that are nickels, (i) **1** (ii) **2** (iii) **3** are 1974 coins. Given coin taken from box is a nickel, chance this coin is a 1974 nickel is
$P(1974|N) =$ (i) $\mathbf{\frac{1}{3}}$ (ii) $\mathbf{\frac{2}{3}}$ (iii) $\mathbf{\frac{3}{3}}$ (iv) $\mathbf{\frac{4}{3}}$.

(c) *Choosing 1974 depends on choosing nickel.*
Unconditional chance coin is "1974", $P(1974) = \frac{2}{10}$, is (i) **equal** (ii) **not equal** to conditional chance coin is "1974, given a nickel", $P(1974|N) = \frac{1}{3}$.
Choosing a "1974" and choosing a "nickel" are *dependent.*

(d) *Choosing cent.*
Chance of choosing a cent is $P(C) =$ (i) $\mathbf{\frac{2}{5}}$ (ii) $\mathbf{\frac{5}{10}}$ (iii) $\mathbf{\frac{2}{10}}$ (iv) $\mathbf{\frac{2}{4}}$.

(e) *Choosing cent, given 1978.*
Of coins that are 1978s, (i) **2** (ii) **4** (iii) **5** are cent coins.
Given a coin is a 1978, chance this coin is a cent is $P(C|1978) =$
(i) $\mathbf{\frac{2}{5}}$ (ii) $\mathbf{\frac{5}{10}}$ (iii) $\mathbf{\frac{2}{10}}$ (iv) $\mathbf{\frac{2}{4}}$.

(f) *Choosing cent independent of choosing 1978.*
Since $P(C) = \frac{5}{10} = P(C|1978) = \frac{2}{4}$, choosing a "cent" and choosing a "1978" are (i) **independent** (ii) **dependent** events.

(g) *In general.*
If $P(E) = P(E|F)$, events E and F are (i) **dependent** (ii) **independent**; otherwise, they are *dependent*. This is a second method to determine independence/dependence.

3. *Independence versus pairwise independence.* A fair 4-sided die is labelled 1, 2, 3, 4; let $A = \{1, 4\}$, $B = \{2, 4\}$, $C = \{3, 4\}$.

(a) $P(A) =$ (i) $\mathbf{\frac{1}{2}}$ (ii) $\mathbf{\frac{1}{3}}$ (iii) $\mathbf{\frac{1}{4}}$ (iv) $\mathbf{\frac{1}{5}}$ (since A contains 2 of 4 outcomes)

(b) $P(B) =$ (i) $\mathbf{\frac{1}{2}}$ (ii) $\mathbf{\frac{1}{3}}$ (iii) $\mathbf{\frac{1}{4}}$ (iv) $\mathbf{\frac{1}{5}}$.

(c) $P(C) =$ (i) $\mathbf{\frac{1}{2}}$ (ii) $\mathbf{\frac{1}{3}}$ (iii) $\mathbf{\frac{1}{4}}$ (iv) $\mathbf{\frac{1}{5}}$.

(d) $A \cap B = \{4\}$, so $P(A \cap B) =$ (i) $\mathbf{\frac{1}{2}}$ (ii) $\mathbf{\frac{1}{3}}$ (iii) $\mathbf{\frac{1}{4}}$ (iv) $\mathbf{\frac{1}{5}}$
$= P(A)P(B) = \frac{1}{2} \cdot \frac{1}{2}$

(e) $B \cap C = \{4\}$, so $P(B \cap C) =$ (i) $\mathbf{\frac{1}{2}}$ (ii) $\mathbf{\frac{1}{3}}$ (iii) $\mathbf{\frac{1}{4}}$ (iv) $\mathbf{\frac{1}{5}}$.
$= P(B)P(C) = \frac{1}{2} \cdot \frac{1}{2}$

(f) $A \cap C = \{4\}$, so $P(A \cap C) =$ (i) $\mathbf{\frac{1}{2}}$ (ii) $\mathbf{\frac{1}{3}}$ (iii) $\mathbf{\frac{1}{4}}$ (iv) $\mathbf{\frac{1}{5}}$.
$= P(A)P(C) = \frac{1}{2} \cdot \frac{1}{2}$

(g) So events A, B, C are pairwise independent
(i) **True** (ii) **False**

(h) $A \cap B \cap C = \{4\}$, so $P(A \cap B \cap C) =$ (i) $\mathbf{\frac{1}{2}}$ (ii) $\mathbf{\frac{1}{3}}$ (iii) $\mathbf{\frac{1}{4}}$ (iv) $\mathbf{\frac{1}{5}}$.
$\neq P(A)P(B)P(C) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$

(i) So events A, B, C are both not triwise independent and also not independent
(i) **True** (ii) **False**

4. *Additive law, multiplicative law and independence: water pipes.* As shown in Figure, two systems of water pipes connecting city A to city B are given below. Water does not flow through a pipe if a valve is closed. Let V_i denote the event a value is *open*. Any value is open with probability $P(V_i) = 0.8$ and so a valve is closed with probability $P(\bar{V_i}) = 0.2$. Whether one valve is open is independent of whether any other value is open.

(a) *Figure (a): Parallel Water Pipes.* Water flows from city A to city B in

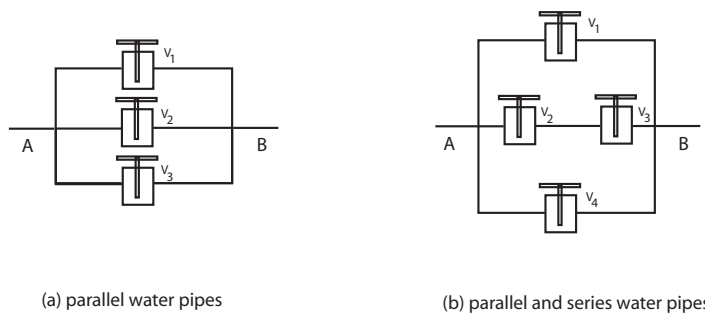

Figure 1.14: Multiplicative and additive rules: water pipes

Figure (a) if either value V_1 *or* value V_2 *or* value V_3 is open.

$$\begin{aligned}
P(V_1 \cup V_2 \cup V_3) &= P(V_1) + P(V_2) + P(V_3) \\
&\quad -P(V_1 \cap V_2) - P(V_1 \cap V_3) - P(V_2 \cap V_3) + P(V_1 \cap V_2 \cap V_3) = \\
&= P(V_1) + P(V_2) + P(V_3) \\
&\quad -P(V_1)P(V_2) - P(V_1)P(V_3) - P(V_2)P(V_3) + P(V_1)P(V_2)P(V_3) = \\
&= 0.8 + 0.8 + 0.8 \\
&\quad -(0.8)(0.8) - (0.8)(0.8) - (0.8)(0.8) + (0.8)(0.8)(0.8) =
\end{aligned}$$

(i) **0.992** (ii) **0.993** (iii) **0.994** (iv) **0.995**.

(b) *Figure (b): Parallel and Series Water Pipes.* Water flows from city A to city B in Figure (b) if either value V_1 *or* values "V_2 *and* V_3" or value V_4 is/are open.

$$\begin{aligned}
P(V_1 \cup (V_2 \cap V_3) \cup V_4) &= P(V_1) + P(V_2 \cap V_3) + P(V_4) \\
&\quad -P(V_1 \cap V_2 \cap V_3) - P(V_1 \cap V_4) \\
&\quad -P(V_2 \cap V_3 \cap V_4) + P(V_1 \cap V_2 \cap V_3 \cap V_4) = \\
&= P(V_1) + P(V_2)P(V_3) + P(V_4) \\
&\quad -P(V_1)P(V_2)P(V_3) - P(V_1)P(V_4) \\
&\quad -P(V_2)P(V_3)P(V_4) + P(V_1)P(V_2)P(V_3)P(V_4) = \\
&= 0.8 + (0.8)(0.8) + 0.8 \\
&\quad -(0.8)(0.8)(0.8) - (0.8)(0.8) \\
&\quad -(0.8)(0.8)(0.8) + (0.8)(0.8)(0.8)(0.8) =
\end{aligned}$$

(i) **0.9592** (ii) **0.9635** (iii) **0.9742** (iv) **0.9856**.

(c) *Figure (b) again: Parallel and Series Water Pipes.* The probability water does *not* flow from city A to city B is given by:

$$P(\overline{V_1 \cup (V_2 \cap V_3) \cup V_4}) = 1 - P(V_1 \cup (V_2 \cap V_3) \cup V_4) =$$

(i) **0.0144** (ii) **0.0185** (iii) **0.0294** (iv) **0.0300**.