1.4 Axioms of Probability and the Addition Rule

A capital letter \( A \), for example, denotes a set of elements (or outcomes). The \( n \) elements in set (or event) \( A \) are denoted \( A = \{a_1, a_2, ..., a_n\} \). The set that contains no elements is called a null or empty set and denoted by \( \emptyset \). The set that contains all elements under consideration is the universal set (or sample space), \( S \). Set \( A \) is a subset of \( B \), \( B \subset A \), if every element in \( A \) is also in \( B \). Events \( A \) and \( B \) are mutually exclusive or disjoint if \( A \cap B = \emptyset \). Mutually exclusive events cannot happen together.

Some ways events can be manipulated:

- **intersection**, described using “and” or “cap” symbol, \( \cap \),
- **union**, described using “or” or “cup” symbol, \( \cup \),
- **complement**, using symbol, \( \bar{A} \) or \( A' \).

Some properties for sets \( A, B \) and \( C \) include

\[
\begin{align*}
A \cup B & = B \cup A \quad \text{(commutative law)} \\
A \cap B & = B \cap A \quad \text{(commutative law)} \\
A \cup (B \cup C) & = (A \cup B) \cup C = A \cup B \cup C \quad \text{(associative law)} \\
A \cap (B \cap C) & = (A \cap B) \cap C \quad \text{(associative law)} \\
A \cap (B \cup C) & = (A \cap B) \cup (A \cap C) \quad \text{(distributive law)} \\
A \cup (B \cap C) & = (A \cup B) \cap (A \cup C) \quad \text{(distributive law)} \\
\overline{A \cup B} & = \bar{A} \cap \bar{B} \quad \text{(DeMorgan’s law)} \\
\overline{A \cap B} & = \bar{A} \cup \bar{B} \quad \text{(DeMorgan’s law)} \\
(A \cup B) \cap C & \neq A \cup (B \cap C) \quad \text{(in general)} \\
A \subset B & \Rightarrow B \subset \bar{A}
\end{align*}
\]

**Axioms of Events.** A collection of events of sample space \( S \) is called a field \( \mathcal{E} \) if

1. \( S \in \mathcal{E} \),

2. \( \mathcal{E} \) is closed under complements: \( A \in \mathcal{E} \Rightarrow \bar{A} \in \mathcal{E} \),

3. \( \mathcal{E} \) is closed under finite unions: \( A, B \in \mathcal{E} \Rightarrow A \cup B \in \mathcal{E} \).

In fact, this collection of events is a \( \sigma \)-field if also closed under countable unions,

\[
A_1, A_2, \ldots \in \mathcal{E} \Rightarrow A_1 \cup A_2 \cup \cdots \in \mathcal{E}.
\]

**Axioms of Probability.** Given sample space \( S \) with \( \sigma \)-field \( \mathcal{E} \), a set function \( P \) defined on \( \mathcal{E} \) is a probability measure if
1. $0 \leq P(A) \leq 1$, $A \in \mathcal{E}$,
2. $P(S) = 1$,
3. and if $A_1, A_2, \ldots \in \mathcal{E}$ is a disjoint sequence, $A_i \cap A_j = \emptyset$, $i \neq j$, then

$$P \left( \bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} P(A_i).$$

Also, $P(\bar{A}) = 1 - P(A)$; if $A \subset B$, then $P(A) \leq P(B)$.

**Frequentist Probability.** If all the outcomes are equally likely, and $S$ is finite sample space, and $A$ is an event, then

$$P(A) = \frac{n(A)}{n(S)}$$

where $n(A)$ and $n(S)$ are number of items in event $A$ and sample space $S$.

The **additive law of probability** for two events is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B),$$

and, more generally, for $n$ events, the **inclusion-exclusion** law is

$$P \left( \bigcup_{i=1}^{n} A_i \right) = \sum_{j=1}^{n} \left( (-1)^{j+1} \sum_{1 \leq i_1 < \cdots < i_j \leq n} P(A_{i_1} \cap \cdots \cap A_{i_j}) \right).$$

**Exercise 1.4 (Axioms of Probability and the Addition Rule)**

1. **Axioms of Events: Three Coins.** Three coins are flipped.

   ![Two Venn diagrams for flipping three coins](image)
(a) A (chance) *experiment* is a process which results in a set of observations where each observation has a chance of occurrence. Flipping a fair coin three times is an example of a chance experiment.

(i) **True**  (ii) **False**

(b) The *sample space*, $S$, is a list of all the possible outcomes (or *sample points*) of a chance experiment. The sample space for flipping a fair coin three times is (choose one)

(i) \{HHH, HHT, HTH, HTT\}
(ii) \{THH, THT, TTH, TTT\}
(iii) \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}

Sample space really just means “set”.

(c) A *discrete sample space* has a finite or countable number of sample points. The sample space for flipping a fair coin three times is an example of a discrete sample space.

(i) **True**  (ii) **False**

(d) An *event* is a *subset* of a sample space. The event “flipping exactly three heads” is $A = \{E_1\} = \{HHH\}$. The event “flipping exactly one head” is $C = (\text{choose one or more})$

(i) \{HTT, THT, TTH\}
(ii) \{E_4, E_6, E_7\}
(iii) $E_4 \cup E_6 \cup E_7$
(iv) $E_4 \cap E_6 \cap E_7$

(e) A *simple event* consists of one sample point. Choose one correct statement.

(i) Events $A$, $B$ and $C$ are all examples of simple events.
(ii) Only events $A$ and $B$ are examples of simple events.
(iii) Only event $A$ is an example of a simple event.

2. **Axioms of Probability: (More) Three Coins.** Assume the coin is fair; that is, the chance of any one of eight mutually exclusive simple events occurring is \(\frac{1}{8}\):

\[ P(E_1) = \frac{1}{8}, P(E_2) = \frac{1}{8}, \ldots, P(E_8) = \frac{1}{8}. \]

(a) Axiom 1 implies any event has a probability between (and including) zero (0) and one (1); for example, $0 \leq P(A) = P(E_1) = \frac{1}{8} \leq 1$.

(i) **True**  (ii) **False**

(b) Since all simple events are mutually exclusive, axiom 3 implies

\[ P(B) = P(E_2 \cup E_3 \cup E_5 \cup E_8) = P(E_2) + P(E_3) + P(E_5) + P(E_8) = \]

(i) $\frac{1}{8}$  (ii) $\frac{3}{8}$  (iii) $\frac{4}{8}$  (iv) $\frac{8}{8}$.

(c) Axiom 3 implies the chance of flipping exactly one head,

\[ P(C) = P(E_4 \cup E_6 \cup E_7) = P(E_4) + P(E_6) + P(E_7) = \]

(i) $\frac{1}{8}$  (ii) $\frac{3}{8}$  (iii) $\frac{4}{8}$  (iv) $\frac{8}{8}$. 
Section 4. Axioms of Probability and the Addition Rule (LECTURE NOTES 2)  

3. Axioms of probability. Consider a random experiment where positive integer $n$ has probability

$$P(\{n\}) = \left(\frac{1}{4}\right)^n.$$ 

(a) $P(\{1\}) = \left(\frac{1}{4}\right)^1 = (i) \frac{1}{4} \quad (ii) \frac{1}{16} \quad (iii) \frac{1}{64} \quad (iv) \frac{21}{64}.$

(b) $P(\{2\}) = \left(\frac{1}{4}\right)^2 = (i) \frac{1}{4} \quad (ii) \frac{1}{16} \quad (iii) \frac{1}{64} \quad (iv) \frac{21}{64}.$

(c) $P(\{3\}) = \left(\frac{1}{4}\right)^3 = (i) \frac{1}{4} \quad (ii) \frac{1}{16} \quad (iii) \frac{1}{64} \quad (iv) \frac{21}{64}.$

(d) For event $A = \{1, 2, 3\} = \{1\} \cup \{2\} \cup \{3\}$,  
$P(\{1\} \cup \{2\} \cup \{3\}) = P(\{1\}) + P(\{2\}) + P(\{3\}) = (i) \frac{1}{4} \quad (ii) \frac{1}{16} \quad (iii) \frac{1}{64} \quad (iv) \frac{21}{64}.$

(e) For event $B = \{4, 5\} = \{4\} \cup \{5\}$,  
$P(\{4\} \cup \{5\}) = P(\{4\}) + P(\{5\}) = (i) 0 \quad (ii) \frac{5}{1024} \quad (iii) \frac{341}{1024} \quad (iv) \frac{683}{1024}.$

(f) For event $A \cap B$, $P(\{A \cap B\}) = (i) 0 \quad (ii) \frac{5}{1024} \quad (iii) \frac{341}{1024} \quad (iv) \frac{683}{1024}.$

(g) For event $A \cup B = \{1, 2, 3, 4, 5\}$,  
$P(A \cup B) = P(A) + P(B) = (i) 0 \quad (ii) \frac{5}{1024} \quad (iii) \frac{341}{1024} \quad (iv) \frac{683}{1024}.$

(h) For event $C = \{n : n \geq 6\}$,  
$P(C) = 1 - P(\overline{C}) = 1 - P(A \cup B) = (i) 0 \quad (ii) \frac{5}{1024} \quad (iii) \frac{341}{1024} \quad (iv) \frac{683}{1024}.$

4. Additive law for two events: dice. In two rolls of fair die, let event $A$ be the “sum of dice is five”. Let event $B$ be the event “no fours are rolled”.

(a) $P(A) = (i) \frac{1}{36} \quad (ii) \frac{2}{36} \quad (iii) \frac{3}{36} \quad (iv) \frac{4}{36}.$

(b) $P(B) = (i) \frac{24}{36} \quad (ii) \frac{25}{36} \quad (iii) \frac{26}{36} \quad (iv) \frac{27}{36}.$

(c) $P(A \cap B) = (i) \frac{1}{36} \quad (ii) \frac{2}{36} \quad (iii) \frac{3}{36} \quad (iv) \frac{4}{36}.$

(d) So $P(A \cup B) = P(A) + P(B) - P(A \cap B) = (i) \frac{26}{36} \quad (ii) \frac{27}{36} \quad (iii) \frac{28}{36} \quad (iv) \frac{29}{36}.$

(e) Event $A$ and event $B$  
(i) are disjoint  
(ii) are not disjoint.

5. Addition rule: fathers, sons and college.  
Data from a sample of 80 families in a midwestern city gives record of college attendance by fathers (F) and their oldest sons (S).
(a) Probability son, in a randomly chosen family, attended college, is 
\[ P(S) = \frac{18}{40} / \frac{18}{25} / \frac{40}{80} / \frac{25}{80}. \]

(b) Probability father, in a randomly chosen family, attended college, is 
\[ P(F) = \frac{18}{40} / \frac{18}{25} / \frac{40}{80} / \frac{25}{80}. \]

(c) Probability son and father both attended college is 
\[ P(S \cap F) = \text{(circle one)} \frac{18}{40} / \frac{18}{25} / \frac{40}{80} / \frac{18}{80}. \]

(d) Probability son or father both attended college is 
\[ P(S \cup F) = P(S) + P(F) - P(S \cap F) = \frac{40}{80} + \frac{25}{80} - \frac{18}{80} = \text{(choose one)} \frac{45}{80} / \frac{46}{80} / \frac{47}{80} / \frac{48}{80}. \]


(a) If \( P(E) = \frac{3}{36}, P(F) = \frac{9}{36} \) and \( P(E \cap F) = \frac{2}{36} \), 
\[ P(E \text{ or } F) = P(E) + P(F) - P(E \cap F) = \frac{10}{36} / \frac{11}{36} / \frac{12}{36}. \]

(b) \( P(E) = 0.25, P(F) = 0.10 \) and \( P(E \cap F) = 0.03 \), 
\[ P(E \cup F) = P(E) + P(F) - P(E \cap F) = 0.30 / 0.32 / 0.33. \]

(c) True / False. Addition rule determines chance of \( E \) “or” \( F \).

(d) True / False. Events \( E \) and \( F \) are disjoint if \( P(E \cap F) = 0 \).
1.5 Conditional Probability and the Multiplication Rule

We will look at conditional probability

\[ P(F|E) = \frac{P(E \text{ and } F)}{P(E)} = \frac{P(F \text{ and } E)}{P(E)}, \]

and general multiplication rule,

\[ P(E \text{ and } F) = P(E) \cdot P(F|E). \]

provided \( P(E) \neq 0. \)

Exercise 1.5 (Conditional Probability and the Multiplication Rule)


<table>
<thead>
<tr>
<th>father attended college, ( F )</th>
<th>son attended college, ( S )</th>
<th>son did not attend college, ( \bar{S} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>18</td>
<td>7</td>
<td>25</td>
</tr>
<tr>
<td>22</td>
<td>33</td>
<td>55</td>
</tr>
<tr>
<td>40</td>
<td>40</td>
<td>80</td>
</tr>
</tbody>
</table>

(a) Probability a son attended college given a father attended college, in a randomly chosen family, is

\[ P(S \mid F) = (i) \frac{18}{40} \quad (ii) \frac{18}{25} \quad (iii) \frac{40}{80} \quad (iv) \frac{25}{80}. \]

(b) Probability a son did not attend college given a father attended college, in a randomly chosen family, is

\[ P(\bar{S} \mid F) = (i) \frac{7}{25} \quad (ii) \frac{18}{25} \quad (iii) \frac{7}{18} \quad (iv) \frac{25}{80}. \]

(c) \( P(S \mid \bar{F}) = (i) \frac{55}{22} \quad (ii) \frac{33}{55} \quad (iii) \frac{22}{33} \quad (iv) \frac{33}{55}. \)

(d) \( P(\bar{S} \mid \bar{F}) = (i) \frac{22}{55} \quad (ii) \frac{33}{80} \quad (iii) \frac{33}{55} \quad (iv) \frac{33}{55}. \)

(e) \( P(F \mid S) = (i) \frac{18}{40} \quad (ii) \frac{18}{25} \quad (iii) \frac{18}{22} \quad (iv) \frac{25}{80}. \)

(i) **equals** (ii) **does not equal** \( P(S \mid F) = \frac{18}{25}. \)

(f) **Using the formula.**

\[ P(S \mid F) = \frac{P(S \cap F)}{P(F)} = \frac{18/80}{25/80} = \]

\[ (i) \frac{18}{40} \quad (ii) \frac{18}{25} \quad (iii) \frac{40}{80} \quad (iv) \frac{25}{80}. \]

2. Using the formula: green grass mowed. Let \( G \) be the event the grass is green and \( M \) the event the grass needs to be mowed and furthermore, \( P(G) = 0.8, P(M) = 0.3 \) and \( P(G \cap M) = 0.2. \)
Chapter 1. Basics of Probability (LECTURE NOTES 2)

(a) Probability grass is green, given it was mowed, is

$$P(G | M) = \frac{P(G \cap M)}{P(M)} \approx \frac{0.2}{0.3} = 0.67.$$

(i) 0.25 (ii) 0.50 (iii) 0.67 (iv) 0.8.

(b) Probability grass is mowed, given it was green, is

$$P(M | G) = \frac{P(G \cap M)}{P(G)} = \frac{0.2}{0.8} = 0.25.$$

(i) 0.25 (ii) 0.50 (iii) 0.67 (iv) 0.8.

3. General multiplication rule. A deck is shuffled and three cards are dealt.

(a) Chance first card dealt is an ace is

$$P(\text{ace}) = (i) \frac{1}{52} \quad (ii) \frac{4}{52} \quad (iii) \frac{3}{51} \quad (iv) \frac{1}{51}.$$

(b) Chance second card dealt is a jack, given first card dealt is an ace, is

$$P(\text{jack} | \text{ace}) = (i) \frac{1}{52} \quad (ii) \frac{4}{50} \quad (iii) \frac{3}{51} \quad (iv) \frac{1}{51}.$$

This is conditional probability since chance of one event depends on occurrence of another event.

(c) Probability first card is an ace and second card is a jack is

$$P(\text{ace} \cap \text{jack}) = P(\text{ace}) \cdot P(\text{jack} | \text{ace}) =$$

(i) \(\frac{1}{52} \times \frac{3}{51}\) (ii) \(\frac{4}{52} \times \frac{3}{51}\) (iii) \(\frac{4}{51} \times \frac{3}{51}\) (iv) \(\frac{1}{51}\).

This is an example of general multiplication rule because it involves product of unconditional probability and conditional probability.

(d) Probability third card dealt is a jack, conditional on first two cards dealt are a jack and an ace, is

$$P(\text{jack} | (\text{ace} \cap \text{jack})) = (i) \frac{1}{50} \quad (ii) \frac{4}{52} \quad (iii) \frac{3}{51} \quad (iv) \frac{3}{50}.$$

This is another example of a conditional probability.

(e) Probability of an ace, jack and another jack is

$$P(\text{ace} \cap \text{jack} \cap \text{jack}) =$$

$$P(\text{ace}) \cdot P(\text{jack} | \text{ace}) \cdot P(\text{jack} | (\text{ace} \cap \text{jack})) =$$

(i) \(\frac{4}{52} \times \frac{3}{51} \times \frac{2}{50}\) (ii) \(\frac{4}{52} \times \frac{4}{51} \times \frac{3}{50}\) (iii) \(\frac{4}{50} \times \frac{3}{49} \times \frac{2}{48}\)

This is another example of the general multiplication rule.

4. Marbles. Two marbles are chosen at random without replacement from a bucket containing 5 red, 4 blue and 4 green marbles. Find following probabilities.

(a) Choosing two red marbles.

$$P(\text{RR}) = P(\text{R})P(\text{R} | \text{R}) = \frac{5}{13} \times \frac{4}{12} =$$

(i) \(\frac{12}{156}\) (ii) \(\frac{20}{44}\) (iii) \(\frac{20}{156}\) (iv) \(\frac{12}{156}\).
Section 5. Conditional Probability and the Multiplication Rule (LECTURE NOTES 2)23

(b) **Choosing two blue marbles.**
\[ P(BB) = P(B)P(B|B) = \frac{4}{13} \times \frac{3}{12} = \]
(i) \(\frac{12}{156}\) (ii) \(\frac{20}{44}\) (iii) \(\frac{20}{156}\) (iv) \(\frac{44}{156}\).

(c) **Choosing two green marbles.**
\[ P(GG) = P(G)P(G|G) = \frac{4}{13} \times \frac{3}{12} = \]
(i) \(\frac{12}{156}\) (ii) \(\frac{20}{44}\) (iii) \(\frac{20}{156}\) (iv) \(\frac{44}{156}\).

(d) **Choosing two marbles of same color.**
\[ P(S) = P( RR) + P( BB) + P( GG) = \frac{20}{156} + \frac{12}{156} + \frac{12}{156} = \]
(i) \(\frac{12}{156}\) (ii) \(\frac{20}{44}\) (iii) \(\frac{20}{156}\) (iv) \(\frac{44}{156}\).

(e) **Choosing two red marbles, given they are of same color.**
\[ P(RR|S) = \frac{P(RR\cap S)}{P(S)} = \frac{P(RR)}{P(S)} = \frac{20/156}{156/44/156} = \]
(i) \(\frac{12}{156}\) (ii) \(\frac{20}{44}\) (iii) \(\frac{20}{156}\) (iv) \(\frac{44}{156}\).

5. **Urns.** Urn A has 6 red and 10 blue marbles; urn B has 4 red and 14 blue marbles. A fair coin is tossed. If the coin comes up heads, a marble from urn A is chosen, otherwise a marble from urn B is chosen.

(a) **Choosing a red marble.**

i. The chance a head is tossed is \(P(H) = \text{circle one}\)
   (i) \(\frac{1}{2}\) (ii) \(\frac{1}{3}\) (iii) \(\frac{1}{4}\) (iv) \(\frac{1}{5}\).

ii. If a head is tossed, the chance a red marble is chosen is \(P(R|H) = \text{circle one}\)
   (i) \(\frac{6}{10}\) (ii) \(\frac{6}{12}\) (iii) \(\frac{6}{14}\) (iv) \(\frac{6}{16}\).

iii. If a tail is tossed, the chance a red marble is chosen is \(P(R|T) = \text{circle one}\)
   (i) \(\frac{4}{10}\) (ii) \(\frac{4}{12}\) (iii) \(\frac{4}{14}\) (iv) \(\frac{4}{18}\).

iv. The chance a red marble is chosen is
   \[ P(R) = P(R|H)P(H) + P(R|T)P(T) = \frac{6}{16} \times \frac{1}{2} + \frac{4}{18} \times \frac{1}{2} = \]
   (i) \(\frac{43}{144}\) (ii) \(\frac{45}{153}\) (iii) \(\frac{46}{165}\) (iv) \(\frac{53}{180}\).

(b) **Choosing a blue marble.**
\[ P(B) = P(B|H)P(H) + P(B|T)P(T) = \frac{10}{16} \times \frac{1}{2} + \frac{14}{18} \times \frac{1}{2} = \]
(i) \(\frac{43}{144}\) (ii) \(\frac{76}{144}\) (iii) \(\frac{97}{144}\) (iv) \(\frac{101}{144}\). (Also, \(P(B) = 1 - P(R) = 1 - \frac{43}{144} = \frac{101}{144}\).)

(c) **Choosing two red marbles, sample with replacement.**
Let \(R_1\) and \(R_2\) represent first and second red marbles.
\[ P(R_1 \cap R_2) = P(R_1)P(R_2) = \frac{43}{144} \times \frac{43}{144} = \]
(i) \(0.012\) (ii) \(0.034\) (iii) \(0.089\) (iv) \(0.123\).

(d) **Choosing two red marbles, sample without replacement.**
\[ P(R_1 \cap R_2) = P(R_1)P(R_2|R_1) = \]
\[ \frac{43}{144} \times \{ \left( \frac{5}{15} \times \frac{1}{2} + \frac{4}{18} \times \frac{1}{2} \right) + \left( \frac{6}{16} \times \frac{1}{2} + \frac{3}{17} \times \frac{1}{2} \right) \} = \]
(i) \(0.042\) (ii) \(0.076\) (iii) \(0.099\) (iv) \(0.165\).
6. Addition and Multiplication rules. **True**  (ii) **False**.
When “and” is involved, “multiply”: \( P(E \cap F) = P(E) \cdot P(F|E) \).
When “or” is involved, “add”: \( P(E \cup F) = P(E) + P(F) - P(E \cap F) \).

1.6 Bayes’ Theorem

The two sets \( \{A_1, A_2\} \) are a **partition** of sample space \( S \) if
\[
A_1 \cup A_2 = S, \quad \text{and} \quad A_1 \cap A_2 = \emptyset,
\]
and so the **law of total probability for two events** is
\[
P(B) = P(B \mid A_1)P(A_1) + P(B \mid A_2)P(A_2),
\]
and corresponding **Bayes Rule for event** \( A_i \), \( i = 1, 2 \), is
\[
P(A_i \mid B) = \frac{P(B \mid A_i)P(A_i)}{P(B \mid A_1)P(A_1) + P(B \mid A_2)P(A_2)}.
\]

In general, let \( A_1, \ldots, A_k \) be mutually exclusive and exhaustive events (a partition of sample space \( S \)) with \( P(A_j) > 0 \) for all \( j \). Then for new evidence \( B \), where \( P(B) > 0 \), the **law of total probability for \( k \) events** is
\[
P(B \mid A_1)P(A_1) + \cdots + P(B \mid A_k)P(A_k),
\]
and corresponding **Bayes Rule for event** \( A_i \), \( i = 1, 2, \ldots, k \), is
\[
P(A_i \mid B) = \frac{P(B \mid A_i)P(A_i)}{P(B \mid A_1)P(A_1) + \cdots + P(B \mid A_k)P(A_k)}.
\]

**Exercise 1.6 (Bayes’ Theorem)**

1. **Sensitivity and specificity of disease.** Suppose a particular test has a 95% chance of detecting a disease if the person has it (this is called the sensitivity of the test), \( P(T^+ \mid D^+) = 0.95 \), and a 90% chance of correctly indicating the disease is absent if the person really does not have the disease (this is called the specificity of the test), \( P(T^- \mid D^-) = 0.90 \). Suppose 8% of the population has the disease, \( P(D^+) = 0.08 \). Consider figure below.

(a) What is the chance a person tests negative given they have the disease?
\[
P(T^- \mid D^+) = 1 - P(T^+ \mid D^+) = 1 - 0.95 = 0.05
\]

(i) **0.004**  (ii) **0.050**  (iii) **0.168**  (iv) **0.452**.
Figure 1.12: Testing diseases

(b) Chance a person has the disease but/and tests negative, a \textit{false negative}?

\[ P(T^- \cap D^+) = P(T^- \mid D^+) P(D^+) \]
\[ = (0.05)(0.08) = \]
\[(i) \ 0.004 \quad (ii) \ 0.050 \quad (iii) \ 0.100 \quad (iv) \ 0.452. \]

(c) Chance a person tests positive given they do not have the disease?

\[ P(T^+ \mid D^-) = 1 - P(T^- \mid D^-) \]
\[ = 1 - 0.90 = \]
\[(i) \ 0.092 \quad (ii) \ 0.100 \quad (iii) \ 0.168 \quad (iv) \ 0.452. \]

(d) Person does not have the disease and tests positive, a \textit{false positive}?

\[ P(T^+ \cap D^-) = P(T^+ \mid D^-) P(D^-) \]
\[ = (0.10)(0.92) = \]
\[(i) \ 0.010 \quad (ii) \ 0.092 \quad (iii) \ 0.168 \quad (iv) \ 0.452. \]

(e) What is the probability a randomly chosen person will test positive?

\[ P(T^+) = P(T^+ \cap D^+) + P(T^+ \cap D^-) \]
\[ = P(T^+ \mid D^+) P(D^+) + P(T^+ \mid D^-) P(D^-) \]
\[ = (0.95)(0.08) + (0.10)(0.92) = \]
\[(i) \ 0.010 \quad (ii) \ 0.092 \quad (iii) \ 0.168 \quad (iv) \ 0.452. \]
(f) Suppose a randomly chosen person does test positive. What is the probability this person really has the disease?

\[
P(D^+ | T^+) = \frac{P(T^+ | D^+)P(D^+)}{P(T^+ | D^+)P(D^+) + P(T^+ | D^-)P(D^-)}
\]

\[
= \frac{(0.95)(0.08)}{(0.95)(0.08) + (0.10)(0.92)} \approx 0.010
\]

(ii) 0.092 (iii) 0.168 (iv) 0.452.

2. Rental agencies. A firm rents cars from three rental agencies where \(P(A_1) = \alpha, P(A_2) = \alpha, P(A_3) = \frac{1}{2}\alpha\). Also, \(P(T | A_1) = 0.09, P(T | A_2) = 0.2\) and \(P(T | A_3) = 0.13\). A car is chosen at random.

(a) The probability a car is rented from agency \(A_1\), given this car needs a tune-up, is

\[
P(A_1 | T) = \frac{P(T | A_1)P(A_1)}{P(T | A_1)P(A_1) + P(T | A_2)P(A_2) + P(T | A_3)P(A_3)}
\]

\[
= \frac{(0.09)(\alpha)}{(0.09)(\alpha) + (0.2)(\alpha) + (0.13)(\frac{1}{2}\alpha)}
\]

\[
= (i) 0.183 \quad (ii) 0.254 \quad (iii) 0.424 \quad (iv) 0.452.
\]

(b) Also,

\[
P(A_3 | T) = \frac{P(T | A_3)P(A_3)}{P(T | A_1)P(A_1) + P(T | A_2)P(A_2) + P(T | A_3)P(A_3)}
\]

\[
= \frac{(0.13)(\frac{1}{2}\alpha)}{(0.09)(\alpha) + (0.2)(\alpha) + (0.13)(\frac{1}{2}\alpha)} \approx
\]

\[
(i) 0.183 \quad (ii) 0.254 \quad (iii) 0.424 \quad (iv) 0.452.
\]

(c) The probability a car comes from agency \(A_1\), given this car does not need a tune-up, is

\[
P(A_1 | T^c) = \frac{P(T^c | A_1)P(A_1)}{P(T^c | A_1)P(A_1) + P(T^c | A_2)P(A_2) + P(T^c | A_3)P(A_3)}
\]

\[
= \frac{(0.91)(\alpha)}{(0.91)(\alpha) + (0.8)(\alpha) + (0.87)(\frac{1}{2}\alpha)} \approx
\]

\[
(i) 0.183 \quad (ii) 0.254 \quad (iii) 0.424 \quad (iv) 0.452.
\]
1.7 Independent Events

The multiplicative law of probability for two events is given by

\[ P(A \cap B) = P(B)P(A|B), \]

and also more generally, for \( n \) events,

\[ P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_n|A_1 \cap \cdots \cap A_{n-1}), \]

and if events \( A_1, \ldots, A_n \) are independent, then, for every subset \( A_{i_1}, \ldots, A_{i_r} \) of them,

\[ P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_r}). \]

So, for example, events \( A, B \) and \( C \) are independent if both pairwise independent,

\[ P(A \cap B) = P(A)P(B), \quad P(A \cap C) = P(A)P(C), \quad P(B \cap C) = P(B)P(C) \]

and also triwise independent,

\[ P(A \cap B \cap C) = P(A)P(B)P(C). \]

Also, \( A \) and \( B \) are independent, then so are \( A \) and \( \bar{B} \).

Exercise 1.7 (Independent Events)

1. Independence (sampling with replacement) versus dependence (sampling without replacement): Box of Tickets.

Two things are independent if the chances for the second given the first are the same, no matter how the first turns out; otherwise, the two things are dependent.

Consider the following box with six tickets.

```
| 1_a | 2_a | 1_b | 3_b | 2_c | 3_c |
```

(a) Two tickets are drawn at random from the box. The first ticket drawn is not replaced in the box; that is, there are only five tickets remaining in the box when the second ticket is drawn. The chance the second ticket is a “2” given that the first ticket is a “2” is (i) \( \frac{1}{5} \) (ii) \( \frac{2}{5} \) (iii) \( \frac{3}{5} \).

(b) Two tickets are sampled without replacement at random from the box. That is, there are only five tickets remaining in the box when the second ticket is drawn. The chance the second ticket is a “2” given that the first ticket is a “1” is (i) \( \frac{1}{5} \) (ii) \( \frac{2}{5} \) (iii) \( \frac{3}{5} \).

(c) When sampling at random without replacement, the chance the second ticket of two drawn from the box is any given number will depend on what number was drawn on the first ticket.

(i) True (ii) False
Chapter 1. Basics of Probability (LECTURE NOTES 2)

(d) Two tickets are sampled with replacement at random from the box. That is, all six tickets remain in the box when the second ticket is drawn. The chance the second ticket is a “2” given that the first ticket is a “1” is (i) $\frac{1}{6}$ (ii) $\frac{2}{6}$ (iii) $\frac{3}{6}$.

(e) When sampling at random with replacement, the chance the second ticket of two drawn from the box is “2”, no matter what the first ticket is, will always be (i) $\frac{1}{6}$ (ii) $\frac{2}{6}$ (iii) $\frac{3}{6}$.

(f) When sampling at random with replacement, the draws are independent of one another; without replacement, the draws are dependent.

(i) True (ii) False

2. Independence and dependence: box of coins.

![Figure 1.13: Conditional probability: box of coins](image)

(a) Choosing 1974.
Chance a coin chosen at random from box is a 1974 coin is

$$P(1974) = (i) \frac{1}{10} \quad (ii) \frac{2}{10} \quad (iii) \frac{3}{10} \quad (iv) \frac{4}{10}.$$  

(b) Choosing 1974, given nickel.
Of three coins that are nickels, (i) 1 (ii) 2 (iii) 3 are 1974 coins. Given coin taken from box is a nickel, chance this coin is a 1974 nickel is

$$P(1974|N) = (i) \frac{1}{3} \quad (ii) \frac{2}{3} \quad (iii) \frac{3}{3} \quad (iv) \frac{4}{3}.$$  

(c) Choosing 1974 depends on choosing nickel.
Unconditional chance coin is “1974”, $P(1974) = \frac{2}{10}$, is (i) equal (ii) not equal to conditional chance coin is “1974, given a nickel”, $P(1974|N) = \frac{1}{3}$. Choosing a “1974” and choosing a ”nickel” are dependent.

(d) Choosing cent.
Chance of choosing a cent is $P(C) = (i) \frac{2}{5} \quad (ii) \frac{5}{10} \quad (iii) \frac{2}{10} \quad (iv) \frac{2}{4}$.

(e) Choosing cent, given 1978.
Of coins that are 1978s, (i) 2 (ii) 4 (iii) 5 are cent coins. Given a coin is a 1978, chance this coin is a cent is $P(C|1978) = (i) \frac{2}{5} \quad (ii) \frac{5}{10} \quad (iii) \frac{2}{10} \quad (iv) \frac{2}{4}$. 
(f) **Choosing cent independent of choosing 1978.**
Since \( P(C) = \frac{5}{10} = P(C|1978) = \frac{2}{4} \), choosing a “cent” and choosing a “1978” are (i) **independent** (ii) **dependent** events.

(g) **In general.**
If \( P(E) = P(E|F) \), events \( E \) and \( F \) are (i) **dependent** (ii) **independent**; otherwise, they are dependent. This is a second method to determine independence/dependence.

3. **Independence versus pairwise independence.** A fair 4-sided die is labelled 1, 2, 3, 4; let \( A = \{1, 4\} \), \( B = \{2, 4\} \), \( C = \{3, 4\} \).

(a) \( P(A) = (i) \frac{1}{2} \) (ii) \( \frac{1}{3} \) (iii) \( \frac{1}{4} \) (iv) \( \frac{1}{5} \) (since \( A \) contains 2 of 4 outcomes)

(b) \( P(B) = (i) \frac{1}{2} \) (ii) \( \frac{1}{3} \) (iii) \( \frac{1}{4} \) (iv) \( \frac{1}{5} \).

(c) \( P(C) = (i) \frac{1}{2} \) (ii) \( \frac{1}{3} \) (iii) \( \frac{1}{4} \) (iv) \( \frac{1}{5} \).

(d) \( A \cap B = \{4\} \), so \( P(A \cap B) = (i) \frac{1}{2} \) (ii) \( \frac{1}{3} \) (iii) \( \frac{1}{4} \) (iv) \( \frac{1}{5} \)

\[ = P(A)P(B) = \frac{1}{2} \cdot \frac{1}{2} \]

(e) \( B \cap C = \{4\} \), so \( P(B \cap C) = (i) \frac{1}{2} \) (ii) \( \frac{1}{3} \) (iii) \( \frac{1}{4} \) (iv) \( \frac{1}{5} \)

\[ = P(B)P(C) = \frac{1}{2} \cdot \frac{1}{2} \]

(f) \( A \cap C = \{4\} \), so \( P(A \cap C) = (i) \frac{1}{2} \) (ii) \( \frac{1}{3} \) (iii) \( \frac{1}{4} \) (iv) \( \frac{1}{5} \)

\[ = P(A)P(C) = \frac{1}{2} \cdot \frac{1}{2} \]

(g) So events \( A, B, C \) are pairwise independent

(i) **True** (ii) **False**

(h) \( A \cap B \cap C = \{4\} \), so \( P(A \cap B \cap C) = (i) \frac{1}{2} \) (ii) \( \frac{1}{3} \) (iii) \( \frac{1}{4} \) (iv) \( \frac{1}{5} \)

\[ \neq P(A)P(B)P(C) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \]

(i) So events \( A, B, C \) are both not triwise independent and also not independent

(i) **True** (ii) **False**

4. **Additive law, multiplicative law and independence: water pipes.** As shown in Figure, two systems of water pipes connecting city \( A \) to city \( B \) are given below. Water does not flow through a pipe if a valve is closed. Let \( V_i \) denote the event a value is open. Any value is open with probability \( P(V_i) = 0.8 \) and so a valve is closed with probability \( P(\overline{V}_i) = 0.2 \). Whether one valve is open is independent of whether any other value is open.

(a) **Figure (a): Parallel Water Pipes.** Water flows from city \( A \) to city \( B \) in
Figure 1.14: Multiplicative and additive rules: water pipes

Figure (a) if either value $V_1$ or value $V_2$ or value $V_3$ is open.

\[
P(V_1 \cup V_2 \cup V_3) = P(V_1) + P(V_2) + P(V_3) \\
- P(V_1 \cap V_2) - P(V_1 \cap V_3) - P(V_2 \cap V_3) + P(V_1 \cap V_2 \cap V_3) = \\
P(V_1) + P(V_2) + P(V_3) - P(V_1)P(V_2) - P(V_1)P(V_3) - P(V_2)P(V_3) + P(V_1)P(V_2)P(V_3) = \\
0.8 + 0.8 + 0.8 \\
-(0.8)(0.8) - (0.8)(0.8) - (0.8)(0.8) + (0.8)(0.8)(0.8) = \]

(i) 0.992  (ii) 0.993  (iii) 0.994  (iv) 0.995.

(b) Figure (b): Parallel and Series Water Pipes. Water flows from city $A$ to city $B$ in Figure (b) if either value $V_1$ or values “$V_2$ and $V_3$” or value $V_4$ is/are open.

\[
P(V_1 \cup (V_2 \cap V_3) \cup V_4) = P(V_1) + P(V_2 \cap V_3) + P(V_4) \\
- P(V_1 \cap V_2 \cap V_3) - P(V_1 \cap V_4) \\
- P(V_2 \cap V_3 \cap V_4) + P(V_1 \cap V_2 \cap V_3 \cap V_4) = \\
P(V_1) + P(V_2)P(V_3) + P(V_4) - P(V_1)P(V_2)P(V_3) - P(V_1)P(V_4) \\
- P(V_2)P(V_3)P(V_4) + P(V_1)P(V_2)P(V_3)P(V_4) = \\
0.8 + (0.8)(0.8) + 0.8 \\
-(0.8)(0.8)(0.8) - (0.8)(0.8) \\
-(0.8)(0.8)(0.8) + (0.8)(0.8)(0.8)(0.8) = \]

(i) 0.9592  (ii) 0.9635  (iii) 0.9742  (iv) 0.9856.

(c) Figure (b) again: Parallel and Series Water Pipes. The probability water does not flow from city $A$ to city $B$ is given by:

\[
P(V_1 \cup (V_2 \cap V_3) \cup V_4) = 1 - P(V_1 \cup (V_2 \cap V_3) \cup V_4) = \]
Section 7. Independent Events (LECTURE NOTES 2)

(i) 0.0144  (ii) 0.0185  (iii) 0.0294  (iv) 0.0300.