Chapter 2

Discrete Random Variables

2.1 Introduction

A random variable, denoted by a capital letter such as $X$, is a function mapping from each outcome in a sample space to the real line:

$$X : S \rightarrow \mathbb{R}.$$ 

Random variable is discrete if its range is either finite or countably infinite.

![Figure 2.1: Examples of random variables](image)

Exercise 2.1 (Introduction)

1. Number of heads, $X$ in three coin tosses: (i) discrete  (ii) continuous  
   domain: (i) $S \in \{TTT, TTH, \ldots, HHH\}$  (ii) $X \in \{0, 1, 2, 3\}$  
   range: (i) $S \in \{TTT, TTH, \ldots, HHH\}$  (ii) $X \in \{0, 1, 2, 3\}$  
   Random variable $X$ (i) is  (ii) not a probability

2. Number of dots, $X$, in roll of a die: (i) discrete  (ii) continuous  
   domain: (i) $S \in \{dots on die faces\}$  (ii) $X \in \{1, 2, 3, 4, 5, 6\}$
range: (i) $S \in \text{dots on die faces}$ (ii) $X \in \{1, 2, 3, 4, 5, 6\}$

Value of random variable $X$ denoted (i) $X = x$ (ii) $X = X$

3. Number of seizures, $Z$, in a year: (i) \textit{discrete} (ii) \textit{continuous}

domain: (i) $S \in \text{patients in study}$ (ii) $Z \in \{0, 1, 2, \ldots\}$

range: (i) $S \in \text{patients in study}$ (ii) $Z \in \{0, 1, 2, \ldots\}$

Patients, $W$, sick (1 or more seizures) or not (0 seizures):

(i) \textit{discrete} (ii) \textit{continuous}

domain: (i) $S \in \text{patients in study}$ (ii) $W \in \{0, 1\}$

range: (i) $S \in \text{patients in study}$ (ii) $W \in \{0, 1\}$

4. Waiting time, $T$, at Burger King up to 1 hour: (i) \textit{discrete} (ii) \textit{continuous}

domain: (i) $S \in \text{waiting customers}$ (ii) $\{T : 0 \leq T \leq 1\}$

range: (i) $S \in \text{waiting customers}$ (ii) $\{T : 0 \leq T \leq 1\}$

2.2 Probability Mass Functions

If $R$ is the range of discrete random variable $X$, the \textit{probability mass function (pmf)} of $X$ is function $f : R \rightarrow \mathbb{R}$ which satisfies:

- $f(x) > 0$, for all $x \in R$
- $\sum_{x \in R} f(x) = 1$
- If $X \subset R$, then $P(X \in A) = \sum_{x \in A} f(x)$

The pmf can be described by table, graph or function, these descriptions are called the \textit{distribution} of the random variable.

Figure 2.2: Examples of random variable and probability mass function (pmf)

The \textit{cumulative distribution function (cdf)} (\textit{distribution function}) of discrete random
variable $X$ is

$$F(b) = P(X \leq b) = \sum_{x \leq b} f(x), \quad \text{for all real numbers } b.$$  

If range $R$ of discrete random variable $X$ has $k$ elements, $X$ has a uniform distribution with pmf

$$f(x) = \frac{1}{k}, \quad \text{for each } x \in R.$$  

The mode of discrete random variable $X$ is the value of $X$ where the pmf is a maximum. The median is the smallest number $m$ such that

$$P(X \leq m) \geq 0.5 \quad \text{and} \quad P(X \geq m) \geq 0.5.$$  

A random variable with two modes is bimodal, with two or more modes, multimodal.

**Exercise 2.2 (Probability Mass Functions)**

1. *Probability mass function for seizures.* The number of seizures, $X$, of a typical epileptic person in any given year is given by the following pmf.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>0.17</td>
<td>0.21</td>
<td>0.18</td>
<td>0.11</td>
<td>0.16</td>
<td>0.17</td>
</tr>
</tbody>
</table>

(a) The chance a person has 8 epileptic seizures is $f(8) =$

(i) 0.11  (ii) 0.16  (iii) 0.17  (iv) 0.21.

(b) The chance a person has at most 4 seizures is

(i) 0.17  (ii) 0.21  (iii) 0.56  (iv) 0.67.

(c) $P(X \leq 4) = ($i) 0.17  (ii) 0.21  (iii) 0.56  (iv) 0.67.

(d) $f(2) = ($i) 0.17  (ii) 0.21  (iii) 0.56  (iv) 0.67.

(e) $f(2.1) = ($i) 0  (ii) 0.21  (iii) 0.56  (iv) 0.67.

(f) $P(X > 2.1) = ($i) 0.21  (ii) 0.38  (iii) 0.56  (iv) 0.62.

(g) $\sum_{x=0}^{10} f(x) = P(X = 0) + P(X = 2) + \cdots + P(X = 10) =$

(i) 0.97  (ii) 0.98  (iii) 0.99  (iv) 1.

(h) *Graph of distribution.* Which of the three graphs best describes the probability distribution of the number of seizures?

(i) (a)  (ii) (b)  (iii) (c).

(i) *Probability mass function.* Which one of the following functions describes the pmf of the number of seizures?
2. Rolling a pair of dice: number of fours rolled.

Let $X$ be the number of 4’s rolled. Assume the dice are fair.

(a) $P(X = 2) = P\{(4, 4)\} = (i) \frac{1}{36}$ \hspace{1em} (ii) $\frac{20}{36}$ \hspace{1em} (iii) $\frac{25}{36}$ \hspace{1em} (iv) $\frac{30}{36}$.

(b) Since $P(X = 1) =
  \begin{align*}
  & (i) \quad P\{(1, 4), (2, 4), (3, 4), (5, 4), (6, 4)\} \\
  & (ii) \quad P\{(4, 1), (4, 2), (4, 3), (4, 5), (4, 6)\} \\
  & (iii) \quad P\{(1, 4), (2, 4), (3, 4), (5, 4), (6, 4), (4, 1), (4, 2), (4, 3), (4, 5), (4, 6)\}
  \end{align*}$

(c) Then $P(X = 1) = (i) \frac{1}{36}$ \hspace{1em} (ii) $\frac{10}{36}$ \hspace{1em} (iii) $\frac{25}{36}$ \hspace{1em} (iv) $\frac{30}{36}$.

(d) $P(X = 0) = 1 - P(X = 1) - P(X = 2) =
  \begin{align*}
  & (i) \frac{11}{36} \hspace{1em} (ii) \frac{20}{36} \hspace{1em} (iii) \frac{25}{36} \hspace{1em} (iv) \frac{30}{36}.
  \end{align*}$

(e) The probability distribution of $X$ is

<table>
<thead>
<tr>
<th>$x$</th>
<th>$P(X = x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\frac{25}{36}$</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{21}{36}$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{1}{36}$</td>
</tr>
</tbody>
</table>

(i) True \hspace{1em} (ii) False
3. **Flipping until a head comes up.** A (weighted) coin has a probability of \( p = 0.7 \) of coming up heads (and so a probability of \( q = 1 - p = 0.3 \) of coming up tails). Let \( X \) be the number of flips until a head comes up or until a total of 4 flips are made.

(a) Head may come up on first flip, \( f(1) = P\{H\} =
\begin{align*}
(i) & \quad 0.3 \\
(ii) & \quad 0.7 \\
(iii) & \quad 0.3(0.7) \\
(iv) & \quad 0.3^2(0.7).
\end{align*}

(b) Head may come up on second flip, \( f(2) = P\{TH\} =
\begin{align*}
(i) & \quad 0.3 \\
(ii) & \quad 0.7 \\
(iii) & \quad 0.3(0.7) \\
(iv) & \quad 0.3^2(0.7).
\end{align*}

(c) \( f(3) = P\{TTH\} =
\begin{align*}
(i) & \quad 0.3 \\
(ii) & \quad 0.7 \\
(iii) & \quad 0.3(0.7) \\
(iv) & \quad 0.3^2(0.7).
\end{align*}

(d) \( f(4) = P\{TTTT, TTTH\} = 1 - p(1) - p(2) - p(3) =
\begin{align*}
(i) & \quad 0.027 \\
(ii) & \quad 0.063 \\
(iii) & \quad 0.210 \\
(iv) & \quad 0.700.
\end{align*}

(e) The probability distribution of \( X \) is
\[
\begin{array}{c|cccc}
 x & 1 & 2 & 3 & 4 \\
 f(x) & 0.700 & 0.210 & 0.063 & 0.027 \\
\end{array}
\]

(i) **True**  (ii) **False**

(f) **Conditional probability.**
\[
P(X \geq 3|X \geq 2) = \frac{P(X \geq 3 \cap X \geq 2)}{P(X \geq 2)} = \frac{P(X \geq 3)}{P(X \geq 2)} = \frac{0.063 + 0.027}{1 - 0.7} =
\begin{align*}
(i) & \quad 0.2 \\
(ii) & \quad 0.3 \\
(iii) & \quad 0.4 \\
(iv) & \quad 0.5.
\end{align*}

4. **Geometric probability mass function: bull’s eye.** There is a 15% \( (p = 0.15) \) chance of hitting a bull’s eye on a dart board. Throws are independent of one another.

(a) The chance the first bull’s eye occurs on the *first* try is, of course, 15%.

The chance the first bull’s eye occurs on the *second* try equals the chance a miss occurs on the first try and a bull’s eye occurs on the second try,
\[
f(2) = (0.85)0.15 =
\begin{align*}
(i) & \quad 0.1155 \\
(ii) & \quad 0.1275 \\
(iii) & \quad 0.1385 \\
(iv) & \quad 0.2515.
\end{align*}

(b) The chance the first bull’s eye occurs on the *third* try is equal to the chance of two misses and then a bull’s eye occurs on the third try,
\[
f(3) = (0.85)(0.85)0.15 = (0.85)^20.15 \approx
\begin{align*}
(i) & \quad 0.078 \\
(ii) & \quad 0.099 \\
(iii) & \quad 0.108 \\
(iv) & \quad 0.158.
\end{align*}

(c) The chance the first bull’s eye occurs on the *fourth* try is
\[
f(4) = (0.85)^30.15 \approx
\begin{align*}
(i) & \quad 0.078 \\
(ii) & \quad 0.092 \\
(iii) & \quad 0.108 \\
(iv) & \quad 0.151.
(d) The chance the first bull’s eye occurs on the eleventh try, \( x = 11 \), is
\[
f(11) = q^{x-1}p = (0.85)^{11-1} \times 0.15 \approx 0.01 \quad \text{(i)}
\]
\[
0.02 \quad \text{(ii)} \quad 0.03 \quad \text{(iii)} \quad 0.04 \quad \text{(iv)}.
\]

(e) Since the chance the first bull’s eye occurs on the \( x \)th try is
\[
f(x) = q^{x-1}p,
\]
then the sum of this series
\[
\sum_{x \in R} f(x) = \sum_{x=1}^{\infty} f(x) = \sum_{x=1}^{\infty} q^{x-1}p = \frac{p}{1-q} = p
\]
\[
\text{(i) } p \quad \text{(ii) } q \quad \text{(iii) } \frac{p}{q} \quad \text{(iv) } 1.
\]

5. Probability mass function and cumulative distribution function: flipping a coin twice. Let the number of heads flipped in two flips of a coin be a discrete random variable \( X \) with pmf \( f(x) \) is given in table and Figure 2.4(a), and cdf \( F(x) = P(X \leq b) \) given in Figure 2.4(b).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( 0 )</th>
<th>( 1 )</th>
<th>( 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) = P(X = x) )</td>
<td>0.25</td>
<td>0.50</td>
<td>0.25</td>
</tr>
</tbody>
</table>

![Density and distribution function: flipping a coin twice](image)

(a) \( f(0) = P(X = 0) =
\[
\text{(i) } 0 \quad \text{(ii) } 0.25 \quad \text{(iii) } 0.50 \quad \text{(iv) } 0.75.
\]

(b) \( f(1) = P(X = 1) =
\[
\text{(i) } 0 \quad \text{(ii) } 0.25 \quad \text{(iii) } 0.50 \quad \text{(iv) } 0.75.
\]

(c) \( f(2) = P(X = 2) =
\[
\text{(i) } 0 \quad \text{(ii) } 0.25 \quad \text{(iii) } 0.50 \quad \text{(iv) } 0.75.
\]

(d) \( F(0) = P(X \leq 0) = P(X = 0) =
\[
\text{(i) } 0 \quad \text{(ii) } 0.25 \quad \text{(iii) } 0.75 \quad \text{(iv) } 1.
\]
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(e) $F(1) = P(X \leq 1) = P(X = 0) + P(X = 1) =$
(i) 0    (ii) 0.25    (iii) 0.75    (iv) 1.

(f) $F(2) = P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) =$
(i) 0    (ii) 0.25    (iii) 0.75    (iv) 1.

(g) Also, if $b < 0$, $F(b) = P(X \leq b) =$
(i) 0    (ii) 0.25    (iii) 0.75    (iv) 1.

(h) And, if $b \geq 2$, $F(b) = P(X \leq b) =$
(i) 0    (ii) 0.25    (iii) 0.75    (iv) 1.

(i) So, in this case,

\[ F(b) = \begin{cases} 
0, & b < 0 \\
0.25, & 0 \leq b < 1 \\
0.75, & 1 \leq b < 2 \\
1, & b \geq 2.
\]

(i) **True**    (ii) **False**

(j) The discontinuous “step function” graph of this $F(b)$ is given in Figure 2.4(b). Notice that $F(b)$ is right continuous, which is indicated by the solid and empty endpoints on the graph of this distribution function. The height of a “step” in distribution $F(b)$ is equal to the height of the corresponding “stick” in density $p(b)$.

(i) **True**    (ii) **False**

(k) $P(X < 1) = P(X = 0) = 0.25 =$
(i) $F(0)$    (ii) $F(1)$    (iii) $F(2)$    (iv) $F(3)$.

(l) $P(X < 2) = P(X = 0) + P(X = 1) = 0.75 =$
(i) $F(0)$    (ii) $F(1)$    (iii) $F(2)$    (iv) $F(3)$.

(m) $P(X > 1) = 1 - P(X \leq 1) = 1 - F(1) = 1 - 0.75 =$
(i) 0    (ii) 0.25    (iii) 0.75    (iv) 1.

6. Another discrete cumulative distribution function.

Let random variable $X$ have distribution,

\[ F(b) = \begin{cases} 
0, & b < 0 \\
\frac{1}{2}, & 0 \leq b < 1 \\
\frac{1}{3}, & 1 \leq b < 2 \\
1, & b \geq 2.
\]

(a) $f(0) = P(X = 0) =$
(i) 0    (ii) $\frac{1}{6}$    (iii) $\frac{1}{3}$    (iv) $\frac{1}{2}$.

(b) $f(1) = P(X = 1) = P(X \leq 1) - P(X \leq 0) = F(1) - F(0) =$
(i) 0    (ii) $\frac{1}{6}$    (iii) $\frac{1}{3}$    (iv) $\frac{1}{2}$. 
(c) \( f(2) = P(X = 2) = P(X \leq 2) - P(X \leq 1) = F(2) - F(1) = \)
\begin{align*}
& (i) \ 0 \\
& (ii) \ \frac{1}{6} \\
& (iii) \ \frac{1}{3} \\
& (iv) \ \frac{1}{2}.
\end{align*}

(d) Random variable \( X \) is discrete, not continuous, because the associated \( F(b) \) is a discontinuous (“step”, in this case) function.

(i) **True** (ii) **False**

(e) Notice that
\begin{align*}
(1) \ \lim_{b \to -\infty} F(b) &= 0, \\
(2) \ \lim_{b \to \infty} F(b) &= 1, \\
(3) \text{if } b_1 < b_2, \text{ then } F(b_1) &\leq F(b_2); \text{ that is, } F \text{ is nondecreasing.}
\end{align*}

(i) **True** (ii) **False**

(f) \( F(1) = P(X \leq 1) = P(X = 0) + P(X = 1) = \)
\begin{align*}
& (i) \ 0 \\
& (ii) \ \frac{1}{6} \\
& (iii) \ \frac{1}{3} \\
& (iv) \ \frac{1}{2}.
\end{align*}

(g) \( P(X < 2) = P(X = 0) + P(X = 1) = \)
\begin{align*}
& (i) \ 0 \\
& (ii) \ \frac{1}{6} \\
& (iii) \ \frac{1}{3} \\
& (iv) \ \frac{1}{2}.
\end{align*}

(h) \( P(X \leq 1.5) = P(X = 0) + P(X = 1) = \)
\begin{align*}
& (i) \ 0 \\
& (ii) \ \frac{1}{6} \\
& (iii) \ \frac{1}{3} \\
& (iv) \ \frac{1}{2}.
\end{align*}

7. **Mode and Median, Discrete.**

(a) **Number of Ears of Corn.** The number of ears of corn, \( Y \), on a typical corn plant has the following probability distribution.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>0.17</td>
<td>0.21</td>
<td>0.18</td>
<td>0.11</td>
<td>0.16</td>
<td>0.17</td>
</tr>
</tbody>
</table>

The mode number of ears of corn with largest probability,

(i) \(0\) (ii) \(2\) (iii) \(4\) (iv) \(6\).

The median number of ears of corn, \( m \), is equal to the smallest number of ears of corn where there is at least a 50% chance of getting less
than or equal to this number and also at least a 50% of getting more than this number), in other words the median $m =$

i. 2 ears of corn since there is a $17\% + 21\% = 38\%$ chance of getting less than or equal to 2 ears and a $21\% + 18\% + 11\% + 16\% + 17\% = 83\%$ chance of getting 2 or more.

ii. between 2 and 4 ears of corn since there is a $17\% + 21\% = 38\%$ chance of getting less than or equal to 2 ears and a $18\% + 11\% + 16\% + 17\% = 62\%$ chance of getting 4 or more.

iii. 4 ears of corn since there is a $17\% + 21\% + 18\% = 56\%$ chance of getting less than or equal to 4 ears and a $18\% + 11\% + 16\% + 17\% = 62\%$ chance of getting 4 or more.

(b) **Swallows.** The number of swallows, $X$, in any group of three birds is given by the following probability distribution.

<table>
<thead>
<tr>
<th>$X$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P(X = x)$</td>
<td>$\frac{1}{8}$</td>
<td>$\frac{3}{8}$</td>
<td>$\frac{3}{8}$</td>
<td>$\frac{1}{8}$</td>
</tr>
</tbody>
</table>

The mode number of swallows is the number with largest probability,

(i) 0 (ii) 1 (iii) 2 (iv) 3.

The median number of swallows, $m$, is smallest number

i. $m = 0$ because

$$P(X \leq 0) = \frac{1}{8} < 0.5 \quad \text{and} \quad P(X \geq 0) = \frac{8}{8} \geq 0.5.$$

ii. $m = 1$ because

$$P(X \leq 1) = \frac{4}{8} \geq 0.5 \quad \text{and} \quad P(X \geq 1) = \frac{7}{8} \geq 0.5.$$
iii. $m = 2$ because

\[ P(X \leq 2) = \frac{7}{8} \geq 0.5 \quad \text{and} \quad P(X \geq 2) = \frac{4}{8} \geq 0.5. \]

### 2.3 The Hypergeometric and Binomial Distributions

The *hypergeometric* probability mass function, the probability of selecting $n$ items at random without replacement from a population of size $N$ with $N_1$ type I objects and $N_2$ type II objects, $N_1 + N_2 = N$, where $x$ of the $n$ objects are type I objects, is

\[ f(x) = P(X = x) = \binom{N_1}{x} \binom{N_2}{n-x} \binom{N}{n}, \quad x = 1, \ldots, n. \]

Random variable $X$ is the number of type I objects from the sample of size $n$; $N_1$, $N_2$, $N$, and $n$ are all parameters of this pmf.

The *Bernoulli experiment* has the following properties:

1. An outcome from any Bernoulli experiment trial is either a success or failure,
2. under repetition, trial outcomes are independent, sampling is with replacement,
3. the probability of success in any trial is constant, is $p$.

Assuming a sequence of $n$ Bernoulli trials, each with probability of success $p$, the *binomial* probability mass function is,

\[ f(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \ldots, n, \]

where random variable $X$ is the number of successes in $n$ trials; $n$ and $p$ are parameters of this pmf and “$X$ is $b(n, p)$” means random variable $X$ is a binomial with parameters $n$ and $p$.

The *hypergeometric can be approximated by the binomial* as long as only a small sample, at most 5%, of the population, $n \leq 0.05N$, is selected without replacement, in which case, let

\[ p = \frac{N_1}{N}. \]

The *unusual-event principle* says if we make an assumption and this assumption leads to only a small chance of an event occurring, the event is unusual, then the assumption
is probably incorrect. For example, if we assume \( X \) is \( b(n, p) \), but find out the observed event of \( x \) successes in \( n \) trials is unusual because

\[
P(X \leq x) \leq 0.05 \quad \text{or} \quad P(X \geq x) \leq 0.05,
\]

then our assumption \( X \) is \( b(n, p) \) is possibly incorrect.

Exercise 2.3 (The Hypergeometric and Binomial Distributions)

1. Hypergeometric: sampling red marbles from an urn. Two marbles are taken, \( n = 2 \), one at a time, without replacement, from an urn which has \( N_1 = 6 \) red and \( N_2 = 10 \) blue marbles. We win $2 for each red marble chosen, lose $1 for each blue marble chosen. Let \( X \) be number of red marbles and \( Y \) be winnings.

   (a) The chance both marbles are red, \( X = 2 \), is

   \[
P\{RR\} = P(X = 2) = \binom{N_1}{x} \binom{N_2}{n-x} \binom{N}{n} = \frac{\binom{6}{2} \binom{10}{2-2}}{\binom{16}{2}} = \frac{(i) \binom{6}{2} \binom{10}{0} \binom{16}{2}}{(ii) \binom{9}{1} \binom{8}{2} \binom{16}{3}} = 0.125.
\]

   \[
dhyper(2,6,10,2) \; \# \; \text{hypergeometric pmf}
\]

\[1\] 0.125

   (b) Alternatively, \( P\{RR\} =
\]

   \[
\begin{align*}
&\frac{6}{16} \times \frac{5}{15} \quad (ii) \frac{10}{16} \times \frac{9}{15} \quad (iii) \frac{12}{16} \times \frac{11}{15} \quad (iv) \frac{16}{16} \times \frac{15}{15} = 0.125.
\end{align*}
\]

   (c) Since the winnings are \( Y = \$4 \) when both marbles are red,

   \[
P(Y = \$4) = P\{RR\} = P(X = 2) = (i) \frac{6}{10} \quad (ii) \frac{10}{15} \quad (iii) \frac{12}{16} \quad (iv) \frac{16}{20}.
\]

   (d) The chance both marbles are blue, \( X = 0 \), is

   \[
P\{BB\} = P(X = 0) = \binom{N_1}{x} \binom{N_2}{n-x} \binom{N}{n} = \frac{\binom{6}{0} \binom{10}{2}}{\binom{16}{2}} = \frac{(i) \binom{6}{2} \binom{10}{0} \binom{16}{2}}{(ii) \binom{9}{1} \binom{8}{2} \binom{16}{3}} = 0.375.
\]
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dhyper(0,6,10,2) # hypergeometric pmf

[1] 0.375

(e) Also \( P\{BB\} = P(Y = -\$2) = \)

(i) \( \frac{6}{16} \times \frac{5}{15} \)

(ii) \( \frac{10}{16} \times \frac{9}{15} \)

(iii) \( \frac{12}{16} \times \frac{11}{15} \)

(iv) \( \frac{16}{16} \times \frac{15}{15} = 0.375. \)

(f) \( P\{RB, BR\} = P(Y = \$1) = 1 - P\{RR\} - P\{BB\} = \)

(i) 0.025

(ii) 0.125

(iii) 0.225

(iv) 0.500.

(g) The hypergeometric pmf of number of red marbles, \( X \), is

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P(X = x) )</td>
<td>0.375</td>
<td>0.500</td>
<td>0.125</td>
</tr>
</tbody>
</table>

(i) True  (ii) False

(h) The hypergeometric pmf of payoffs, \( Y \), is

<table>
<thead>
<tr>
<th>( y )</th>
<th>-$2</th>
<th>$1</th>
<th>$4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P(Y = y) )</td>
<td>0.375</td>
<td>0.500</td>
<td>0.125</td>
</tr>
</tbody>
</table>

(i) True  (ii) False

2. Hypergeometric: televisions. Seven television \((n = 7)\) tubes are chosen at random from a shipment of \( N = 240 \) television tubes of which \( N_1 = 15 \) are defective.

(a) The probability that \( X = 4 \) of the chosen televisions are defective is

\[
f(4) = \frac{\binom{N_1}{x} \binom{N_2}{n-x}}{\binom{N}{n}} = \]

(i) \( \frac{15}{4} \times \frac{225}{3} \)

(ii) \( \frac{15}{3} \times \frac{225}{3} \)

(iii) \( \frac{15}{4} \times \frac{225}{7} \)

(b) The probability \( X = 4 \) of the chosen televisions are defective is

\( f(4) = \)

(i) 0.0003069

(ii) 0.0005069

(iii) 0.0006069

(iv) 0.0007069.

dhyper(4,15,225,7) # hypergeometric pmf

[1] 0.0003069143

(c) The probability \( X = 5 \) of the chosen televisions are defective is

\( f(5) = \)

(i) 0.00003069

(ii) 0.00009084

(iii) 0.00010069

(iv) 0.00013059.

dhyper(5,15,225,7) # hypergeometric pmf
Section 3. The Hypergeometric and Binomial Distributions (LECTURE NOTES 3)

(d) The probability at most \( X = 5 \) of the chosen televisions are defective is 
\[
P(X \leq 5) = \]
(i) 0.900 (ii) 0.925 (iii) 0.950 (iv) 0.999.

\[
\text{phyper}(5, 15, 225, 7, \text{lower.tail=}\text{TRUE}) \# \text{hypergeometric cdf}
\]
[1] 0.9999999

(e) The probability at least \( X = 1 \) of the chosen televisions are defective is 
\[
P(X \geq 1) = 1 - P(X < 1) = 1 - P(X = 0) =
\]
(i) 0.367 (ii) 0.435 (iii) 0.545 (iv) 0.633.

\[
\text{phyper}(0, 15, 225, 7, \text{lower.tail=}\text{FALSE}) \# 1 - \text{hypergeometric cdf}
\]
[1] 0.3672717

3. Hypergeometric: capture–recapture. To determine approximate number of perch, \( N \), in Lake Fishalot, \( n = 45 \) are captured at random from the lake, tagged and let go back into the lake. A short while later, another \( n = 32 \) perch are captured, of which \( x = 2 \) are found to be tagged. Approximately how many perch are in Lake Fishalot?

(a) The chance two of the second group of captured fish are tagged is

\[
f(2) = \left( \frac{N_1}{x} \right) \left( \frac{N_2}{n-x} \right) = \]

\[
(\text{i}) \quad \left( \frac{32}{2} \right) \times \left( \frac{N}{43} \right) = \left( \frac{N}{3200} \right)
\]

\[
(\text{ii}) \quad \left( \frac{45}{2} \right) \times \left( \frac{N - 45}{32 - 2} \right) = \left( \frac{N}{32} \right)
\]

\[
(\text{iii}) \quad \left( \frac{45}{43} \right) \times \left( \frac{N}{2} \right) = \left( \frac{N}{3200} \right)
\]

(b) Guess \( N = 500 \). In this case, chance two of 32 fish chosen are tagged is

\[
f(2) = \left( \frac{N_1}{x} \right) \left( \frac{N_2}{n-x} \right) = \left( \frac{45}{2} \right) \times \left( \frac{500 - 45}{32 - 2} \right) = \]

\[
(\text{i}) \quad 0.24 \quad (\text{ii}) \quad 0.26 \quad (\text{iii}) \quad 0.29 \quad (\text{iv}) \quad 0.32.
\]

\[
\text{dhyper}(2, 45, 455, 32) \# \text{hypergeometric pmf}
\]
[1] 0.2405091
(c) Guess $N = 750$. In this case, chance two of 32 fish chosen are tagged is

$$f(2) = \binom{N_1}{x} \binom{N_2}{n-x} \times \binom{45}{2} \times \binom{750-45}{32-2} =$$

(i) 0.24  (ii) 0.26  (iii) 0.29  (iv) 0.32.

dhyper(2,45,705,32) # hypergeometric pmf

(d) Guess $N = 1000$. In this case, chance two of 32 fish chosen are tagged is

$$f(2) = \binom{45}{2} \times \binom{750}{32} =$$

(i) 0.24  (ii) 0.26  (iii) 0.29  (iv) 0.32.

dhyper(2,45,955,32) # hypergeometric pmf

(e) A summary of results are given in the following table.

<table>
<thead>
<tr>
<th>$N$</th>
<th>500</th>
<th>750</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(2)$</td>
<td>0.24</td>
<td>0.29</td>
<td>0.26</td>
</tr>
</tbody>
</table>

Since the largest chance that 2 of 32 fish chosen are tagged is 0.29, then it seems from the choices given, the number of fish in Lake Fishalot is $N = (i)$ 500  (ii) 750  (iii) 1000  (iv) 1250.

(f) The approximation to $N$ would improve if we had more than three $f(2)$ to choose from; however, more effort would be required in calculating the extra $f(2)$. Differentiating

$$\binom{45}{2} \times \binom{N-45}{32-2} \binom{N}{32}$$

with respect to $N$ and then setting to zero, to locate the maximum $N$ is also possible, but difficult to do.

(i) True  (ii) False

4. Bernoulli: Flipping a coin. The number of heads, $X$, in one flip of a coin, is given by the following probability function,

$$f(x) = (0.25)^x(0.75)^{1-x}, \ x = 0, 1.$$ 

(a) The chance of flipping 1 head ($X = 1$) is

$$f(1) = (0.25)^1(0.75)^{1-1} = (i) \ 0 \ (ii) \ 0.25 \ (iii) \ 0.50 \ (iv) \ 0.75.$$ 

(b) This coin is (i) fair  (ii) unfair.
(c) The chance of flipping no heads \((X = 0)\) is
\[
f(0) = (0.25)^0(0.75)^{1-0} = (i) \ 0 \quad (ii) \ 0.25 \quad (iii) \ 0.50 \quad (iv) \ 0.75.
\]

(d) A “tabular” version of this probability distribution of flipping a coin is

i. Distribution A.

<table>
<thead>
<tr>
<th>(x)</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(x))</td>
<td>0.25</td>
<td>0.75</td>
</tr>
</tbody>
</table>

ii. Distribution B.

<table>
<thead>
<tr>
<th>(x)</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(x))</td>
<td>0.75</td>
<td>0.25</td>
</tr>
</tbody>
</table>

iii. Distribution C.

<table>
<thead>
<tr>
<th>(x)</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f(x))</td>
<td>0.50</td>
<td>0.50</td>
</tr>
</tbody>
</table>

(e) The number of different ways of describing a distribution include (choose one or more) (i) function (ii) table (iii) graph.

5. Airplane engines. Each engine of four \((n = 4)\) on an airplane fails 11% \((p = 0.11, q = 1 - p = 0.89)\) of the time. Assume this problem obeys the conditions of a binomial experiment.

(a) The chance two engines fail is
\[
f(2) = \binom{4}{2} 0.11^2 0.89^2 = (i) \ 0.005 \quad (ii) \ 0.011 \quad (iii) \ 0.058 \quad (iv) \ 0.157.
\]
\[
dbinom(2,4,0.11) \quad \# \ binomial \ pmf
\]
\[
[1] \ 0.05750646
\]

(b) The chance three engines fail is
\[
f(2) = \binom{4}{3} 0.11^3 0.89^1 = (i) \ 0.005 \quad (ii) \ 0.011 \quad (iii) \ 0.040 \quad (iv) \ 0.057.
\]
\[
dbinom(3,4,0.11) \quad \# \ binomial \ pmf
\]
\[
[1] \ 0.00473836
\]

(c) The chance at most two engines fail is
\[
\Pr(X \leq 2) = \sum_{x=0}^{2} \binom{4}{x} 0.11^x 0.89^{4-x} \approx
\]
\[
(i) \ 0.991 \quad (ii) \ 0.995 \quad (iii) \ 0.997 \quad (iv) \ 0.999.
\]
\[
pbinom(2,4,0.11) \quad \# \ binomial \ cdf
\]
\[
[1] \ 0.9951152
\]

(d) The chance at most three engines fail is
\[
\Pr(X \leq 3) = \sum_{x=0}^{3} \binom{4}{x} 0.11^x 0.89^{4-x} \approx
\]
\[
(i) \ 0.991 \quad (ii) \ 0.995 \quad (iii) \ 0.997 \quad (iv) \ 0.999.
\]
\[
pbinom(3,4,0.11,\text{lower.tail=TRUE}) \quad \# \ binomial \ cdf
\]
Chapter 2. Discrete Random Variables (LECTURE NOTES 3)

(e) The chance at least three engines fail is
\[ P(X \geq 3) = 1 - P(X \leq 2) \approx \]
(i) 0.005 (ii) 0.010 (iii) 0.016 (iv) 0.023.

(f) Since
\[ P(X \leq 3) \approx 0.999 > 0.05 \quad \text{and} \quad P(X \geq 3) \approx 0.005 < 0.05, \]
it (i) is (ii) is not unusual 3 engines fail, assuming 11% engine failure.

Let \( 0 \leq x \leq 1, 0 \leq y \leq 1 \), where part of the square, \( x < y \), is shaded.

(a) If points are taken at random from this square, determine the probability of choosing a point from the shaded region, which is half the square.
\[ p = (i) \, 0.025 \quad (ii) \, 0.050 \quad (iii) \, 0.069 \quad (iv) \, 0.500. \]

(b) If two points are taken at random from a square, what is the chance both are from the shaded region? Clearly,
\[ p^2 = 0.5^2 = (i) \, 0.25 \quad (ii) \, 0.50 \quad (iii) \, 0.069 \quad (iv) \, 0.119 \]
or, using binomial pmf, since \( p = 0.5, n = 2, \) and \( x = 2 \), then
\[ f(2) = \binom{2}{2} 0.5^2 0.5^0 = (i) \, 0.25 \quad (ii) \, 0.050 \quad (iii) \, 0.069 \quad (iv) \, 0.119. \]

(c) If ten points are taken at random from a square, what is the chance seven are from the shaded region? Using binomial pmf, since \( p = 0.5, n = 10, \) and \( x = 7 \), then
\[ f(7) = \binom{10}{7} 0.5^7 0.5^3 = (i) \, 0.117 \quad (ii) \, 0.252 \quad (iii) \, 0.369 \quad (iv) \, 0.419. \]
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\[ \text{dbinom}(7,10,0.5) \]  
\[ \text{# binomial pmf} \]  
\[ 0.1171875 \]

(d) If \( n = 11, p = 0.5, x = 3, \) then  
\[ f(3) = \binom{11}{3} 0.5^3 0.5^8 = (i) 0.081 \]  
\[ (ii) 0.231 \]  
\[ (iii) 0.258 \]  
\[ (iv) 0.319. \]

\[ \text{dbinom}(3,11,0.5) \]  
\[ \text{# binomial pmf} \]  
\[ 0.08056641 \]

7. **Multiple choice questions.** On a multiple–choice exam with 4 possible answers for each of the 5 questions, what is the probability that a student should get at most 3 correct answers just by guessing?

(a) Since there are five questions,  
\[ n = (i) 2 \]  
\[ (ii) 3 \]  
\[ (iii) 4 \]  
\[ (iv) 5. \]

(b) Since a student wants 3 or more correct answers,  
\[ x = (i) 2 \]  
\[ (ii) 2, 3 \]  
\[ (iii) 2, 3, 4 \]  
\[ (iv) 3, 4, 5. \]

(c) Since the student is choosing from 4 answers at random,  
\[ p = (i) \frac{1}{2} \]  
\[ (ii) \frac{1}{3} \]  
\[ (iii) \frac{1}{4} \]  
\[ (iv) \frac{1}{5}. \]

(d) The chance a student should get at most 3 correct answers is  
\[ P(X \leq 3) = \sum_{x=0}^{3} \binom{5}{x} \left( \frac{1}{4} \right)^x \left( \frac{3}{4} \right)^{5-x} \approx \]
\[ (i) 0.697 \]  
\[ (ii) 0.704 \]  
\[ (iii) 0.812 \]  
\[ (iv) 0.984. \]

\[ \text{pbinom}(3,5,0.25) \]  
\[ \text{# binomial cdf} \]  
\[ 0.984375 \]

(e) The chance a student should get at least 3 correct answers is  
\[ P(X \geq 3) = 1 - P(X \leq 2) \approx \]
\[ (i) 0.097 \]  
\[ (ii) 0.104 \]  
\[ (iii) 0.112 \]  
\[ (iv) 0.284. \]

\[ 1 - \text{pbinom}(2,5,0.25) \]  
\[ \text{# 1 - binomial cdf} \]  
\[ 0.1035156 \]

(f) Since  
\[ P(X \leq 3) = 0.984 > 0.05 \]  
\[ \text{and} \]  
\[ P(X \geq 3) = 0.104 > 0.05, \]

it (i) **is**  
(ii) **is not** unusual to score 3 correct answers, guessing at random.

8. **Lawyer.** A lawyer estimates she wins 40% \( (p = 0.4) \) of her cases. Assume each trial is independent of one another and, in general, this problem obeys the conditions of a binomial experiment. The lawyer presently represents 10 \( (n = 10) \) defendants.

(a) The tabular form of this distribution is given by,
Chapter 2. Discrete Random Variables (LECTURE NOTES 3)

where, for example, the chance of her winning 6 of 10 cases is 0.111. In a similar way, the chance of her winning 4 of 10 cases is

(i) 0.121  (ii) 0.215  (iii) 0.251  (iv) 0.351.

(b) The graphical form of the probability distribution is given in Figure 2.8.

The number of cases the lawyer has most chance of winning is

(i) **one**  (ii) **two**  (iii) **three**  (iv) **four**.

(c) The binomial distribution, in this case, is

(i) **skewed left**  (ii) **skewed right**  (iii) **more or less symmetric**.

(d) In general, the binomial distribution will sometimes, but not always, be symmetric. If her chance of winning was $p = 0.1$ instead of $p = 0.4$, the binomial distribution would be

(i) **skewed left**  (ii) **skewed right**  (iii) **more or less symmetric**.

(e) **Understanding the binomial formula.**

Chance of nine wins and then a lose would be $p^9q = 0.4^9 0.6^1$, eight wins, one lose, one win would be: $p^8qp = p^9q = 0.4^9 0.6^1$, seven wins, one lose, two wins would be: $p^7qp^2 = p^9q = 0.4^9 0.6^1$, . . . . Since nine wins occurs ten different ways, chance nine wins would be:

$$10 \times p^9q = 10 \times 0.4^9 0.6^1 = \binom{10}{9} 0.4^9 0.6^1.$$  

In general, chance of $x$ wins in $n$ trials is binomial formula,

$$\binom{n}{x} p^x q^{n-x}, \ x = 0, 1, \ldots, n$$

(i) **True**  (ii) **False**

(f) **Conditional versus unconditional binomial.**
Since $P(X > 3) = 1 - P(X \leq 3) \approx 0.46$ (ii) 0.58 (iii) 0.62 (iv) 0.67,
$1 - \text{pbinom}(3,10,0.4) \# 1 - \text{binomial cdf}$

and

$$P(X > 4 + 3 | X > 4) = \frac{P(X > 4 + 3, X > 4)}{P(X > 4)} = \frac{P(X > 4 + 3)}{P(X > 4)} \approx$$

(i) 0.034 (ii) 0.048 (iii) 0.075 (iv) 0.089;

$(1 - \text{pbinom}(7,10,0.4))/(1 - \text{pbinom}(4,10,0.4)) \# \text{conditional probability}$

in other words, $P(X > 3) = 0.62 \neq P(X > 4 + 3 | X > 4) = 0.03$. The chance the lawyer wins at least 4 cases is not equal to the chance she wins at least 8 cases, given winning at least 5 cases.

9. Binomial approximation to the hypergeometric: televisions. Seven television tubes are chosen at random from a shipment of $N = 240$ television tubes of which $N_1 = 15$ are defective.

(a) We sample without replacement; that is, every time a TV is chosen, we do not replace it to be potentially chosen again. In other words, the chance of choosing a defective TV, every time a TV is chosen, changes or depends on the number of defective TVs that were chosen before it.

(i) True (ii) False

(b) If the sample size, $n$, is small relative to the number of televisions, $N$, $n < 0.05N$, say, the hypergeometric can be approximated by a binomial. The chance, $p = \frac{N_1}{N}$, of choosing a defective TV, every time a TV is chosen, does not change “that much” when $\frac{n}{N} < 0.05$. Since $\frac{n}{N} = \frac{7}{240} \approx 0.029 < 0.05$, the binomial will probably approximate the hypergeometric

(i) very closely. (ii) somewhat closely. (iii) not closely at all.

(c) (Exact) hypergeometric. Probability $x = 4$ of $n = 7$ chosen TVs are defective is

$$f(4) = \frac{\binom{N_1}{x} \binom{N_2}{n-x}}{\binom{N}{n}} = \frac{\binom{15}{4} \times \binom{225}{3}}{\binom{240}{7}} =$$

(i) 0.0003069 (ii) 0.0004400 (iii) 0.0006069 (iv) 0.0007069.

dhyper(4,15,225,7) \# exact hypergeometric pmf
Chapter 2. Discrete Random Variables (LECTURE NOTES 3)

(d) Approximate binomial. Since \( p = \frac{N}{N} = \frac{15}{240} = 0.0625 \) and so a binomial approximation to probability \( x = 4 \) of \( n = 7 \) chosen TVs are defective is:

\[
f(4) = \binom{7}{4} 0.0625^4 (1 - 0.0625)^{7-4} =
\]

(i) 0.0003069 (ii) 0.0004400 (iii) 0.0006069 (iv) 0.0007069.

dbinom(4,7,0.0625) # approximate binomial pmf

[1] 0.0004400499

10. Multinomial: faculty and subjects. There are 9 different faculty members and 3 subjects: mathematics, statistics and physics. There is a 50%, 35% and 15% chance a faculty member teaches mathematics, statistics and physics, respectively.

(a) Chance 4, 3 and 2 faculty members teach mathematics, statistics and physics, respectively, is

\[
p(4,3,2) = \frac{9!}{4!3!2!} 0.5^4 0.35^3 0.15^2 \approx
\]

(i) 0.055 (ii) 0.067 (iii) 0.076 (iv) 0.111

dmultinom(c(4,3,2), prob = c(0.5,0.35,0.15)) # multinomial pmf

[1] 0.07596914

(b) Chance 4, 4 and 1 faculty members teach mathematics, statistics and physics, respectively, is

\[
p(4,4,1) = \frac{9!}{4!4!1!} 0.5^4 0.35^4 0.15^1 \approx
\]

(i) 0.089 (ii) 0.098 (iii) 0.108 (iv) 0.131

dmultinom(c(4,4,1), prob = c(0.5,0.35,0.15)) # multinomial pmf

[1] 0.08863066

2.4 The Poisson Distribution

Poisson probability mass function, used to describe probabilities of count, \( X \), of occurrences (successes) of an event in a specified time (or space) interval, is

\[
f(x) = P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!} \quad x = 0, 1, \ldots,
\]

where \( \lambda > 0 \) is average number of occurrences of an event; also, random variable \( X \) obeys conditions of Poisson process:
Section 4. The Poisson Distribution (LECTURE NOTES 3)

- zero probability of two or more successes in sufficiently small subinterval,
- probability of success same if two intervals are of equal length,
- number successes in one interval independent of number of successes in any other nonoverlapping interval.

If \( n \geq 100, np \leq 100 \), Poisson pmf can be used to approximate binomial pmf where \( \lambda = np \).

**Exercise 2.4 (The Poisson Distribution)**

1. Poisson: number of photon hits.

Piece of iron bombarded with electrons releases a number of photons, \( X \), to a surrounding magnetic detection field. An average of \( \lambda = 5 \) particles hit magnetic detection field *per (one) microsecond*. Assume this is a Poisson process.

(a) Chance \( x = 0 \) photons hit field in one microsecond

\[
    f(0) = \frac{\lambda^x}{x!} e^{-\lambda} = \frac{5^0}{0!} e^{-5} \approx 0.006737947
\]

   (i) 0.007 (ii) 0.008 (iii) 0.009.

   dpois(0,5) # Poisson pmf

   [1] 0.006737947

(b) Chance \( x = 2 \) photons hit field in one microsecond

\[
    f(2) = \frac{\lambda^x}{x!} e^{-\lambda} = \frac{5^2}{2!} e^{-5} \approx 0.08422434
\]

   (i) 0.06 (ii) 0.07 (iii) 0.08.

   dpois(2,5) # Poisson pmf

   [1] 0.08422434

(c) Chance at most \( x = 2 \) photons hit field in one microsecond

\[
    P(X \leq 2) = \frac{5^0}{0!} e^{-5} + \frac{5^1}{1!} e^{-5} + \frac{5^2}{2!} e^{-5} \approx 0.124652
\]

   (i) 0.08 (ii) 0.12 (iii) 0.18.

   ppois(2,5) # Poisson cdf

   [1] 0.124652
(d) Chance at least \( x = 2 \) photons hit field in one microsecond

\[
P(X \geq 2) = \frac{5^2}{2!}e^{-5} + \frac{5^3}{3!}e^{-5} + \cdots \text{ forever}
= 1 - P(X < 2) = 1 - P(X \leq 1)
= 1 - [P(0) + P(1)]
= 1 - \left[\frac{5^0}{0!}e^{-5} + \frac{5^1}{1!}e^{-5}\right] \approx
\]

(i) 0.91 (ii) 0.93 (iii) 0.96.

1 - ppois(1,5) # 1 - Poisson cdf
ppois(1,5,lower.tail=FALSE) # 1 - Poisson cdf

[1] 0.9595723

(e) Chance \( x = 2 \) photons hit field in two microseconds

\[
P(2) = \frac{(5 \cdot 2)^2}{2!}e^{-5 \cdot 2} \approx
\]

(i) 0.002 (ii) 0.005 (iii) 0.006.

dpois(2,10) # Poisson pmf, 2 x 5 = 10

[1] 0.002269996

(f) Chance at most \( x = 3 \) photons hit field in two microseconds

\[
P(X \leq 3) \approx
\]

(i) 0.01 (ii) 0.06 (iii) 0.08.

ppois(3,10) # Poisson cdf, 2 x 5 = 10

[1] 0.01033605

(g) Chance at least \( x = 21 \) photons hit field in four microseconds

\[
P(X \geq 21) = 1 - P(X < 21) = 1 - P(X \leq 20) \approx
\]

(i) 0.44 (ii) 0.52 (iii) 0.68.

1 - ppois(20,20) # 1 Poisson cdf, 4 x 5 = 20

[1] 0.4409074

(h) Poisson process? Match columns.

<table>
<thead>
<tr>
<th>Poisson process</th>
<th>photon example</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) zero chance more than one success in small subinterval</td>
<td>(A) assume “zero” chance two hits at same time</td>
</tr>
<tr>
<td>(b) chance of success same if two intervals are of equal length</td>
<td>(B) assume chance number hits same per microsecond</td>
</tr>
<tr>
<td>(c) independence of successes in different intervals</td>
<td>(C) assume hits independent of one another</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Poisson process</th>
<th>(a)</th>
<th>(b)</th>
<th>(c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>photon example</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
2. **Poisson: Bryozoan count.**

During a biology study, 250 bryozoans are found attached on a submerged 10 centimeter by 10 centimeter (100 centimeters$^2$) plate, an average of $\lambda = \frac{250}{100} = 2.5$ bryozoans per one centimeter$^2$. Assume this is a Poisson process.

(a) Chance $x = 0$ bryozoans attached in one centimeter$^2$

$$f(0) = \frac{\lambda^x}{x!} e^{-\lambda} = \frac{2.5^0}{0!} e^{-2.5} \approx 0.07 \quad (ii) \ 0.08 \quad (iii) \ 0.09.$$

dpois(0,2.5) \ # Poisson pmf

(b) Chance at most $x = 2$ bryozoans attached in one centimeter$^2$

$$P(X \leq 2) \approx$$

(i) 0.54 \quad (ii) 0.62 \quad (iii) 0.78.

ppois(2,2.5) \ # Poisson cdf

(c) Chance between $x = 2$ and $x = 4$ bryozoans in one centimeter$^2$, inclusive:

$$P(2 \leq X \leq 4) = P(X \leq 4) - P(X < 2) = P(X \leq 4) - P(X \leq 1) \approx$$

(i) 0.60 \quad (ii) 0.73 \quad (iii) 0.76.

ppois(4,2.5) - ppois(1,2.5) \ # between Poisson cdfs

(d) Chance between $x = 2$ and $x = 4$ bryozoans in three centimeters$^2$, inclusive:

$$P(2 \leq X \leq 4) = P(X \leq 4) - P(X < 2) = P(X \leq 4) - P(X \leq 1) \approx$$

(i) 0.13 \quad (ii) 0.23 \quad (iii) 0.36.

ppois(4,7.5) - ppois(1,7.5) \ # between Poisson cdfs, 3 x 2.5 = 7.5

[1] 0.1273606

3. **Poisson approximation of the binomial: photons.** What is the chance $x = 15$ particles hit in one microsecond? The Poisson distribution can be used to approximate the binomial distribution by letting $\lambda = np$. This is a fairly good approximation if $n \geq 100$ and $np \leq 100$.
(a) Assume $n = 2000$ particles are released by iron per microsecond, and there is a chance $p = 0.005$ that a particle hits the surrounding field per microsecond. Since $n = 2000$ and $p = 0.005$, then $n = 2000 > 100$ and $np = 2000(0.005) = 10 \leq 100$ and so the conditions for approximation are satisfied (ii) violated.

(Exact) binomial

\[ f(15) = \frac{2000!}{15!(2000-15)!} \times (0.005)^{15} \times (0.995)^{1985} \approx \]

(i) 0.00234  (ii) 0.0346  (iii) 0.0445  (iv) 0.0645.

dbinom(15,2000,0.005) # exact binomial pmf

[1] 0.03463059

Approximate Poisson.

since $\lambda = np = 2000(0.005) =$ (i) 5  (ii) 10  (iii) 15  (iv) 20, then

\[ f(15) = \frac{10^{15}}{15!}e^{-10} \approx \]

(i) 0.00234  (ii) 0.0347  (iii) 0.0445  (iv) 0.0645

dpois(15,10) # approximate Poisson pmf

[1] 0.03471807

(b) If $n = 1500$ and $p = 0.01$, then $n = 1500 > 100$ and $np = 1500(0.01) = 15 \leq 100$ and so the conditions for approximation are satisfied (ii) violated.

(Exact) binomial

\[ f(15) = \frac{1500!}{15!(1500-15)!} \times (0.01)^{15} \times (0.99)^{985} \approx \]

(i) 0.1024  (ii) 0.1030  (iii) 0.1245  (iv) 0.1345.

dbinom(15,1500,0.01) # exact binomial pmf

[1] 0.1029519

Approximate Poisson

since $\lambda = np = 1500(0.01) =$ (i) 5  (ii) 10  (iii) 15  (iv) 20, then

\[ f(15) = \frac{15^{15}}{15!}e^{-15} \approx \]

(i) 0.1024  (ii) 0.1030  (iii) 0.1345  (iv) 0.1445

dpois(15,15) # approximate Poisson pmf
4. Poisson process: photons. A random variable has a Poisson distribution if the Poisson process conditions are satisfied. For the photons example, this would mean the following assumptions are satisfied.

- $P$(no “hit” occurs in time subinterval)$= 1 - p$
- $P$(one “hit” occurs in time subinterval)$= p$
- $P$(two or more “hits” occurs in time subinterval)$= 0$
- Occurrence of hit in each time subinterval is independent of occurrence of event in other nonoverlapping subintervals.

![Figure 2.9: Poisson process: sequence of photon hits](image)

(a) The Poisson process assumes all time subintervals are created so small one and only one photon could hit the magnetic detection field during each subinterval. During seven time subintervals shown in Figure 2.9, three hits occur in time subintervals 1, 2 and 7 and four misses occur in time subintervals 3, 4, 5 and 6.
(i) True  (ii) False

(b) The Poisson process assumes the chance a photon hits the magnetic detection field during any one time subinterval is $p$ and the chance it misses is $q = 1 - p$. It is impossible to have more than one photon hit the magnetic detection field during any time subinterval. Consequently, the time subintervals are infinitesimally small.
(i) True  (ii) False

(c) The Poisson process assumes a hit in each time subinterval is independent of any other hit in other nonoverlapping subintervals.
(i) True  (ii) False

(d) If there was a 20% chance, $p = 0.2$, a photon hit the magnetic detection field during any time subinterval then the chance of $y = 3$ hits in the $n = 7$ time subintervals is given by the binomial,

$$f(3) = \binom{n}{x} p^x q^{n-x} = \binom{7}{3} 0.2^3 0.8^{7-3} \approx$$

(i) 0.089  (ii) 0.115  (iii) 0.124  (iv) 0.134.
(e) If a large number of time subintervals are considered; in other words, as, \( n \) gets bigger, in fact, as \( n \to \infty \), it becomes increasingly difficult to calculate the probability of a number of hits using the binomial. In this case, the poisson is used instead.

(i) **True**  (ii) **False**

(f) In general,

\[
\lim_{n \to \infty} \binom{n}{x} p^x q^{n-x} = \frac{\lambda^x}{x!} e^{-\lambda}
\]

where \( \lambda = np \).

(i) **True**  (ii) **False**

5. **Poisson: stars on tiles.**

Twenty-four (24) stars are placed on a tiled floor with 90 tiles and it is found, for example, 70 of tiles had no (0) stars, 17 tiles had 1 star and so on. Does this distribution of stars approximately follow a Poisson distribution?

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>frequency</td>
<td>70</td>
<td>17</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

(a) Fill in the blank for the relative frequency of stars per tile table:
Section 4. The Poisson Distribution (LECTURE NOTES 3)

The Poisson pmf (i) is (ii) is not a close approximation to the relative frequency of stars per tiles.

(b) Average number of stars per tile is

\[ \lambda = \frac{24}{90} \]

(i) 0.21  (ii) 0.24  (iii) 0.27  (iv) 0.29.

(c) Assuming Poisson where \( \lambda = 0.27 \), chance \( x = 0 \) stars on one tile is

\[ f(0) = \frac{\lambda^x}{x!}e^{-\lambda} = \frac{0.27^0}{0!}e^{-0.27} \approx \]

(i) 0.01  (ii) 0.02  (iii) 0.19  (iv) 0.76.

\[ \text{dpois}(0, 0.27) \# \text{Poisson pmf} \]
[1] 0.7633795

(d) Fill in the blank for the Poisson pmf \( f(x) \) of stars per tile table:

\[
\begin{array}{c|ccc}
 x & 0 & 1 & 2 & 3 \\
 \hline
 \text{frequency} & 70 & 17 & 2 & 1 \\
 \text{relative frequency} & \frac{70}{90} \approx 0.78 & 0.19 & 0.02 & 0.01 \\
 f(x) & 0.76 & 0.03 & 0.00 & \\
\end{array}
\]

The Poisson pmf (i) is (ii) is not a close approximation to the relative frequency of stars per tiles.

\[ \text{dpois}(0:3, 0.27) \# \text{Poisson pmf for 0,1,2,3} \]
[1] 0.763379494 0.206112463 0.027825183 0.002504266